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ON THE RIGIDITY OF PUNCTURED OVALOIDS. II

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A C^k isometric embedding $\phi: M \to R^n$ of a Riemannian manifold M into n-dimensional euclidean space R^n is said to be *rigid in the class of C^k isometric embeddings* if, corresponding to each C^k isometric embedding $\psi: M \to R^n$, there is a rigid motion $T_\phi: R^n \to R^n$ such that $\psi = T_\phi \circ \phi$. A theorem of Cohn-Vossen [3] is that any C^3 isometric embedding of a compact 2-dimensional Riemannian manifold of everywhere positive Gaussian curvature in R^3 is rigid in the class of C^3 isometric embeddings. Earlier, Hadamard [5] had proven that a compact C^2 submanifold of R^3 having everywhere positive Gaussian curvature was a convex surface, that is, the boundary of a convex body in R^3. Using this convexity property, Herglotz [6](see also Hicks [7]) gave a brief new proof of the rigidity theorem of Cohn-Vossen; Wintner [20] showed using a refinement of Herglotz's approach that the theorem of Cohn-Vossen remains true if C^3 is replaced by C^2 throughout its statement. In the course of their work on the total curvature of submanifolds of euclidean space, Chern and Lashof [2] proved that a compact C^2 surface in R^3 with everywhere nonnegative Gaussian curvature was necessarily a convex surface. Using this generalization of the convexity result of Hadamard, Voss [18] and independently Sacksteder [15] extended Herglotz's rigidity argument to show that any C^2 isometric embedding of a compact 2-dimensional Riemannian manifold of everywhere nonnegative Gaussian curvature is rigid.

It was shown in [4], the first paper of this series, that, if M is a compact orientable 2-dimensional Riemannian manifold with a C^5 metric of everywhere positive Gaussian curvature and if M' is the manifold obtained from M by deleting a finite number of points $p_1, \cdots, p_n$, then any C^2 isometric embedding $\phi': M' \to R^3$ is rigid in the class of C^2 isometric embeddings. In fact, it was shown that ϕ' is necessarily the restriction to M' of a C^2 isometric embedding $\phi: M \to R^3$, and the rigidity of ϕ' is then a consequence of the rigidity theorem for C^2 isometric embeddings of compact manifolds of positive curvature. The purpose of the present paper is to prove a similar rigidity and regularity result for compact orientable 2-dimensional Riemannian manifolds of everywhere nonnegative curvature with a finite number of points deleted, at each of which

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points the Gaussian curvature is positive. This rigidity result is stated and proved in § 2. § 1 contains the statement and proof of a proposition concerning the planar point set of an isometric embedding in R^3 of such a manifold. Because the manifold is not assumed to be complete, this proposition is not a special case of Sacksteder's Theorem 1 in [14]; however, the present proposition is proved here using Sacksteder's methods. Some related results regarding the type of local convexity used in the proof of the proposition can be found in [16].

1. The structure of the set of planar points

Let V be a 2-dimensional manifold and $\psi\colon V \to R^3$ be an embedding. A *planar point* of ψ is a point $p \in V$ such that the second fundamental form of $\psi\colon V \to R^3$ at p is zero. If V is oriented, the *Gauss map* $\eta\colon V \to S^2$ (corresponding to a fixed orientation of R^3) is the map which assigns to $p \in V$ the unit vector at the origin in R^3 parallel to the positively oriented normal to $\psi(V)$ at $\psi(p)$. The Gauss map has rank zero at $p \in V$ if and only if p is a planar point.

Proposition. *If M' is the manifold obtained by deleting a finite number of points $p_1, \cdots, p_n$ from a (C^∞) manifold M homeomorphic to S^2, and $\psi\colon M' \to R^3$ is a C^3 embedding of M' having the Gaussian curvature of the induced metric on M' nonnegative everywhere and positive in a neighborhood of the points $p_1, \cdots, p_n$, then each component T_0 of the set T of planar points of ψ has the property that $\psi(T_0)$ is a compact convex subset of a plane in R^3.*

Proof. Fix orientations of R^3 and M. The corresponding Gauss map $\eta\colon M' \to S^2$ is C^2 since ψ is C^3. Since the planar points of ψ are precisely those points of M' at which η has rank zero, a theorem of Sard [17] implies that η is constant on each component T_β of the set of planar points. If the unit vector $\vec{N}$ is the common value of η at the points of T_β, then the C^3 function $p \to \vec{N}\cdot\overrightarrow{\psi(p)}$ from M' to R has derivative zero at every point of T_β. By a theorem of A. P. Morse [9], $p \to \vec{N}\cdot\overrightarrow{\psi(p)}$ is constant on T_β. Thus $\psi(T_\beta)$ lies in a (uniquely determined) plane perpendicular to $\vec{N}$. In particular, $\psi(T_0)$ lies in a plane which is perpendicular to $\vec{N}_0$, where $\vec{N}_0$ is the normal vector to $\psi(M')$ at every point of $\psi(T_0)$.

