

THE SIGNATURE AND G -SIGNATURE OF MANIFOLDS WITH BOUNDARY

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1. The signature theorem

Let M be a compact oriented C^∞ manifold¹ of dimension $4k$ with boundary B of dimension $4k - 1$. The oriented B is called a *reflecting boundary* of M if it admits an orientation-reversing involution π . A simple example of the reflecting boundaries of M is a $(4k - 1)$ -sphere. For convenience and simplicity, we shall always denote such as (B, π) a reflecting boundary with its involution together but with its dimension omitted. The following problem seems to be of interest: Is there any manifold M with a reflecting boundary (B, π) on which the involution π cannot be extended to the interior of the manifold M ?

Now let \tilde{M} with boundary \tilde{B} be a C^∞ homeomorphic copy of (M, B) with the same orientation, and μ be the homeomorphism so that $\mu(M, B) = (\tilde{M}, \tilde{B})$. Then we can define the double of M with a reflecting boundary (B, π) to be a C^∞ closed oriented manifold N such that $N = M \cup \tilde{M}$ and that $\tilde{B} = \pi(B)$ by identifying $\mu\pi(x) \in \tilde{B}$ with x for all $x \in B$. Thus on the double N we can define a homeomorphism $\nu: N \rightarrow N$ by:

$$(1.1) \quad \nu(x) = \begin{cases} \mu(x), & \text{for } x \in M, \\ \mu^{-1}(x), & \text{for } x \in \tilde{M}. \end{cases}$$

To see that this is well-defined, at first we notice that $\nu(x) = \mu(x) \in \tilde{B}$ for $x \in \tilde{B}$. Since x is identified with $\mu\pi(x) \in \tilde{B}$, $\nu(x) = \nu(\mu\pi(x)) = \mu^{-1}(\mu\pi(x)) = \pi(x) \in B$, and therefore $\mu(x)$ is identified with $\pi(x)$; this is indeed true by the definition of our identification and the assumption $\pi^2 = 1$. Clearly, ν is an involution. (It should be noted that the definition of doubling a manifold M here is somewhat different from the ordinary one under which M and \tilde{M} are of opposite orientations so that every point of B is a fixed point under the involution ν ; for the latter see, for instance, [3].) Alternatively, we may regard doubling the manifold M with a reflecting boundary (B, π) as finding a C^∞ homeomorphism

$$h: M \rightarrow N$$

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¹ Throughout this paper all manifolds are differentiable.

where N is a C^∞ closed manifold with an involution

$$\nu: N \rightarrow N$$

such that $\nu h = h\pi: B \rightarrow N$. Assume h maps M into a fundamental domain of the involution ν in such a way that B is mapped onto itself. We shall identify M with the fundamental domain henceforth, so that we may regard the double N as composed of two halves M and \tilde{M} with the same orientation such that M is mapped onto \tilde{M} by ν , $M \cap \tilde{M} = B$, and $\nu|_B = \pi$.

A Riemannian metric on the double N , for which the involution ν is an isometry, is said to be *symmetric*, and the restriction of a symmetric metric on N to M is also called a symmetric metric on M . Now there arises the problem of deriving a C^∞ symmetric Riemannian metric on N from a C^∞ Riemannian metric on M . For this problem at first we are naturally tempted to prolong to N a differentiable metric g on some manifold containing M by setting

$$(1.2) \quad g(x) = g(\nu(x))$$

for $x \in M$. Although (1.2) is well defined, the difficulty is that the resulting metric will, in general, not be differentiable across B . However, on the other hand, for a given C^∞ Riemannian metric g everywhere defined on N we may obtain from it a C^∞ symmetric metric g by setting

$$(1.3) \quad g(x) = \frac{1}{2}[g(x) + g(\nu(x))] , \quad \text{for } x \in N .$$

Now we consider a C^∞ symmetric Riemannian metric g on N . Then by Hirzebruch signature theorem [5] the signature of N is given by

$$(1.4) \quad \text{sign}(N) = \int_N L_k(p_1, \dots, p_k)(\Omega_g) ,$$

where p_j is the j -th Pontrjagin class of N , and $\{L_k(p_1, \dots, p_k)(\Omega_g)\}$ is the Hirzebruch's multiplicative sequence of polynomials with each p_i expressed in terms of the curvature 2-forms Ω_{j_k} of the Riemannian metric g by a theorem of Chern [2]. However,

$$(1.5) \quad \begin{aligned} \int_N L_k(p_1, \dots, p_k)(\Omega_g) &= \left(\int_M + \int_{\tilde{M}} \right) L_k(p_1, \dots, p_k)(\Omega_g) \\ &= 2 \int_M L_k(p_1, \dots, p_k)(\Omega_g) , \end{aligned}$$

since $L_k(p_1, \dots, p_k)(\Omega_g)$ depends only on the Riemannian metric g , and the metric g is symmetric.

