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SINGULAR HOMOLOGY ON AN UNTRIANGULATED MANIFOLD

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1. Objectives

This paper is concerned with singular homology over $\boldsymbol{Z}$ on a compact, connected differentiable manifold M_n of class C^∞. We suppose that there is given on M_n a polar nondegenerate[1] function f of class C^∞, and that $n > 1$.

This paper continues a program with two objectives. The *first* objective is to relate the existence and characteristics of critical points of f to invariants (the Betti numbers and torsion coefficients of the different dimensions) of the singular homology groups of M_n sufficient to determine these homology group up to an isomorphism. The *second* objective is to accomplish this without any global triangulation of M_n. This is a prelude to a similar study of topological manifolds which admit no triangulatoin.

The cogency of the second objective became evident in Morse's study of global variational analysis. The function spaces thereby arising are in general not even locally compact. To make the global theory depend on triangulations imposes difficulties which obscure the relations between the critical elements and the topology. This first historical reason was reinforced by the conviction that topological manifolds which admit topologically *ND* functions (see [3]) are more general than those which admit triangulations (see [1]). This last conviction is being further substantiated by current research of R. C. Kirby and L. C. Siebenmann. See [2].

The present paper continues the development in [5] of singular homology over $\boldsymbol{Z}$ on M_n. In [5] the following condition was imposed on f.

Condition C_0 on f. *Under condition C_0, f has different values a at different critical points.*

The following theorem was proved in [5]. Its terms are there defined.

Theorem 0.1 of [5]. *Under Condition C_0 on f there exists an inductive group-theoretic mechanism by virtue of which relative numerical invariants, associated with the critical points of f on each subset*

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[1] We abbreviate the word nondegenerate by *ND*. Polar *ND* functions are shown to exist on M_n in [4].

$$f_c = \{x \in |M_n| \mid f(x) \leq c\} \tag{1.1}$$

of the carrier $|M_n|$ *of* M_n, *determine, up to an isomorphism, the singular homology groups over* $\boldsymbol{Z}$ *of the subspace* f_c *of* M_n.

Paper [6] is concerned with the "orientability" of M_n. A current definition affirms that M_n is *orientable* if and only if its n-th Betti number is 1. We term such orientability "homological," and introduce what we term *geometric orientability*, defining such orientability without reference to homology or triangulation of M_n. A fundamental theorem of [6] follows.

Corollary 9.1 of [6]. *The manifold* M_n *is geometrically orientable if and only if its n-th Betti number is* 1.

In treating certain aspects of singular homology theory over $\boldsymbol{Z}$ the Condition C_0 on f is too restrictive. In this paper we shall replace Condition C_0 by the following condition.

Condition C_1 on f. *Under Condition* C_1, *critical points of f with different indices shall have different critical values. Critical points with the same index may or may not have the same critical values.*

If Condition C_0 is satisfied, Condition C_1 is satisfied. We shall review some of the theorems of [5] established under Condition C_0 on f, and give these theorems new forms under Condition C_1 on f. In § 6 we shall return to a study of orientability and prove the following without making use of any triangulation of M_n.

Theorem 1.0. *The torsion subgroup* $\mathscr{T}_{n-1}(|M_n|)$ *of* $H_{n-1}(|M_n|, \boldsymbol{Z})$ *is trivial or of order* 2 *according as* M_n *is geometrically orientable or not.*

We prepare for the topological analysis of § 2 by Lemma 1.1 below. "Coset-contracting" isomorphisms are characterized in Theorem 1.2 of [5].

Coset-contracting isomorphisms. Extensive use of such isomorphisms will be made in § 2. We shall here prove a useful lemma.

Let χ be a Hausdorff space and A a subspace of χ, possibly empty. If $A \neq \chi$ we term (χ, A) an *admissible set pair*, and A a *modulus* of χ. Let $q \geq 0$ be an integer. Let (χ, A) and (χ', A') be admissible set pairs with (χ, A) "*preceding*" (χ', A'), in the sense that $\chi \supset \chi'$ and $A \supset A'$. A coset-contracting isomorphism

$$H_q(\chi, A, \boldsymbol{Z}) \underset{\approx}{\rightarrow} H_q(\chi', A', \boldsymbol{Z}) \tag{1.2}$$

is defined in Theorem 1.2 of [5]. The arrow $\rightarrow$ above $\approx$ indicates that a relative homology class U of the group $Z_q(\chi, A, \boldsymbol{Z})$ of singular q-cycles on χ mod A over $\boldsymbol{Z}$ corresponds under the isomorphism (1.2) to a relative homology class U' of $Z_q(\chi', A', \boldsymbol{Z})$, such that $U \supset U'$. The conditions (a) and (b) of Theorem 1.2 of [5] are necessary and sufficient that (1.2) hold.

To formulate Lemma 1.1 let there be given admissible set pairs

$$(\chi, A)\ , \quad (\chi', A')\ , \quad (\chi'', A'') \tag{1.3}$$

with order of "precedence" the order of writing in (1.3).

Lemma 1.1. *Sufficient conditions that* (1.2) *hold are that*

$$H_q(\chi, A, Z) \underset{\approx}{\rightarrow} H_q(\chi'', A'', Z) , \tag{1.4}$$

$$H_q(\chi', A', Z) \underset{\approx}{\rightarrow} H_q(\chi'', A'', Z) . \tag{1.5}$$

The reader can show that the coset-contracting isomorphisms (1.4) and (1.5) imply that conditions (a) and (b) of Theorem 1.2 of [5] are satisfied and hence that (1.2) holds.

2. f-Saddles

Let a be a critical value of f and f_a the corresponding closed sublevel set defined in (1.1). Let $p_a^1, p_a^2, \cdots, p_a^{r_a}$ be the critical points at the f-level a. By Condition C_1 on f these points have the same index k. We say that a then has the *index* k. If a is the minimum or maximum of f on $|M_n|$, then $r_a = 1$. We introduce the subspace

$$f_a^- = f_a - p_a^1 - \cdots - p_a^{r_a} \qquad \text{(cf. (2.4) of [5])} \tag{2.1}$$

of f_a. When $r_a = 1$ we may denote p_a^1 by p_a and f_a^- by $\dot{f}_a$ as in [5]. The principal use of f_a^- is as a *modulus* associated with f_a in the set pair (f_a, f_a^-) when $0 < k \leq n$.

The sets N_a^i. Suppose that $0 <$ (index a) $\leq n$. With the critical points $p_a^1, \cdots, p_a^{r_a}$ we associate open subsets

$$N_a^1, \cdots, N_a^{r_a} \tag{2.2}$$

of f_a which contain the respective points $p_a^1, \cdots, p_a^{r_a}$ and have disjoint closures. For each i we set $\dot{N}_a^i = N_a^i - p_a^i$.

Definition 2.1. *f-saddles.* Set (index a) $= k$ and suppose that $0 < k \leq n$. A C^∞-manifold L_k^i which is the C^∞-diffeomorph of an open euclidean k-ball B_k^e of radius e and which is C^∞-embedded in M_n so as to meet p_a^i, will be termed an *f-saddle L_k^i at p_a^i* if the following is true.