Since by assumption the curvature is positive in a neighborhood in M' of each of the points $p_1, \cdots, p_n$, it follows that T is bounded away from $p_1, \cdots, p_n$ in M and hence that T is a closed subset not only of M' but also of M. Thus T is compact, and consequently every component of T is compact. In particular, T_0 is compact and hence $\psi(T_0)$ is compact.

Let V be a component of the open set $M - T$ and $\{U_\alpha\}$ be the components of $M - V$. Note that no p_i, $i = 1, \cdots, n$, is contained in the boundary of any U_α since for each i the boundary of V, which is a subset of T, is bounded away from p_i. Also, each U_α contains only one component of the boundary of $M - V$

[10, p. 124]. Therefore the boundary of each U_α is a connected subset of T, and thus by the previous remarks the normal to $\psi(M')$ is constant on the boundary of U_α. Let $\vec{N}_\alpha$ be its constant value there.

For the remainder of the proof, a special type of local convexity property called here (as in [14]) W-convexity will be used; Let $\vec{v}$ be any unit vector at the origin in R^3, P be the plane through the origin orthogonal to $\vec{v}$, and π_P: $R^3 \to P$ be the orthogonal projection on P. Define W to be any (open) connected component of the set $\{p \in M' \mid \overrightarrow{\eta(p)} \cdot \vec{v} \neq 0\}$. A subset $U \subset W$ is said to be *W-convex* if every two points p, q of U, which are joined by an arc in W whose orthogonal projection on P is the line segment from $\pi_P(p)$ to $\pi_P(q)$, are joined by an arc in U whose orthogonal projection on P is the line segment from $\pi_P(p)$ to $\pi_P(q)$. (Since $\pi_P\psi \mid W$ is a local homeomorphism, such an arc in W is unique up to parametrization, it being supposed here and henceforth that the arc is transversed without reversals.) The following properties of W-convex sets are easily derived from the definition: a) Any component of a W-convex set is W-convex. b) The intersection of any family of W-convex sets is W-convex.

The following lemma will be used in proving that $\pi_P\psi(T_0)$ is convex:

Lemma 1. *Let V be a component of $M - T$. If U_{α_0} is a component of $M - V$ such that U_{α_0} contains T_0, then $U_{\alpha_0} \cap W$ is W-convex for $W =$ the component of $\{p \in M' \mid \overrightarrow{\eta(p)} \cdot \vec{N}_0 \neq 0\}$ containing T_0.*

Proof. Without loss of generality it can be supposed that the unit vector $\vec{N}_0$ is along the positive z-axis so that the perpendicular plane P through the origin is the xy-plane. Let $x(p) = [x(p), y(p), z(p)] \in R^3$ be the coordinate representation of ψ. To prove that $U_{\alpha_0} \cap W$ is W-convex, it is sufficient to show that if γ is an arc in W from p to q, $p, q \in U_{\alpha_0} \cap W$, with the projection of γ the line segment L form $\pi_P\psi(p)$ to $\pi_P\psi(q)$ in P, then γ lies in $U_{\alpha_0} \cap W$. Since $U_{\alpha_0} \cap W$ is closed in W, $\gamma \cap (W - (U_{\alpha_0} \cap W))$ is a (possibly empty) union of disjoint open curve segments in W whose endpoints lie in $U_{\alpha_0} \cap W$. Suppose that there is at least one such segment γ_1. Denote the closed arc which consists of γ_1 together with its endpoints by $\bar{\gamma}_1$. Note that $\pi_P\psi(\bar{\gamma}_1)$ is a closed line segment in P with endpoints in $\pi_P\psi(U_{\alpha_0})$.

If U_α is any component of $M - V$ with $U_\alpha \cap W \neq \emptyset$, then the boundary of U_α intersects W since otherwise W would be contained in the interior of U_α, contradicting the fact that the boundary of U_{α_0} is contained in W. Thus, if $U_\alpha \cap W \neq \emptyset$ then $\vec{N}_\alpha \cdot \vec{N}_0 \neq 0$. Let P_α be the (uniquely determined) plane perpendicular to $\vec{N}_\alpha$ which contains the boundary of U_α. For each α, define the function $z_\alpha \colon P \to R$ by taking $z_\alpha(x, y)$ to be the unique real number satisfying $[x, y, z_\alpha(x, y)] \in P_\alpha$. ($\vec{N}_\alpha \cdot \vec{N}_0 \neq 0$ insures that such a $z_\alpha(x, y)$ exists and is uniquely determined.) Note that $z_{\alpha_0}(x, y)$ is constant since $\vec{N}_0$ is perpendicular to the xy-plane. Define $F \colon W \to R$ by

$$\begin{aligned} F(p) &= z(p) - z_{\alpha_0}(x(p), y(p)) && \text{if } p \in W \cap V \\ &= z_\alpha(x(p), y(p)) - z_{\alpha_0}(x(p), y(p)) && \text{if } p \in W \cap U_\alpha \,. \end{aligned}$$

F is C^2, since the boundary of each U_α is contained in T and $\psi(M')$ coincides with its tangent plane up to third order at each point of $\psi(T)$. Now define a mapping $\Phi\colon W \to R^3$ by

$$\Phi(p) = [x(p), y(p), F(p)] \,.$$

Φ is a C^2 immersion, and its planar points are precisely the points of $W - V$. The second fundamental form of ψ on V can be supposed without loss of generality to be everywhere positive semidefinite (it is either everywhere positive semidefinite or everywhere negative semidefinite on V because V is a connected set of non-planar points). Then the second fundamental form of Φ is positive semidefinite.