On the other hand, the following theorem was first observed by S. P. Novikov and proved jointly by Atiyah and Singer [1, Prop. (7.1)]:

Theorem 1.1. *Suppose that two compact oriented manifolds M_1 and M_2 have a common boundary B with opposite orientations. Then*

$$(1.6) \quad \text{sign}(M_1 \cup M_2) = \text{sign}(M_1, B) + \text{sign}(M_2, B) ,$$

where $\text{sign}(M_i, B)$ denotes the signature of the manifold M_i with boundary B for $i = 1, 2$.

By applying Theorem 1.1 to our case we obtain

$$(1.7) \quad \text{sign}(N) = \text{sign}(M, B) + \text{sign}(\tilde{M}, B) = 2 \text{sign}(M, B) ,$$

since M and \tilde{M} are homeomorphic with the same orientation. Combination of (1.4), (1.5), (1.7) thus gives

$$(1.8) \quad \text{sign}(M, B) = \int_M L_k(p_1, \dots, p_k)(\Omega_g) .$$

Hence we arrive at

Theorem 1.2. *The signature of a compact oriented C^∞ manifold M of dimension $4k$ with a reflecting boundary B of dimension $4k - 1$ is given by (1.8) where g is a symmetric metric on M .*

When B is empty, Theorem 1.2 reduces to Hirzebruch signature theorem.

2. The G -signature and signature-defect

We first state the following theorem of Atiyah and Singer [1, p. 588], which is a generalization of Theorem 1.1 on the additivity property of the signature:

Theorem 2.1. *Suppose that two compact oriented manifolds M_1 and M_2 have a common boundary B with opposite orientations, and that a compact Lie group acts differentiably on M_1 and M_2 preserving the orientations. Then*

$$(2.1) \quad \text{sign}(G; M_1 \cup M_2) = \text{sign}(G; M_1, B) + \text{sign}(G; M_2, B) ,$$

where $\text{sign}(G; M_1 \cup M_2)$ and $\text{sign}(G; M_i, B)$ denote, respectively, the G -signatures of $M_1 \cup M_2$ and M_i with boundary B for $i = 1, 2$.

As in § 1 let M be a compact oriented manifold of dimension $2l$ with a reflecting boundary (B, π) , and $N = M \cup \tilde{M}$ be the double of M with an involution ν defined by (1.1). Then we naturally intend to extend any automorphism g of M to an automorphism of N by

$$(2.2) \quad g(x) = \begin{cases} g(x) , & \text{for } x \in M , \\ \nu(g\nu^{-1}(x)) , & \text{for } x \in \tilde{M} , \end{cases}$$

so that

$$(2.3) \quad g(\nu\pi(x)) = \nu g\nu^{-1}(\nu\pi(x)) = \nu g\pi(x) \quad \text{for } x \in B .$$

On the other hand, for $x \in B$ since $x = \nu\pi(x)$ by our identification for the reflecting boundary (B, π) , we have

$$(2.4) \quad g(x) = \nu \pi g(x),$$

which, together with (2.3), implies immediately that *to well define an automorphism of N by (2.2) it is necessary that*

$$(2.5) \quad \pi g = g\pi \quad \text{on } B.$$

Now let M be a compact oriented manifold of dimension $2l$ without boundary, and suppose that there is a compact Lie group G acting on M preserving the orientation. For expressing $\text{sign}(g; M)$ for an element g of G , in [1] Atiyah and Singer obtained the G -signature Theorem (6.12), Corollary (6.13), Proposition (6.15), Corollary (6.16) (which was proved by Conner and Floyd [4, Cor. (27.4)] by a different method), and Proposition (6.18). If M has a reflecting boundary (B, π) , by using our method of doubling a manifold in § 1 and Theorem 2.1, we can easily show that *the above mentioned expressions of Atiyah and Singer also hold for $\text{sign}(g; M, B)$ provided that on the boundary B , g has no fixed point and satisfies the condition (2.5).*

Very recently Hirzebruch [6] defined the signature-defect of a finite group acting effectively on a connected compact oriented manifold M without boundary, and obtained some interesting relationships between number theory and the signature-defect at some special points of a four-dimensional M .

Now let M be a connected compact oriented manifold of dimension 4 with a reflecting boundary (B, π) , and suppose that there is a finite group G acting by orientation-preserving diffeomorphisms effectively on M and freely on B such that $\pi G = G\pi$ on B . By following Hirzebruch we can easily generalize his definition of the signature-defect of a group action on a manifold without boundary to the G -action on M , and show that *his relationships [6, § 5] between number theory and the signature-defect also hold for the signature-defect of the G -action on the manifold M with boundary (B, π) by using the conditions of G on (B, π) and the extension in § 1 of Proposition (6.18) of Atiyah and Singer [1].*

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