(i) *The point p_a^i is a ND critical point of $f | L_k^i$ of index k;*

(ii) $\dot{N}_a^i \supset |\dot{L}_k^i|$, *where* $|\dot{L}_k^i| = |L_k^i| - p_a^i$.

One should compare Definition 36.2 of [7] with this definition.

Subsaddles of L_p^i. For fixed a and i an "f-saddle $\mathscr{L}_k^i$ at p_a^i" such that $|L_k^i| \supset |\mathscr{L}_k^i|$ will be called a *subsaddle* of L_k^i. If $\mathscr{L}_k^i$ is a subsaddle of L_k^i then for each integer $q \geq 0$ there exists a coset-contracting isomorphism

$$H_q(|L_k^i|, |\dot{L}_k^i|, Z) \underset{\approx}{\rightarrow} H_q(|\mathscr{L}_k^i|, |\dot{\mathscr{L}}_k^i|, Z) . \tag{2.3}$$

The Excision Theorem 1.3 of [3] implies (2.3) on setting

$$\chi = |L_k^i| , \quad A = |\dot{L}_k^i| , \quad A^* = |L_k^i| - |\mathscr{L}_k^i| . \tag{2.4}$$

Lemma 2.1, which follows, is an extension of (2.6) of [5]. In Lemma 2.1 the right member of (2.5) is the "external direct sum" of the groups indexed by i. The range of i is $1, 2, \cdots, r_a$.

Lemma 2.1. *If* (*index* a) $= k$ *and if* $0 < k \leq n$, *there exists for each integer* $q \geq 0$ *an isomorphism*

$$H_q(f_a, f_a^-, Z) \approx \bigoplus_{i=1}^{r_a} H_q(|L_k^i|, |\dot{L}_k^i|, Z) \; . \tag{2.5}$$

The relation (2.5) is a consequence of the coset-contracting isomorphism,

$$H_q(f_a, f_a^-, Z) \underset{\approx}{\rightarrow} H_q(\bigcup_{i=1}^{r_a} N_a^i, \bigcup_{i=1}^{r_a} \dot{N}_a^i, Z) \; , \tag{2.6}$$

and of r_a coset-contracting isomorphisms,

$$H_q(N_a^i, \dot{N}_a^i, Z) \underset{\approx}{\rightarrow} H_q(|L_k^i|, |\dot{L}_k^i|, Z) \qquad (i = 1, \cdots, r_a) \tag{2.7}$$

now to be established.

Proof of (2.6). The Excision Theorem 1.3 of [5] implies (2.6) on setting

$$\chi = f_a \; , \quad A = f_a^- \; , \quad A^* = f_a - \bigcup_{i=1}^{r_a} N_a^i \; . \tag{2.8}$$

Proof of (2.7). We shall make use of Lemma 1.1 to prove (2.7). The analysis on page 330 of [7] shows[2] that for fixed a and i there exists an open neighborhood[2] Y^i of p_a^i with $f_a \cap Y^i \subset N_a^i$ and a subsaddle[2] $\mathscr{L}_k^i$ of L_k^i so small that the following is true:

Proposition 2.1. *There exists a deformation retracting* $f_a \cap Y^i$ *onto* $|\mathscr{L}_k^i|$ *and* $f_a^- \cap Y^i$, *onto* $|\dot{\mathscr{L}}_k^i|$.

For fixed i Proposition 2.1 is similar to Proposition 36.1 of [7] and is proved similariy. It follows as in the proof of (36.1) of [7] that

$$H_q(f_a \cap Y^i, f_a^- \cap Y^i, Z) \underset{\approx}{\rightarrow} H_q(|\mathscr{L}_k^i|, |\dot{\mathscr{L}}_k^i|, Z) \; . \tag{2.9}$$

Since $f_a \cap Y^i \subset N_a^i$ the Excision Theorem 1.3 of [5] implies that

$$H_q(N_a^i, \dot{N}_a^i, Z) \underset{\approx}{\rightarrow} H_q(f_a \cap Y^i, f_a^- \cap Y^i, Z) \; . \tag{2.10}$$

The isomorphism (2.10) followed by the isomorphism (2.9) implies that

$$H_q(N_a^i, \dot{N}_a^i, Z) \underset{\approx}{\rightarrow} H_q(|\mathscr{L}_k^i|, |\dot{\mathscr{L}}_k^i|, Z) \; . \tag{2.11}$$

Relations (2.11) *and* (2.3) *yield* (2.7) *in accord with Lemma* 1.1.

[2] In [7], one is concerned with one critical point p_a and one neighborhood Y of p_a in f_a, and L_k is a subsaddle which could have been denoted by $\mathscr{L}_k$.

The proof of Lemma 2.1 *concluded.* The fact that the sets $N_a^1, \cdots, N_a^{r_a}$ have disjoint closures implies that the right member of (2.6) is isomorphic to the direct sum,

$$\bigoplus_{i=1}^{r_a} H_q(N_a^i, \dot{N}_a^i, Z) \ . \qquad \text{(cf. (2.3) in [8].)} \tag{2.12}$$

With this understood (2.6) and (2.7) imply (2.5). Thus Lemma 2.1 is true.

3. Universal k-caps

Definition 3.0. Let p_a^i, $i = 1, 2, \cdots \cdot r_a$, be the set of critical points of index k at a level a of f, with $0 < k \leq n$. A singular k-cell σ^k which is simply-carried, in the sense of Definition 26.2b of [7], by an f-saddle L_k^i of p_a^i with p_a^i interior to $|\sigma^k|$ will be denoted by $\kappa_a^{k,i}$ and termed a *universal k-cap at* p_a^i.

The k-cap $\kappa_a^{k,i}$ is termed "universal" because it is a k-cap of p_a over each field $\mathscr{K}$, as the Carrier Theorem 36.2 of [7] implies. See Definition 29.1 of [7] of k-caps over $\mathscr{K}$.

We supplement Definition 3.0 by the convention that when a is the absolute minimum of f on $|M_n|$ the 0-cell carired by the critical point p_a at the level a is a *universal* 0-cap.

Given a universal k-cap $\kappa_a^{k,i}$ we shall set

$$|\kappa_a^{k,i}| - p_a^i = |\dot{\kappa}_a^{k,i}| \tag{3.1}$$

and verify the following theorem.

Theorem 3.1. *If* (*index a*) $= k$ *and* $0 < k \leq n$ *then, for each integer* $q \geq 0$, *there exists an isomorphism*

$$H_q(f_a, f_a^-, Z) \approx \bigoplus_{i=1}^{r_a} H_q(|\kappa_a^{k,i}|, |\dot{\kappa}_a^{k,i}|, Z) \ . \tag{3.2}$$

Let L_k^i be an f-saddle at p_a^i such that

$$|\kappa_a^{k,i}| \subset |L_k^i| \qquad (i = 1, \cdots, r_a) \ . \tag{3.3}$$

The Excision Theorem then implies that

$$H_q(|L_k^i|, |\dot{L}_k^i|, Z) \underset{\approx}{\rightarrow} H_q(\kappa_a^{k,i}|, |\dot{\kappa}_a^{k,i}|, Z) \tag{3.4}$$

so that (3.2) follows from (2.5).