Return now to the consideration of the curve segment $\bar{\gamma}_1 \subset W$ whose endpoints lie in U_α. Since $\pi_P\psi\,|\,\bar{\gamma}_1$ is a one-to-one map of $\bar{\gamma}_1$ onto L_1, $\pi_P\psi$ and hence Φ are each one-to-one in some neighborhood U of $\bar{\gamma}$ in W. Thus $\Phi(U)$ can be considered to be the graph over $\pi_P\psi(U)$ of the function F. Then F is convex on $\pi_P\psi(U)$ because of the positive semidefiniteness of the second fundamental form of Φ. It may be assumed (by a linear change in the x, y coordinates) that the segment L_1 is a portion of the x-axis, say from $[a, 0, 0]$ to $[b, 0, 0]$, $b > a$. $F_{xx} \geq 0$; but since the endpoints of L_1 lie in $\pi_P\psi(T)$, $F_x([a, 0, 0]) = F_x([b, 0, 0]) = 0$ and $F([a, 0, 0]) = F([b, 0, 0]) = 0$. Hence $F = 0$ everywhere on L_1.

$F_{yy} \geq 0$ everywhere on $\pi_P\psi(U)$; since $F = F_y = 0$ at every point of L_1, it follows that there is some $\varepsilon > 0$ such that $\{[x, y, 0]\,|\,a \leq x \leq b,\ |y| \leq \varepsilon\} \subset \pi_P\psi(U)$ and such that $F([x, y, 0]) \geq 0$ for $a \leq x \leq b$, $|y| \leq \varepsilon$. Choose such an ε. Then, for $a \leq x \leq b$ and $|y| \leq \varepsilon$,

$$0 \leq F([x, y, 0]) \leq \frac{b - x}{b - a} F([a, y, 0]) + \frac{x - a}{b - a} F([b, y, 0]) \,,$$

the second inequality following directly from the convexity of F. Since $F = F_y = F_{yy} = 0$ at $[a, 0, 0]$, $\lim_{y \to 0} F([a, y, 0])/y^2 = 0$; similarly, $\lim_{y \to 0} F([x, y, 0])/y^2 = 0$ for $a \leq x \leq b$. The inequalities for $F([x, y, 0])$ then imply that $\lim_{y \to 0} F([x, y, 0])/y^2 = 0$ for $a \leq x \leq b$. Thus $F_{yy}([x, 0, 0]) = 0$ for $a \leq x \leq b$. Since also $F_{xx}([x, 0, 0]) = 0$ for $a \leq x \leq b$, the semidefinite second fundamental form of the graph F is 0 everywhere on L_1. Thus $\bar{\gamma}_1$ lies entirely in the set of planar points of Φ, and hence $\bar{\gamma}_1 \cap V = \emptyset$. It follows that $\bar{\gamma}_1 \subset U_{\alpha_0}$, and thus the proof of the lemma is complete.

To continue the proof of the proposition, consider the intersection $\bigcap_{U_\alpha}(W \cap U_\alpha)$ with W of every component U_α of $M - V$ which contains T_0 for all components V of $M - T$. By Lemma 1 and property (b) of W-convexity, this intersection is W-convex; and hence by property (a) of W-convexity the component of this intersection which contains T_0 is W-convex. This component is a connected subset of T, which contains T_0 and hence equals T_0. Thus T_0 is W-convex.

To complete the proof that $\psi(T_0)$ is convex, let $S =$ the set of all pairs of points $(p, q) \in T_0 \times T_0$ such that the (closed) line segment from $\psi(p)$ to $\psi(q)$ lies in $\psi(T_0)$. $\psi(T_0)$ is convex if and only if $S = T_0 \times T_0$. Since $\{(p, p) \mid p \in T_0\} \subset S$ and T_0 is non-empty, S is non-empty. From the compactness (and consequent closedness) of $\psi(T_0)$ in R^3, it follows easily that S is closed in $T_0 \times T_0$. To see that S is open in $T_0 \times T_0$, observe that since $\psi(T_0)$ lies in a plane parallel to P, $\pi_P\psi$ is a diffeomorphism of some open subset U_0 of M' with $T_0 \subset U_0 \subset W$ onto an open subset of the plane P. If $(p, q) \in S$, then the line segment L from $\pi_P\psi(p)$ to $\pi_P\psi(q)$ lies in $\pi_P\psi(U_0)$ and hence some convex open neighborhood W_L of L in P lies in $\pi_P\psi(U_0)$. Put $U_L = U_0 \cap (\pi_P\psi)^{-1}(W_L)$. U_L is an open subset of W, and if $p', q' \in U_L \cap T_0$ then the line segment from $\pi_P\psi(p')$ to $\pi_P\psi(q')$ lies in W_L and hence in $\pi_P\psi(U)$. Then there exists an arc γ' in U with the image of this arc under $\pi_P\psi$ equal to this line segment in P. Since $U \subset W$, the W-convexity of T_0 implies that γ' lies in T_0, and therefore the line segment from $\pi_P\psi(p')$ to $\pi_P\psi(q')$ lies in $\pi_P\psi(T_0)$. Since $\pi_P \mid \psi(T_0)$ is a translation, it follows that the line segment from $\psi(p')$ to $\psi(q')$ lies in $\psi(T_0)$. Thus $(U_L \times U_L) \cap (T_0 \times T_0)$ is an open neighborhood of $(p, q) \in S$, and so S is open in $T_0 \times T_0$. Since S is also closed in $T_0 \times T_0$ and is nonempty, the connectedness of $T_0 \times T_0$ implies that $S = T_0 \times T_0$. The proof of the convexity of $\psi(T_0)$ is thus complete.