Corollary 3.1. *Under the hypotheses of Theorem* 3.1 *the group* $H_q(f_a, f_a^-,$ $Z)$ *is finitely generated and free. When* $q \neq k$ *this group is trivial and when* $q = k$ *has the set*

$$\kappa_a^{k,i}, \cdots, \kappa_a^{k,r_a} \tag{3.5}$$

of universal k-caps as a prebase.

The corollary is a consequence of (3.2) and a lemma concerning the i-th summand in the right member of (3.2). This lemma is derived from Theorem 2.2 of [5] with $\kappa_a^{k,i}$ replacing κ_a^k therein. The lemma follows.

Lemma 3.0. *If $\kappa_a^{k,i}$ is a universal k-cap of p_a^i with $0 < k \leq n$, then for each $q \geq 0$ the group*

$$(3.6) \qquad H_q(|\kappa_a^{k,i}|, |\dot{\kappa}_a^{k,i}|, \boldsymbol{Z})$$

is a finitely generated, free, abelian group whose dimension is δ_q^k and for which $\kappa_a^{k,i}$ is a prebase when $q = k$.

Corollary 3.2. *If under the hypotheses of Theorem* 3.1 *and Corollary* 3.1, *y^k and z^k are universal k-caps of p_a^i, then*[3] *for some choice of e as* ± 1

$$(3.7) \qquad y^k \sim ez^k \qquad (on\ f_a \bmod f_a^-) .$$

a-**Level** $(n - 1)$-**caps.** When a is a critical value of index $n - 1$ there are special $(n - 1)$-caps of each critical point p_a^i whose carriers are simply-carried $(n - 1)$-cells on the level set f^a. To describe these cells we shall recall the form taken by Theorem 3.1 of [6] when $k = n - 1$. To that end let D_σ be an open origin-centered n-ball of radius σ in a euclidean space of rectangular coordinates $u_1, \cdots, u_n$. Let

$$(3.8) \qquad (I_a^i : D_\sigma, X_a^i) \in \mathscr{D}M_n \qquad (I_a^i(\boldsymbol{O}) = p_a^i)$$

be a presentation of a neighborhood X_a^i of p_a^i on M_n. Theorem 3.1 of [6] permits us to affirm the following.

Lemma 3.1. *Corresponding to a sufficiently small positive constant σ and to the i-*th *critical point p_a^i of index $n - 1$ on f^a, the Riemaniann metric on M_n may be supposed such that there exist isometric mappings I_a^i of form* (3.8) *of D_σ onto respective neighborhoods X_a^i of p_a^i on M_n such that*

$$(3.9) \qquad f(I_a^i(u)) - a = -u_1^2 - \cdots - u_{n-1}^2 + u_n^2 \qquad (u \in D_\sigma) .$$

We can and will suppose that σ is so small that the closures of the neighborhoods X_a^i are *disjoint*

The cone Λ_{n-1}. The cone

$$(3.10) \qquad \Lambda_{n-1} = \{u \in E_n \,|\, u_n^2 = u_1^2 + \cdots + u_{n-1}^2\}$$

has subsets Λ_{n-1}^+ and Λ_{n-1}^- on which $u_n \geq 0$ and $u_n \leq 0$ respectively. The subsets Λ_{n-1}^+ and Λ_{n-1}^- intersect only in the origin. Set

$$(3.11) \qquad I_a^i(\Lambda_{n-1}^- \cap D_\sigma) = T_a^i , \qquad I_a^i(\Lambda_{n-1}^+ \cap D_\sigma) = \mathscr{T}_a^i .$$

Definition 3.1. *Opposite* $(n - 1)$*-faces of f^a at p_a^i.* Given a critical point

[3] Unless otherwise specified, chains, cycles and homologies shall be over $\boldsymbol{Z}$ in this paper.

p_a^i of index $n - 1$, the topological $(n - 1)$-balls $\mathscr{T}_a^i$ and T_a^i on f^a will be called *opposite* $(n - 1)$-*faces* of f^a at p_a^i. These faces are carried by f^a and intersect in p_a^i.

The "universal k-caps" previously defined have been simply-carried by "f-saddles". This made a proof of the "Saddle Theorem" (Corollary 36.1 of [7]) possible. We have need of $(n - 1)$-caps not simply-carried by f-saddles but rather by singular level sets f^a. Each such k-cap will be associated with a critical point p_a^i of index $n - 1$ on f^a and defined as follows.

Definition 3.2. *a-Level* $(n - 1)$-*caps.* Given "opposite $(n - 1)$-faces" T_a^i and $\mathscr{T}_a^i$ of f^a at a critical point p_a^i, a singular $(n - 1)$-cell σ^{n-1} which is simply-carried by T_a^i or $\mathscr{T}_a^i$ with p_a^i on the interior of $|\sigma^{n-1}|$ will be termed an *a-level* $(n - 1)$-*cap* of p_a^i and will be denoted by $K_a^{n-1,i}$ or $\mathscr{K}_a^{n-1,i}$ respectively.

The following lemma is essential.

Lemma 3.2. *Let p_a^i be the i-th critical point of index $n - 1$ at the f-level a. With p_a^i suppose that there is associated a universal $(n - 1)$-cap $\kappa_a^{n-1,i}$ together with a-level $(n - 1)$-caps $K_a^{n-1,i}$ and $\mathscr{K}_a^{n-1,i}$ on opposite faces of f^a at p_a^i. Then for suitable choices of e and ε as ±1*

$$K_a^{n-1,i} \sim e\kappa_a^{n-1,i}\ , \quad \mathscr{K}_a^{n-1,i} \sim \varepsilon\kappa_a^{n-1,i} \qquad (\text{on } f_a, \bmod f_a - p_a^i)\ . \tag{3.12}$$

We shall establish this lemma with the aid of two deformations d and D.

The deformation d. In the n-space E_n of coordinates $u_1, \cdots, u_n$ of the domain D_σ of the presentation (3.8), let E_{n-1} be the coordinate $(n - 1)$-plane on which $u_n = 0$ and let π be the orthogonal projection of E_n onto E_{n-1}. Under π the point $u = (u_1, \cdots, u_n) \in E_n$ goes into the point $\pi(u) = (u_1, \cdots, u_{n-1}) \in E_{n-1}$. A deformation

$$(u, t) \to d(u, t)\colon E_n \times [0, 1] \to E_n \tag{3.13}$$

retracting E_n onto E_{n-1} (cf. Def. 23.1 of [7]) is defined by setting

$$d(u, t) = (u_1, \cdots, u_{n-1}, (1 - t)u_n) \qquad (0 \leq t \leq 1)$$

for each $u \in E_n$. The partial mapping $u \to d(u, t)$ is denoted by d_t. Let B_r be the origin-centered $(n - 1)$-ball in E_{n-1} of radius r. The images under d_1 of both $\Lambda_{n-1}^- \cap D_\sigma$ and $\Lambda_{n-1}^+ \cap D_\sigma$ is B_ρ with $\rho = \sigma/\sqrt{2}$. Under d the sets $\Lambda_{n-1}^- \cap D_\sigma$ and $\Lambda_{n-1}^+ \cap D_\sigma$ are isotopically deformed onto B_ρ holding the origin fast.

The deformation D. The presentations (3.8) characterized in Lemma 3.1 have been denoted by I_a^i.

The image of the origin under I_a^i is p_a^i. The range of I_a^i is X_a^i. Set $X = \bigcup_{i=1}^{r_a} X_a^i$. A deformation

$$(x, t) \to D(x, t)\colon X \times [0, 1] \to X \tag{3.14}$$

retracting X_a^i onto $I_a^i(B_\sigma)$ for each i is defined by setting

$$D(x, t) = I_a^i(d(u, t)) \qquad (i = 1, \cdots, r_a) \tag{3.15}$$

for $0 \leq t \leq 1$ and for each pair (x, u) such that $u \in D_\sigma$ and $x = I_a^i(u)$. Note that the set $I_a^i(B_\sigma)$ is the carrier of an f-saddle L_{n-1}^i at p_a^i.

The $(n-1)$*-cell* $t^i = K_a^{n-1,i}$. Under D, $|t^i|$ is deformed on $X_a^i \cap f_a$ onto $D_1(|t^i|)$. The mapping D_1 induces a chain transformation $\widehat{D}_1$ (cf. Def. 26.5 of [7]) which maps each singular cell on X_a^i into a singular cell on $D_1(X_a^i)$. In particular, $\widehat{D}_1(t^i)$ is a singular $(n-1)$-cell y_i^{n-1}, simply-carried on $|L_{n-1}^i|$ with p_a^i on the interior of $|y_i^{n-1}|$. Hence y_i^{n-1} is a universal $(n-1)$-cap at p_a^i. It follows from Corollary 3.2 that for some choice of e as ± 1

$$y_i^{n-1} \sim e\kappa_a^{n-1,i} \qquad (\text{on } f_a \bmod (f_a - p_a^i)) \ . \tag{3.16}$$

$D\,|\,(X_a^i \times [0, 1])$ is a deformation retracting X_a^i onto $D_1(X_a^i)$ and since $|t^i| \subset X_a^i$, it follows from Theorem 1.4 of [5] that for $i = 1, \cdots, r_a$,

$$t^i \sim \widehat{D}_1(t^i) = y_i^{n-1} \qquad (\text{on } f_a \bmod (f_a - p_a^i)) \ . \tag{3.17}$$

The first homology in (3.12) follows from (3.16) and (3.17). The second homology in (3.12) follows similarly. This completes the proof of Lemma 3.2.

4. Lemmas on singular homology

Let M be the maximum of f on $|M_n|$ and p_M be the unique critical point of index n at the f-level M. The n-caps associated with p_M play a special role in the study of the orientability of M_n, as the proof of Theorem 9.1 of [6] shows. We shall construct a universal n-cap κ_M^n associated with p_M.

The level manifold f^β. To that end let β be an ordinary value of f such that the open interval (β, M) contains no critical values of f. The set

$$f_{[\beta, M]} = \{x \in |M_n| \,|\, \beta \leq x \leq M\} \tag{4.1}$$

is a topological n-disc Δ_n on $|M_n|$, bounded on $|M_n|$ by the topological $(n-1)$-sphere f^β. It follows that there is a universal n-cap κ_M^n defined by an equivalence class of homeomorphic maps onto Δ_n of vertex-ordered n-simplices. (See p. 371 of [7] or Definition 2.2 of [5].) We set

$$\partial\kappa_M^n = y_\beta^{n-1} \tag{4.2}$$

and note that y_β^{n-1} is carried by f^β.

We shall verify the following lemma.

Lemma 4.1. *The* $(n-1)$*-cycle* y_β^{n-1} *of* (4.2) *is an* $(n-1)$ st *IHP*[4] *of* f^β.

[4] IHP abbreviates the term "integral homology prebase". An IHP of f^β by definition is a prebase over Z of a Betti subgroup of $H_{n-1}(f^\beta, Z)$.

Proof. Let z^{n-1} be an arbitrary $(n-1)$-cycle on f^β. There exists an integral n-chain z^n on $|\kappa_M^n| = \Delta_n$ such that $z^{n-1} = \partial z^n$. The chain z^n is thus an n-cycle on Δ_n mod $\dot{\Delta}$ where $\dot{\Delta}_n = \Delta_n - p_M$. It follows from Lemma 3.0, with $k = n$, that for some integer μ

$$z^n \sim \mu\kappa_M^n \qquad (\text{on } \Delta_n \bmod \dot{\Delta}_n) \ . \tag{4.3}$$

The application of ∂ to the members of (4.3) gives the homology

$$z^{n-1} \sim \mu\partial\kappa_M^n \qquad (\text{on } \dot{\Delta}_n) \ . \tag{4.4}$$

Since there exists an f-deformation retracting $\dot{\Delta}_n$ onto f^β, (4.2) and (4.4) imply that

$$z^{n-1} \sim \mu y_\beta^{n-1} \qquad (\text{on } f^\beta)$$

thereby establishing the lemma.

We shall make use of the following lemma.

Lemma 4.2. *Let χ be a Hausdorff space and r a positive integer such that $H_r(\chi, \boldsymbol{Z})$ is torsion-free. A nontrivial integral r-cycle z^r on χ such that $z^r \sim 0$ over $\boldsymbol{Q}$ on χ is such that $z^r \sim 0$ on χ over $\boldsymbol{Z}$.*

By hypothesis there exists a rational chain c^{r+1} on χ such that $z^r = \partial c^{r+1}$. For a suitably chosen positive integer m, mc^{r+1} will be an integral chain w^{r+1}, so that $mz^r = \partial w^{r+1}$ and hence $mz^r \sim 0$ over $\boldsymbol{Z}$ on χ. It follows that $z^r \sim 0$ on χ. Otherwise, z^r would be in the torsion subgroup of $H_r(\chi, \boldsymbol{Z})$, contrary to hypothesis.

Thus Lemma 4.2 is true.

Lemma 4.3. *If c is an ordinary value of f such that the critical points on f_c have indices less than some positive integer μ, then the following is true:*

(i) *The homology group $H_\mu(f_c, \boldsymbol{Z})$ is trivial.*

(ii) *The homology group $H_{\mu-1}(f_c, \boldsymbol{Z})$ is torsion free.*

We shall prove this lemma by means of theorems in [5]. Since the function f was subject to the Condition C_0 in [5] we shall here suppose that Condition C_0 is satisfied. Were Condition C_0 not satisfied a slight alteration of f near the critical points of f can be made so that Condition C_0 is satisfied. This alteration of f can be made in accord with Lemma 22.4 of [7] in such a manner that the set f_c is unaltered as well as the critical points on f_c and their indices.