2. The rigidity theorem

Theorem. *If M' is the manifold obtained by deleting a finite number of points $p_1, \cdots, p_n$ from a compact, orientable 2-manifold M with a C^5 Riemannian metric whose Gaussian curvature is everywhere nonnegative, and the Gaussian curvature of this metric on M is positive at each of the points $p_1, \cdots, p_n$, then any C^3 isometric embedding ϕ': $M' \to R^3$ is the restriction to M' of a C^2 isometric embedding ϕ: $M \to R^3$, and the isometric embedding ϕ' is rigid in the class of C^3 isometric embeddings of M'.*

Proof. Since ϕ': $M' \to R^3$ is an isometric embedding, the Riemannian distance between two points q_1, q_2 of M' is greater than or equal to the distance between $\phi'(q_1)$ and $\phi'(q_2)$ in R^3. Hence ϕ' maps Cauchy sequences in M' to Cauchy sequences in R^3, and consequently there is a unique continuous extension ϕ: $M \to R^3$ of ϕ' to all of M. ϕ' is by assumption an embedding: the topology induced on M' by ϕ' from the topology of R^3 agrees with the manifold topology. This induced topology is thus Hausdorff; it follows that ϕ: $M \to R^3$

is one-to-one except for possible identifications of the points $p_1, \cdots, p_n$ to each other under ϕ.

Lemma 2. *$R^3 - \phi(M)$ has precisely two components, and $\phi(M)$ is their common boundary.*

Proof (after Alexander's proof of the Jordan-Brouwer theorem). Choose a triangulation of M with $p_1, \cdots, p_n$ among the vertices. The image under ϕ of this triangulation is a triangulation of $\phi(M)$. It is easy to verify using these triangulations that

$$H_2(\phi(M); Z_2) \cong H_2(M; Z_2) \cong Z_2 .$$

Then $H^2(\phi(M); Z_2) = Z_2$. From Alexander duality, $H_0(R^3 - \phi(M); Z_2) = H^2(\phi(M); Z_2) \oplus Z_2$. Thus $H_0(R^3 - \phi(M); Z_2) = Z_2 \oplus Z_2$, and $R^3 - \phi(M)$ has exactly two components, say I and E.

$\mathrm{Cl}\, I \cap \mathrm{Cl}\, E \subseteq \phi(M)$. Suppose that $\phi(M) - (\mathrm{Cl}\, I \cap \mathrm{Cl}\, E)$ is not empty. Then $\phi^{-1}(\phi(M) - (\mathrm{Cl}\, I \cap \mathrm{Cl}\, E))$ is open (and nonempty) in M. Hence there is a set $D \subset M'$ such that D is homeomorphic to the closed unit disc and such that $\phi(\mathring{D}) \subset \phi(M) - (\mathrm{Cl}\, I \cap \mathrm{Cl}\, E)$. ($\mathring{D}$ = interior of D.) Since $\phi(\mathring{D}) \subset \phi(M) - (\mathrm{Cl}\, I \cap \mathrm{Cl}\, E)$, $R^3 - \phi(M - \mathring{D})$ is not connected. But $H^2(M - \mathring{D}; Z_2) \cong H_2(M - \mathring{D}; Z_2) = 0$ and as before $H^2(\phi(M - \mathring{D}); Z_2) \cong H^2(M - \mathring{D}; Z_2)$. Hence, again applying Alexander duality, $H_0(R^3 - \phi(M - \mathring{D}); Z_2) = Z_2$ so that $R^3 - \phi(M - \mathring{D})$ is connected. This contradiction completes the proof. (For a more detailed version of this argument, see the Appendix of [4].)

Note that the compactness of $\phi(M)$ implies that $R^3 - \phi(M)$ has only one unbounded component. According to Lemma 2, $R^3 - \phi(M)$ has then exactly one bounded component and one unbounded component; hereafter I denotes the bounded component of $R^3 - \phi(M)$ (the "interior" of $\phi(M)$) and E the unbounded component (the "exterior" of $\phi(M)$).