Notation. In accord with the notation in [5], for each critical value a of f and integer $q \geq 0$ we shall set

$$H_q^a = H_q(f_a, \boldsymbol{Z}) \ , \tag{4.5}$$

and when (index a) is positive set

$$\dot{H}_q^a = H_q(\dot{f}_a, \boldsymbol{Z}) \ . \tag{4.6}$$

Let $\mathscr{T}_q^a$ and $\dot{\mathscr{T}}_q^a$ denote the torsion subgroups of H_q^a and $\dot{H}_q^a$ respectively.

With f altered as above, the critical values of f less than c form a sequence,

$$a_0 < a_1 < a_2 < \cdots < a_m < c \,. \tag{4.7}$$

We shall examine the sequence

$$H_q^{a_0};\ \dot{H}_q^{a_1}, H_q^{a_1};\ \dot{H}_q^{a_2}, H_q^{a_2}; \cdots ;\ \dot{H}_q^{a_m}, H_q^{a_m};\ H_q(f_c, \boldsymbol{Z}) \tag{4.8q}$$

of homology groups.

Proof of (i). To establish (i) we show inductively that the homology groups of the sequence (4.8μ) are trivial.

This is true of $H_\mu^{a_0}$, since $\mu > 0$. Let s have the range $1, 2, \cdots, m-1$. If $H_\mu^{a_{s-1}}$ is trivial then $\dot{H}_\mu^{a_s}$ is trivial, since there exists an "f-deformation"[5] retracting $\dot{f}_{a_s}$ onto $f_{a_{s-1}}$. Similarly, if $H_\mu^{a_m}$ is trivial $H_\mu(f_c, \boldsymbol{Z})$ is trivial since there exists an f-deformation retracting f_c onto f_{a_m}. Moreover,

$$\dot{H}_\mu^{a_s} \approx H_\mu^{a_s} \qquad (s = 1, \cdots, m) \,. \tag{4.9}$$

Proof of (4.9). The Betti number $\beta_\mu(\dot{f}_{a_s}) = \beta_\mu(f_{a_s})$ since (index a_s) $< \mu$ (see (7.11) of [5]), and the torsion group $\dot{\mathscr{T}}_\mu^{a_s} \approx \mathscr{T}_\mu^{a_s}$ by Theorem 7.3(i) of [5].

Lemma 4.3(i) follows.

Proof of (ii). To establish (ii) we show inductively that the groups in the sequence $\mathscr{T}_{\mu-1}^{a_0};\ \dot{\mathscr{T}}_{\mu-1}^{a_1}, \mathscr{T}_{\mu-1}^{a_1}; \cdots ;\ \dot{\mathscr{T}}_{\mu-1}^{a_m}, \mathscr{T}_{\mu-1}^{a_m};\ \mathscr{T}_{\mu-1}(f_c, \boldsymbol{Z})$ are trivial.

It is clear that $\mathscr{T}_{\mu-1}^{a_0}$ is trivial. Let s be on the range $1, \cdots, m-1$. If $\mathscr{T}_{\mu-1}^{a_{s-1}}$ is trivial, then $\dot{\mathscr{T}}_{\mu-1}^{a_s}$ is trivial, since there exists an f-deformation retracting $\dot{f}_{a_s}$ onto $f_{a_{s-1}}$. If $\mathscr{T}_{\mu-1}^{a_m}$ is trivial $\mathscr{T}_{\mu-1}(f_c, \boldsymbol{Z})$ is trivial for similar reasons. For s on the range $1, \cdots, m$, $\dot{\mathscr{T}}_{\mu-1}^{a_s} \approx \mathscr{T}_{\mu-1}^{a_s}$ by virtue of Theorem 7.3(i) of [5], since (index a_s) $\leq \mu - 1$.

Lemma 4.3(ii) follows.

Lemma 4.3 implies following.

Lemma 4.4. *Let a be a critical value of f of positive index μ such that critical points on f_a^- have indices less than μ, then the following is true*:

(i) *The homology group $H_\mu(f_a^-, \boldsymbol{Z})$ is trivial.*

(ii) *The torsion group of $H_{\mu-1}(f_a^-, \boldsymbol{Z})$ is trivial.*

Let c be an ordinary value of f such that (c, a) is an interval of ordinary values of f. For this c Lemma 4.3 is true as stated and implies Lemma 4.4.

A corollary on orientability of M_n. In [6] we have proved the following.

Theorem 4.1. *The manifold M_n is geometrically orientable or nonorientable according as the connectivity $R_n(|M_n|, \boldsymbol{Q}) = 1$ or 0.*

Notation for Corollary 4.1. Recall that a critical point p_a of positive index

[5] See Cor. 23.1 of [7]. Retracting deformations whose "trajectories" are ortho-f-arcs will be called *f-deformations*.

k, unique at an f-level a, is said to be of *linking* type over a field $\mathscr{K}$ if for *some* k-cap u^k associated with p_a, $\partial u^k \sim 0$ on $\dot{f}_a$ over $\mathscr{K}$. (Cf. [7, p. 259].) It was shown in [7] that if p_a is of linking type, then for *each* k-cap v^k associated with p_a, $\partial v^k \sim 0$ on $\dot{f}_a$ over $\mathscr{K}$.

We state a corollary of Theorem 4.1.

Corollary 4.1. *The manifold M_n is geometrically orientable or nonorientable according as the critical point p_M is or is not of linking type over Q.*

Proof. As in Lemma 4.1 let β be an ordinary value of f such that the interval (β, M) contains no critical value of f. The critical values a of f less than M have indices less than n. It follows from Lemma 4.3(i) that the Betti number $\beta_n(f_\beta)$ vanishes. By a classical theorem the connectivity $R_n(f_\beta, Q) = \beta_n(f_\beta)$ so that $R_n(f_\beta, Q)$ also vanishes. Moreover, $R_n(\dot{f}_M, Q) = 0$ since there exists an f-deformation retracting $\dot{f}_M$ onto f_β. Since $f_M \doteq |M_n|$ it follows from Theorem 29.2 of [7] that $R_n(M_n, Q) = 1$ or 0 according as the critical point p_M is or is not of linking type.

Corollary 4.1 now follows Theorem 4.1.

In (4.2) we have introduced a universal n-cap κ_M^n with algebraic boundary y_β^{n-1}. The critical point p_M is of linking type if and only if $y_\beta^{n-1} \sim 0$ on $\dot{f}_M$ over Q. For future use we formulate a consequence of this fact and of Corollary 4.1.

Corollary 4.2. *The manifold M_n is geometrically orientable or nonorientable according as the integral cycle y_β^{n-1} is or is not rationally bounding on $\dot{f}_M$.*

5. The homology class of y_β^{n-1} on $\dot{f}_M$

In this section we suppose that M_n in *nonorientable.*

The $(n-1)$-cycle y_β^{n-1} was introduced in (4.2). Its carrier is the topological $(n-1)$-sphere f^β.

Subdivisions of y_β^{n-1}. Let $y_{\beta,\mu}^{n-1}$ denote the μ-th "barycentric subdivision" of y_β^{n-1}. (See p. 217 of [7].) The cycle $y_{\beta,\mu}^{n-1}$ has a "reduced form,"

$$y_{\beta,\mu}^{n-1} = e_1\sigma_1^{n-1} + \cdots + e_m\sigma_m^{n-1} \tag{5.0}$$

where the cells σ_i^{n-1} of this "reduced form" are simply-carried by f^β, where $e_i = \pm 1$ and for $i \neq j$, $|\sigma_i^{n-1}| \cap |\sigma_j^{n-1}|$ includes no open subset of f^β. We seek an integral linear combination u^{n-1} of elements of a prebase of $H_{n-1}(f_\beta, Z)$ such that $u^{n-1} \sim y_\beta^{n-1}$ on f_β, or equivalently on $\dot{f}_M$. Since M_n is assumed nonorientable, y_β^{n-1} is rationally nonbounding on f_β in accord with Corollary 4.2.

How large the "index μ of subdivision" of y_β^{n-1}, should be, will presently be indicated.

A condition Ω on f. Because of the hypothesis of § 5 that M_n is nonorientable, there must be at least one critical value of f of index $n-1$. Otherwise, the group $H_{n-1}(f_\beta, Z)$ would be trivial by Lemma 4.3 (i), contrary to Corollary 4.2. Under the *condition Ω* on f, *all* critical points of f of index $n-1$

shall be at *one* f-level, a level ω greater than each critical value of f with a smaller index. The reasoning of § 4 of [6] shows that this condition is either satisfied by f or will be satisfied after a suitable modification of f that leaves the set f_β invariant.

Notation. Under condition Ω on f let

$$p_\omega^1, \cdots, p_\omega^r \qquad (r > 0) \tag{5.1}$$

be the critical points of f of index $n-1$ at the f-level ω and let

$$(u_1, \cdots, u_r) = (\kappa_\omega^{n-1,1}, \cdots, \kappa_\omega^{n-1,r}) \tag{5.2}$$

be a set of universal $(k-1)$-caps associated with the respective critical points (5.1) and, as in § 2, carried in open subsets

$$N_\omega^1, \cdots, N_\omega^r \tag{5.3}$$

of f_ω with disjoint closures.

A retracting deformation δ. The subspace f_β of $|M_n|$ admits an f-deformation

$$(x, t) \to \delta(x, t) \colon f_\beta \times [0, 1] \to f_\beta \tag{5.4}$$

retracting f_β onto f_ω. Under δ each point $x \in f_\beta - f_\omega$ descends on an ortho-f-arc to a limiting end point on f^ω. The terminal mapping δ_1 of δ maps f^β biuniquely onto f^ω, except that each critical point p_ω^i, $i = 1, \cdots, r$ of f on f^ω has two antecedents, say

$$q_1^i, \qquad q_2^i, \tag{5.5}$$

on f^β. The 1-bowl B^i ascends from p_ω^i to meet f^β in the two points (5.5).

A condition Ω_1 on μ and on f. We can suppose that the "index μ of subdivision" of y_β^{n-1} is *so large* that as i ranges over the set $1, \cdots, r$, no two of the $2r$ points (5.5) are carried by the same $(n-1)$-cell of $y_{\beta,\mu}^{n-1}$. We suppose further that f is modified, if necessary, on $f_{(\omega,\beta)}$ so that the points (5.5) are in the *interiors* of $(n-1)$-cells of $y_{\beta,\mu}^{n-1}$ (are represented in (5.0)). This will occur after a suitable modification of the 1-bowls B^i ascending from the points p_ω^i. Let

$$\tau_1^{n-1,i}, \qquad \tau_2^{n-1,i} \tag{5.6}$$

by the $(n-1)$-cells in the reduced form (5.0) of $y_{\beta,\mu}^{n-1}$ whose carriers contain the points q_1^i and q_2^i, respectively.

The terminal mapping δ_1 of δ. According to Definition 26.5 of [3], the terminal mapping δ_1 of δ induces a chain transformation $\widehat{\delta}_1$ of chains y^m on $f_{[\omega,\beta]}$ into chains $\widehat{\delta}_1 y^m$ on f^ω. According to Theorem 1.4 of [5], and Corollary 27.3 of [7],

$$y_\beta^{n-1} \sim \hat{\delta}_1 y_{\beta,\mu}^{n-1} = z^{n-1} \qquad (\text{on } f_{[\omega,\beta]}) \tag{5.7}$$

introducing the $(n-1)$-cycle z^{n-1}. It is clear that $|z^{n-1}| = f^\omega$. For i on the range $1, \cdots, r$, set

$$\hat{\delta}_1 \tau_1^{n-1,i} = \eta_i^{n-1}\,, \qquad \hat{\delta}_1 \tau_2^{n-1,i} = \zeta_i^{n-1}\,. \tag{5.8}$$

If the "index μ of subdivision" of y_β^{n-1} is sufficiently large (as we suppose the case) η_i^{n-1} and ζ_i^{n-1} are simply-carried by "opposite faces" of f^ω at p_ω^i, and are "*ω-level $(n-1)$-caps*" of p_ω^i. (Cf. Def. 3.2.) If for i on the range $1, \cdots, r$, e_i and e_i' have suitable values ± 1 one sees that

$$z^{n-1} = e_i \eta_i^{n-1} + e_i' \zeta_i^{n-1} + c^{n-1}\,, \tag{5.9}$$

where the repeated index i indicates summation of the corresponding terms over the range $1, \cdots, r$ of i, and c^{n-1} is an $(n-1)$-chain on f^ω whose carrier meets the interior of none of the carriers $|\eta_i^{n-1}|$ and $|\zeta_i^{n-1}|$.

In (5.9) the chain c^{n-1} is on f_ω^-, as defined in (2.1). Taking account of this fact and of Lemma 3.2 we are led to the following lemma.

Lemma 5.1. *Under the hypothesis that M_n is nonorientable the critical points p_ω^i, $i = 1, \cdots, r$, of f of index $n-1$, given in* (5.1), *can be reordered, together with their respective universal caps* (5.2), *so that the following is true.*

For a suitable positive integer $\nu \le r$ and proper choices of integers ρ_i as ± 1,

$$y_\beta^{n-1} \sim 2\rho_1 \kappa_\omega^{n-1,1} + \cdots + 2\rho_\nu \kappa_\omega^{n-1,\nu} \qquad (\text{on } f_\beta \bmod f_\omega^-)\,. \tag{5.10}$$

The homology (5.10) is an immediate consequence of (5.9), (5.7) and Lemma 3.2 on "ω-level $(n-1)$-caps" (such as η_i^{n-1} and ζ_i^{n-1}) provided that one excludes the possibility that *all* the coefficients $2\rho_i$ in (5.10) are 0. That is, one must exclude the homology

$$y_\beta^{n-1} \sim 0 \qquad (\text{on } f_\beta \bmod f_\omega^-) \tag{5.11}$$

or eqnivalently the homology

$$y_\beta^{n-1} \sim c_-^{n-1} \qquad (\text{on } f_\beta)\,, \tag{5.12}$$

where c_-^{n-1} is an $(n-1)$-cycle on f_ω^-. However, Lemma 4.4 (i) implies that an $(n-1)$-cycle c_-^{n-1} which is on f_ω^- is bounding on f_ω^-. If then (5.12) held, $y_\beta^{n-1} \sim 0$ on f_β, contrary to Corollary 4.2.