Lemma 3. *The complement in M' of the set of planar points of ϕ' is connected.*

Proof. Let T be the set of planar points of φ'. As noted in the proof of the proposition of § 1, T is a compact subset of M' and hence a closed subset of M. A closed subset of the sphere separates the sphere only if one of its components separates the sphere [10, p. 123]. Since $T \cup \{p_1, \cdots, p_n\}$ is closed and has as its components the components of T together with the one-point sets $\{p_1\}, \cdots, \{p_n\}$, it suffices for the proof of the present lemma to show that no component of T separates M. If a component T_0 of T separated M, then any subset of M homeomorphic to this component would also separate M [8, p. 101]. Every component of T is homeomorphic to a compact convex plane set by the proposition of § 1. Hence T_0 would be homeomorphic to a point, a closed straight line segment, or the closed unit disc. Since each of these three clearly has a homeomorphic image in M which fails to separate M, the lemma follows.

It follows immediately by continuity considerations from Lemma 3 that the semidefinite second fundamental form of $\phi'\colon M' \to R^3$ (relative to either of the two continuous unit normal fields defined on all of $\phi'(M)$) is of constant sign, that is, is either positive semidefinite everywhere or negative semidefinite everywhere in M'. This conclusion will be used in the proofs of Lemmas 4 and 5.

For a given vector $\vec{N} \in S^2$, the function $h_N\colon M \to R$ defined by $h_N(p) = \overrightarrow{\phi(p)} \cdot \vec{N}$ is C^3 on M', and its critical points in M' are precisely those points $p \in M'$ at which the tangent plane to $\phi(M')$ at $\phi(p)$ is perpendicular to $\vec{N}$. Such a critical point can be degenerate only if $\vec{N}$ or $-\vec{N}$ is a critical value of the Gauss map $\eta\colon M' \to S^2$. By Sard's theorem [17], the set Ⓒ of critical values of the C^2 map η is of measure 0 in S^2. Thus Ⓒ $\cup$ $-$ Ⓒ($-$Ⓒ $= \{\vec{v} \in S^2 \mid -\vec{v} \in$ Ⓒ$\}$) is also of measure 0 in S^2. The set Ⓐ of vectors $\vec{N} \in S^2$ for which h_N has only nondegenerate critical points, which set contains the complement in S^2 of Ⓒ $\cup$ $-$ Ⓒ, is therefore everywhere dense in S^2.

Lemma 4. *If $\vec{N} \in$ Ⓐ, then, for any $t \in R$, the intersection of the set $P_t = \{\vec{v} \in R^3 \mid \vec{v} \cdot \vec{N} = t\}$ with $\phi(M)$ is either empty or connected.*

Proof. Define a C^k *generalized circle* to be either a point or a C^k diffeomorph of a circle; for brevity, a C^0 generalized circle will be referred to hereafter simply as a generalized circle. A generalized circle which is not a single point will be called a *nondegenerate generalized circle*. Clearly, it suffices to show that $P_t \cap \phi(M)$ consists of at most a single generalized circle. Since the proof of this fact is very nearly identical to the proof of the Lemma in [4], only an outline of the argument will be given here together with a description of the modifications needed to fit the argument to the present situation. The reader is then referred to [4] for the remaining details.

Throughout the following discussion, denote h_N by h and let $t_1, \cdots, t_m$ be an ordered listing ($t_1 < \cdots < t_m$) of the finite set $\{h\phi(p_1), \cdots, h\phi(p_n)\}$ so that $\{\phi(p_1), \cdots, \phi(p_n)\} \subset (P_{t_1} \cap \phi(M)) \cup \cdots \cup (P_{t_m} \cap \phi(M))$ but no such inclusion holds for any smaller set of P_t's.

A number of preliminary conclusions will now be stated and discussed:

(A) If $t \notin \{t_1, \cdots, t_m\}$, then $P_t \cap \phi(M)$ is a finite disjoint union of generalized circles of class C^3.

This assertion follows easily from the facts that $P_t \cap \phi(M) \subset \phi(M')$ and that h has as critical points in $\phi(M')$ at most nondegenerate and hence isolated maxima and minima.

(B) Each generalized circle of (A) is a convex curve in P_t (i.e., it lies entirely on one side of each of its tangents and thus bounds a convex domain in P_t).

This assertion is an essential point in the proof of the lemma; the convexity of these curves of intersection is precisely the property which makes it possible to deduce enough information about the behavior of $\phi(M)$ in a neighborhood of each $\phi(p_i)$ to complete the proof. The fact that the intersection curves are convex is a consequence of the constancy of sign of the second fundamental form of $\phi(M')$, which was deduced from Lemma 3, and of Meusnier's theorem (Willmore [19, p. 96]). Since $P_t \cap \phi(M) \subset \phi(M')$, this constancy of sign implies that the sign of the curvature of each C^3 generalized circle (which is not a point) is constant when the generalized circle is considered as a plane curve in P_t. It is a standard result that a plane curve whose curvature has constant sign is convex.

(C) If the critical points of h and the points $\phi(p_1), \cdots, \phi(p_n)$ are removed from $P_{t_j} \cap \phi(M)$, the remainder consists of a disjoint union of C^3 diffeomorphs of straight lines and circles. The critical points of h form a discrete set in $P_{t_j} \cap \phi(M)$.