We infer the truth of Lemma 5.1.

In the proof of Theorem 5.1 which follows we shall make use of a lemma in abelian group theory formulated as Lemma 3.1 in [8].

Introduction to Lemma 5.2. Let A be an arbitrary finitely generated abelian group. If $\mathscr{T}$ is the torsion subgroup of A it is well-known that there exists a free subgroup $\mathscr{B}$ of A (termed complementary to $\mathscr{T}$) such that

$$A = \mathscr{B} \oplus \mathscr{T} . \tag{5.13}$$

We term $\mathscr{B}$ a *Betti subgroup* of A. The group $\mathscr{B}$ has a finite base $u_1, \cdots, u_m$, possibly empty. Any unimodular transform of a base of $\mathscr{B}$ is again a base of $\mathscr{B}$. In formulating Lemma 5.2 we shall write $x \equiv y \bmod \mathscr{T}$ whenever x and y are elements in A such that $x - y$ is in $\mathscr{T}$.

Lemma 5.2. *Corresponding to a prescribed element $w \in A$ of infinite order there exists a unique positive integer s such that*

$$w = su \qquad \bmod \mathscr{T} \tag{5.14}$$

for some element u in a base of a Betti subgroup of A.

A proof of this lemma in the form of Lemma 3.1 of [8] is given in [8].

We conclude this section with the following theorem.

Theorem 5.1. *When M_n is nonorientable the cycle y_β^{n-1} introduced in* (4.2) *satisfies a homology*

$$y_\beta^{n-1} \sim 2\lambda^{n-1} \qquad (on\ \dot{f}_M) \tag{5.15}$$

where λ^{n-1} is an element in a prebase of a Betti subgroup of $H_{n-1}(\dot{f}_M, Z)$.

Proof. Turning to (5.10) we set

$$\rho_1\kappa_\omega^{n-1,1} + \cdots + \rho_\nu\kappa_\omega^{n-1,\nu} = c_\omega^{n-1} \tag{5.16}$$

obtaining thereby a chain c_ω^{n-1} on f_ω. According to (5.10) and (5.16)

$$y_\beta^{n-1} = 2c_\omega^{n-1} + \partial c_\beta^n + c_-^{n-1} \tag{5.17}$$

where c_β^n is a chain on f_β and c_-^{n-1} a chain on f_ω^-. From (5.17) we infer that

$$2\partial c_\omega^{n-1} = -\partial c_-^{n-1} \tag{5.18}$$

so that ∂c_ω^{n-1} is rationally bounding on f_ω^-. It follows from Lemma 4.4 (ii)[6] that $H_{n-2}(f_\omega^-, Z)$ is torsion-free and then from Lemma 4.2 that ∂c_ω^{n-1} is integrally bounding on f_ω^-. That is, $\partial c_\omega^{n-1} = \partial w_-^{n-1}$, where w_-^{n-1} is a chain on f_ω^-. We now set

$$\lambda^{n-1} = c_\omega^{n-1} - w_-^{n-1} \tag{5.19}$$

so that λ^{n-1} is an $(n-1)$-cycle on f_ω, and verify statements (i), (ii), (iii) below.

(i) *The cycle λ^{n-1} satisfies the homology* (5.15).

Proof of (i). It follows from (5.17) and (5.19) that the chain

[6] With $a = \omega$ and $\mu = n - 1$ in Lemma 4.4.

$$y_{\beta}^{n-1} - 2\lambda^{n-1} - \partial c_{\beta}^{n} = z_{-}^{n-1} \qquad \text{(introducing } z_{-}^{n-1}\text{)} \tag{5.20}$$

is an $(n-1)$-cycle on f_{ω}^{-}, and from Lemma 4.4 (i)[6] that $z_{-}^{n-1} \sim 0$ on f_{ω}^{-}, so that $y_{\beta}^{n-1} \sim 2\lambda^{n-1}$ on $\dot{f}_M$. Thus statement (i) is true.

We note that λ^{n-1}, as defined by (5.9), satisfies the homology

$$\lambda^{n-1} \sim \rho_1 \kappa_{\omega}^{n-1,1} + \cdots + \rho_{\nu} \kappa_{\omega}^{n-1,\nu} \qquad (\text{on } f_{\omega} \bmod f_{\omega}^{-}) . \tag{5.21}$$

(ii) *The cycle λ^{n-1} is an element in a prebase of a Betti subgroup of $H_{n-1}(\dot{f}_M, \boldsymbol{Z})$.*

The cycle $\lambda^{n-1} \not\sim 0$ on $\dot{f}_M$, since $y_{\beta}^{n-1} \not\sim 0$ on $\dot{f}_M$ by Corollary 4.2 and (5.15) holds. Hence $\lambda^{n-1} \not\sim 0$ on f_{ω}. To establish (ii) it is sufficient to establish the following.

(iii) *The cycle λ^{n-1} is an element in a prebase of a Betti subgroup of $H_{n-1}(f_{\omega}, \boldsymbol{Z})$.*

To establish (iii) we shall apply Lemma 5.2 to the abelian group $A = H_{n-1}(f_{\omega}, \boldsymbol{Z})$, taking w of Lemma 5.2 as the homology class on f_{ω} of λ^{n-1}. The group $H_{n-1}(\dot{f}_M, \boldsymbol{Z})$ is torsion free by Lemma 4.4 (ii). Hence the isomorph $H_{n-1}(f_{\omega}, \boldsymbol{Z})$ of $H_{n-1}(\dot{f}_M, \boldsymbol{Z})$ is torsion-free. In applying Lemma 5.2 to $A = H_{n-1}(f_{\omega}, \boldsymbol{Z})$ we can accordingly suppose that $\mathscr{T} = 0$. According to Lemma 5.2 there then exists a positive integer s such that

$$\lambda^{n-1} \sim s v^{n-1} \qquad (\text{on } f_{\omega}) \tag{5.22}$$

for some element v^{n-1} in a prebase of $H_{n-1}(f_{\omega}, \boldsymbol{Z})$. Hence to prove (iii) it is sufficient to show that $s = 1$ in (5.22).

Proof that $s = 1$. We shall apply Corollary 3.1 with $a = \omega$ and $r_a = r$ therein. Corollary 3.1 implies that for suitable integers $n_1, \cdots, n_r$

$$v^{n-1} \sim \sum_{i=1}^{r} n_i \kappa_{\omega}^{n-1,i} \qquad (\text{on } f_{\omega} \bmod f_{\omega}^{-}) . \tag{5.23}$$

From (5.23), (5.22) and (5.21) we infer that

$$\sum_{i=1}^{\nu} \rho_i \kappa_{\omega}^{n-1,i} \sim s \sum_{n=1}^{r} n_i \kappa_{\omega}^{n-1,i} \qquad (\text{on } f_{\omega}, \bmod f_{\omega}^{-}) . \tag{5.24}$$

Since $\kappa_{\omega}^{n-1,1}, \cdots, \kappa_{\omega}^{n-1,r}$ is a prebase of $H_{n-1}(f_{\omega}, f_{\omega}^{-}, \boldsymbol{Z})$ by Corollary 3.1, (5.24) is possible only if for $i = 1, \cdots, r$ the coefficients of $\kappa_{\omega}^{n-1,i}$ are the same in the two members of the homology (5.24). In particular, s must be 1.