These facts are immediate consequences respectively of the implicit function theorem and of the fact that the critical points of h in $\phi(M')$ are nondegenerate and hence isolated.

$P_{t'} \to P_t$ as $t' \to t$; it is intuitively clear that the intersection $P_{t'} \cap \phi(M)$ converges to $P_t \cap \phi(M)$ in a uniform fashion as $t' \to t$ because of the facts that the critical points of h are nondegenerate so that the level sets of h on $\phi(M')$ vary uniformly and that the value of ϕ at each p_i is uniquely determined by the values of ϕ in a (deleted) neighborhood of p_i in M'. To make this observation precise, an explicit description of the variation of $P_t \cap \phi(M)$ with t is needed:

Let grad h be the gradient vector field of the function h relative to the induced metric on $\phi(M')$ and set $V = \operatorname{grad} h / \|\operatorname{grad} h\|^2$. V is defined everywhere on $\phi(M)$ except at the points $\phi(p_1) \cdots \phi(p_n)$ and the critical points of h. On its domain of definition, V generates a local one-parameter group of local diffeomorphisms η_t; explicitly, if V is defined in a neighborhood of $p \in \phi(M')$, and $\gamma\colon [0, \varepsilon) \to \phi(M)$ is the integral curve of V issuing from p, then $\eta_t(p) = \gamma(t)$. Observe that, if $h(p) = t'$, then $h(\gamma(t)) = t + t'$ for all t for which $\gamma(t)$ is defined; equivalently, if $p \in P_{t'} \cap \phi(M)$ and $\eta_t(p)$ is defined, then $\eta_t(p) \in P_{t+t'} \cap \phi(M)$.

Suppose that $t \notin \{t_1, \cdots, t_m\}$, and let C be a C^2 diffeomorph of the unit circle in $P_t \cap \phi(M)$. Suppose further that there is a t' such that $\eta_s(p)$ is defined for all $p \in C$ and all s such that $0 \leq s < t'$. Then it can be shown as in [4] that $\eta_s(p)$ approaches a limit, to be called $\eta_{t'}(p)$, as $s \to t'$ and that the mapping $p \to \eta_{t'}(p)$ is a continuous mapping of C. Symbolically, write $C \uparrow \eta_{t'}(C)$. Similarly, if $C \subseteq P_t \cap \phi(M)$ is a C^2 diffeomorph of the circle and the integral curves $\{\xi_p(x)\}$ of $-V$ issuing from points p of C are defined for all $p \in C$ and all s such that $0 \leq s < t'$, then $\eta_{-t'}(p) = \lim_{s \uparrow t'} \xi_p(s)$ exists for all $p \in C$, and the mapping $p \to \eta_{-t'}(p)$ is continuous. Again, write symbolically $C \downarrow \eta_{-t'}(C)$. The following statement (D) describes the situation in which the hypotheses required for the definition of the maps $\eta_{t'}$ and $\eta_{-t'}$ are satisfied (only the case of $\eta_{t'}$, $t' > 0$, will be treated explicitly; the case of $\eta_{-t'}$ is obtainable by obvious minor modifications).

(D) Let $t_{j-1} < t < t_j$ and $t' = t_j - t$. If $C \subseteq P_t \cap \phi(M)$ is a C^2 diffeomorph of the unit circle and γ_p are the integral curves of V issuing from $p \in C$

with $\gamma_p(0) = p$, then one but not both of the following conditions (α), (β) is satisfied: (α) There is some $p \in C$ such that, for some u with $0 < u < t'$, $\gamma_p(u)$ is not defined but $\gamma_p(s)$ is defined for $0 \leq s < u$. Then, for all $q \in C$, $\gamma_q(s)$ is defined for $0 \leq s < u$, but $\gamma_q(u)$ is undefined, and there exists a local maximum Q of h such that $\lim_{s \uparrow u} \gamma_q(s) = Q$ for all $q \in C$. (β) For all $p \in C$, $\gamma_p(s)$ is defined for $0 \leq s < t'$. Then $C \uparrow \eta_{t'}(C)$.

(E) If $Q \in P_{t_j} \cap \phi(M)$ is an isolated point in $P_{t_j} \cap \phi(M)$, then for some $t \neq t_j$ and some $C \subseteq P_t \cap \phi(M)$, C is a C^2 diffeomorph of the circle, $C \uparrow Q$ (if $t < t_j$) or $C \downarrow Q$ (if $t > t_j$).

This assertion follows immediately if Q is a critical point of h in $\phi(M')$. In case $Q \in \phi(M')$, a more intricate argument is needed and is given in [4], where it is also shown that:

(F) If $Q \in P_{t_j} \cap \phi(M)$ is not an isolated point, then for some $\varepsilon > 0$ there are C^2 diffeomorphs C_1 and C_2 of the circle with $C_1 \subseteq P_{t_j-\varepsilon} \cap \phi(M)$ and $C_2 \subseteq P_{t_j+\varepsilon} \cap \phi(M)$ such that $C_1 \uparrow \eta_\varepsilon(C_1)$ with $Q \in \eta_\varepsilon(C_1)$ and $C_2 \downarrow \eta_{-\varepsilon}(C_2)$ with $Q \in \eta_{-\varepsilon}(C_2)$.