Thus (iii) is true and hence (ii). Theorem 5.1 follows from (i) and (ii).

6. Proof of Theorem 1.0

Our proof of Theorem 1.0 without use of a triangulation of $|M_n|$ depends on the concept of the "free index" s of an element w in an arbitrary finitely generated abelian group A. The group A is a direct sum

$$A = \mathscr{B} \oplus \mathscr{T} \tag{6.1}$$

of its torsion group $\mathscr{T}$ and a free subgroup $\mathscr{B}$ "complementary" to $\mathscr{T}$ in A. The following definition is given in § 3 of [8].

Definition 6.1. *The free index s of $w \in A$.* If $w \in \mathscr{T}$ the free index of w shall be 0. If $w \notin \mathscr{T}$ the free index of w shall be the integer s affirmed to exist in Lemma 5.2.

We shall apply this definition. To that end let p_a be a critical point of f unique among critical points of f at the f-level a. Suppose that the index k of p_a is positive. Let κ_a^k be a universal k-cap at p_a, and let w_a^{k-1} be the homology class of $\partial\kappa_a^k$ on $\dot{f}_a$. We note that $w_a^{k-1} \in H_{k-1}(\dot{f}_a, Z)$. If κ_a^k is replaced by any other universal k-cap of p_a, the homology class of $\partial\kappa_a^k$ on $\dot{f}_a$ remains unchanged or is multiplied by -1. (See Theorem 2.3 of [5].) Set

$$A = H_{k-1}(\dot{f}_a, Z) \ . \tag{6.2}$$

Definition 6.2. *The free index s^a of p_a.* Under the conditions of the preceding paragraph the *free index s^a* of the critical point p_a is taken as the free index of $\pm w_a^{k-1}$.

The value of s^a is independent of the choice of κ_a^k as a universal k-cap of p_a since the free index of w_a^{k-1} equals the free index of $-w_a^{k-1}$. (See definition of s^a in § 4 of [5].)

The critical point p_a of Definition 6.2 is of "linking" or "nonlinking" type over the field Q of rational numbers in the sense of Definition 29.1 [7]. We shall verify the following lemma.

Lemma 6.1. *The critical point p_a of Definition 6.2 is of linking or nonlinking type over Q according as the free index s^a of p_a is zero or positive.*

In terms of the connectivities $R_q(f_a)$ and $R_q(\dot{f}_a)$ over Q, set

$$\Delta R_q = R_q(f_a) - R_q(\dot{f}_a) \qquad (q = 0, 1, \cdots) \ . \tag{6.3}$$

According to Theorem 29.2 of [7], when p_a has the index k, $\Delta R_k = 1$ or $\Delta R_{k-1} = -1$ according as p_a is of linking or nonlinking type over Q. In terms of Betti numbers $\beta_q(f_a)$ and $\beta_q(\dot{f}_a)$ set

$$\Delta\beta_q = \beta_q(f_a) - \beta_q(\dot{f}_a) \qquad (q = 0, 1, \cdots) \ .$$

It is well-known that Betti numbers and connectivities over Q, indexed by the same dimension, are equal when finite. Thus $\Delta\beta_q = \Delta R_q$. According to Theorem 7.2 of [5], $\Delta\beta_k = 1$ or $\Delta\beta_{k-1} = -1$ according as the free index s^a of p_a is zero positive.

Lemma 6.1 follows.

Proof of Theorem 1.0. The proof of this theorem will be based on Proposition 7.1 of [5]. We state Proposition 7.1 as follows.

Theorem 6.1. *Let p_a be a critical point of positive index k, with p_a unique among critical points at the f-level a. When $\dot{H}^a_{k-1}$ is torsion-free, H^a_{k-1} is torsion-free unless $s^a > 1$. If $s^a > 1$, H^a_{k-1} has a unique torsion coefficient s^a.*

Theorem 1.0 breaks down into two theorems, Theorem 1.0a and Theorem 1.0b. Recall that $f_M = |M_n|$.

Theorem 1.0a. *The torsion group $\mathscr{T}_{n-1}(f_M)$ is trivial if M_n is geometrically orientable.*

Theorem 1.0b. *The torsion group $\mathscr{T}_{n-1}(f_M)$ has the order 2 if M_n is geometrically nonorientable.*

Proof of Theorem 1.0a. We shall apply Lemma 4.4 with $a = M$ therein. The critical points of f other than p_M have indices at most $n - 1$. It follows from Lemma 4.4 that $H_{n-1}(\dot{f}_M, \boldsymbol{Z})$ is torsion-free. By hypotheses of Theorem 1.0a, M_n is geometrically orientable so that p_M is of linking type over $\boldsymbol{Q}$ by Corollary 4.1. Hence the free index $s^M = 0$ by Lemma 6.1. It follows from Theorem 6.1 with $a = M$ that $H_{n-1}(f_M, \boldsymbol{Z})$ is torsion-free.

Thus Theorem 1.0a is true.

Proof of Theorem 1.0b. We have just seen that $\mathscr{T}_{n-1}(\dot{f}_M)$ is trivial. According to Theorem 6.1, to prove Theorem 1.0b it then suffices to verify that the free index s^M of the critical point p_M is 2. That this is the case is an immediate consequence of Theorem 5.1 and the definition of s^M. One has merely to recall that the $(n - 1)$-cycle y^{n-1}_β is defined by (4.2) to infer from Theorem 5.1 that $s^M = 2$.

That Theorem 1.0b is true follows now from Theorem 6.1 with $k = n$.

References

[1] J. Eells & N. H. Kuiper, *Manifolds which are like projective planes,* Publ. Inst. Hautes Études Sci. No. 14 (1962) 181–222.

[2] R. C. Kirby & L. C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung,* Bull. Amer. Math. Soc. **75** (1969) 742–749.

[3] M. Morse, *Topologically non-degenerate functions on a compact* n*-manifold* M, J. Analyse Math. **7** (1959) 189–208.

[4] ——, *The existence of polar non-degenerate functions on differentiable manifolds,* Ann. of Math. **71** (1960) 352–383.

[5] M. Morse & S. S. Cairns, *Singular homology over* **Z** *on topological manifolds,* J. Differential Geometry **3** (1969) 257–288.

[6] ——, *Orientation of differentiable manifolds,* J. Differential Geometry **6** (1971) 1–31.

[7] ——, *Critical point theory in global analysis and differential topology,* Academic Press, New York, 1969.

[8] ——, *Elementary quotients of abelian groups, and singular homology on manifolds,* Nagoya Math. J. **39** (1971) 167–198.

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