Statements (E) and (F) together assert that all of $P_{t_j} \cap \phi(M)$ is an upward or downward limit of $P_t \cap \phi(M)$, $t \notin \{t_1, \cdots, t_m\}$. Statement (G) asserts that these limits are uniform:

(G) Let C be a C^2 diffeomorph of the circle in $P_t \cap \phi(M)$, $t_{j-1} < t < t_j$, such that all the integral curves $\gamma_p(s)$ of V issuing from points $p \in C$ are defined for $0 \leq s < t'$. Let C_s be the set $\{\gamma_p(s) \colon p \in C\}$. Then C_s converges uniformly to $\eta_{t'}(C)$, i.e., for every neighborhood U of $\eta_{t'}(C)$ there is some s_1 such that $C_s \subset U$ if $s_1 < s < t'$.

(H) For all t, $P_t \cap \phi(M)$ is a union of (C^0) generalized circles; each generalized circle is either an isolated point or a convex (C^0) curve, and no one of the generalized circles intersects the interior of any other.

To verify (H), recall statement (B): if $t \notin \{t_1, \cdots, t_m\}$, then $P_t \cap \phi(M)$ is a union of disjoint convex plane curves. The uniform limit of convex plane curves is a convex plane curve and hence a (C^0) generalized circle (note that convexity is essential for this conclusion concerning the uniform limit: the uniform limit of arbitrary generalized circles is not necessarily a generalized circle). Then (G) combined with (E) and (F) implies that, for any t, $P_t \cap \phi(M)$ is a union of convex generalized circles. That no one of these convex curves intersects the interior of any other is a consequence of Lemma 2 and the fact, easily derived from the observations used to prove (B), that the interior in P_t of any of the generalized circles in $P_t \cap \phi(M)$, $t \notin \{t_1, \cdots, t_m\}$, is contained in the interior of $\phi(M)$. Since every point of $\phi(M)$ is a boundary point of the interior of $\phi(M)$ by Lemma 2, these generalized circles cannot intersect one another's interiors in P_t. The same conclusion then follows for $t \in \{t_1, \cdots, t_m\}$ by a limit argument using (G).

(I) Any two of the generalized circles of (H) can intersect only at one of the $\phi(p_1), \cdots, \phi(p_n)$.

To verify this assertion, it suffices to observe that an intersection point in $\phi(M')$ would have to be a critical point of h, but that each critical point of h in $P_t \cap \phi(M')$, being a nondegenerate maximum or minimum of h, is an isolated point in $P_t \cap \phi(M')$.

Now a relation on the nondegenerate generalized circles in all the level sets $P_t \cap \phi(M)$ can be defined as follows:

Two nondegenerate generalized circles C' and C'' in $P_{s_1} \cap \phi(M)$ and $P_{s_2} \cap \phi(M)$, respectively, are *deformation-related* if there is a chain $C' = C_1$, $C_2, \cdots, C_{l-1}, C_l = C''$ of nondegenerate convex curves satisfying

(1) $C_1 \subseteq P_{s'} \cap \phi(M)$, $C_2 \subseteq P_{s''} \cap \phi(M), \cdots, C_l \subseteq P_{s^{(l)}} \cap \phi(M)$ with either $s_1 = s' > s'' > \cdots > s^{(l)} = s_2$ or $s_1 = s' < s'' < \cdots < s^{(l)} = s_2$.

(2) If $s_1 > s_2$, then for all i either $C_i \downarrow \eta_t(C_i) = C_{i+1}$ or $C_{i+1} \uparrow \eta_t(C_{i+1}) = C_i$ for some t. If $s_1 < s_2$, then for all i either $C_i \uparrow \eta_t(C_i) = C_{i+1}$ or $C_{i+1} \downarrow \eta(C_{i+1}) = C_i$ for some t.

The symmetry and reflexiveness of this relation are obvious; that the relation is transitive and hence is an equivalence relation follows from the observation that by (H) the generalized circles in each $P_t \cap \phi(M)$ have disjoint interiors in P_t and hence that a given nondegenerate generalized circle $C_{t'} \subseteq P_{t'} \cap \phi(M)$ cannot be the limit from below (and similarly from above) of two distinct families of generalized circles $C_t \subseteq P_t \cap \phi(M)$ and $C_t^* \subseteq P_t \cap \phi(M)$ as $t \uparrow t'$.

Let $I_1, \cdots, I_\alpha, \cdots$ be the subsets of $\phi(M)$ obtained by taking, for each distinct equivalence class of nondegenerate generalized circles under the deformation-relatedness equivalence relation, the union of the sets of points of the generalized circles. It is then easily verified from statements (A) and (I) that $I_\alpha \cap I_\beta \subseteq \{\phi(p_1), \cdots, \phi(p_n)\}$ if $I_\alpha \neq I_\beta$.

To complete the proof that $P_t \cap \phi(M)$ consists of at most a single generalized circle, consider the set A obtained by deleting from $\phi(M)$ the critical points of h in $\phi(M')$ and the points $\phi(p_1), \cdots, \phi(p_n)$. The deleted set is countable because the critical points of h are isolated, and hence it has topological dimension 0, while $\phi(M)$ has topological dimension 2; therefore, A is connected by a theorem of Hurewicz and Wallman [8, p. 48]. Moreover, $I_\alpha \cap A$ is open in A by statement (D) and the existence of box-like neighborhoods in $\phi(M)$ of points in $I_\alpha \cap A$. But $I_\alpha \cap I_\beta \subseteq \{\phi(p_1), \cdots, \phi(p_n)\}$ for $I_\alpha \neq I_\beta$, so that $(I_\alpha \cap A) \cap (I_\beta \cap A) = \phi$ for $I_\alpha \neq I_\beta$. Thus the connectedness of A implies that there is at most one distinct nonempty $I_\alpha \cap A$. If $I_\alpha \cap A$ is empty so is I_α since I_α is either empty or nondenumerable. Thus there is only one nonempty I_α, say I. Here for any t, $P_t \cap \phi(M)$ contains at most one generalized circle which is nondegenerate.

Suppose that, for some $t \neq t_1, \cdots, t_m$, $P_t \cap \phi(M)$ consisted of more than one generalized circle. If one of these circles were nondegenerate, then, for some $t' \neq t_1, \cdots, t_m$ near t, $P_{t'} \cap \phi(M)$ would contain two nondegenerate generalized circles. Thus, if $P_t \cap \phi(M)$ contains more than one generalized circle, then $P_t \cap \phi(M)$ most consist only of isolated points, say $q_1, \cdots, q_k$, $k \geq 2$, each q

being either a maximum or a minimum of h. If two of the q's are maxima, then for some $t' \neq t_1, \cdots, t_m$ slightly less than t, $P_{t'} \cap \phi(M)$ would contain two nondegenerate generalized circles; thus there cannot be two maxima among the q's. Similarly there cannot be two minima among the q's. Thus there can be but two q's, one a maximum and one a minimum of h. But then the connected set $\phi(M) - \{q_1, q_2\}$ is separated by the plane P_t. This contradiction shows that $P_t \cap \phi(M)$ contains at most one generalized circle and is hence connected for $t \neq t_1, \cdots, t_m$. Since the limit of a single family of convex generalized circles can be only a single generalized circle, it follows then from statements (E), (F), and (G) that $P_t \cap \phi(M)$ is at most a single generalized circle and is hence connected for $t = t_1, \cdots, t_m$ as well. Hence the proof of Lemma 4 is complete.

To resume the proof of the theorem, observe that if $p \in M'$ and U is a sufficiently small neighborhood in R^3 of $\phi(p)$, then $U \cap \phi(M') = U \cap \phi(M)$. If the Gaussian curvature is positive at p, then such a U can be so chosen as to satisfy the additional condition that the intersection of the tangent plane of $\phi(M')$ at $\phi(p)$ with $U \cap \phi(M')$ (and hence with $U \cap \phi(M)$) contains only the single point p. If in addition $\eta(p)$ is in the set Ⓐ of vectors $\vec{N}$ in S^2 for which the function h_N has only nondegenerate critical points, it follows from Lemma 4 that the intersection of the tangent plane of $\phi(M')$ at $\phi(p)$ with all of $\phi(M)$ contains only the single point p. From the connectedness of $M - \{p\}$, it follows then that $\phi(M)$ lies entirely in one of the (closed) half-spaces of R^3 determined by the tangent plane of $\phi(M')$ at $\phi(p)$. Let Ⓑ be the set of points $p \in M'$ such that the Gaussian curvature of M' is positive at p and $\eta(p) \in$ Ⓐ. Note that Ⓑ is not empty; in fact, since Ⓐ is everywhere dense in S^2 and η is an open mapping wherever the Gaussian curvature is positive, Ⓑ is dense in the set of points of M' at which the Gaussian curvature is positive. If $p \in$ Ⓑ, then since $\phi(M)$ lies entirely in one of the closed half-spaces determined by the tangent plane of $\phi(M')$ at $\phi(p)$, so does $I \cup \phi(M)$.

Lemma 5. *$I \cup \phi(M)$ is a compact convex body in R^3.*

Proof. A plane P in R^3 containing a point x in the boundary of a set in R^3 is called a *local support plane* of the set at x if there is some neighborhood of x in R^3 such that the intersection of the set with the neighborhood lies entirely in one of the closed half-spaces determined by P. A theorem of E. Schmidt is that if a connected set with nonempty interior has a local support plane at each of its boundary points then the set is convex. Thus for the proof of the lemma it suffices (since I is open and nonempty) to show that $I \cup \phi(M)$ has a local support plane at each point of its boundary $\phi(M)$.

Denote by $P_p (p \in M')$ the tangent space in R^3 of $\phi(M')$ at $\phi(p)$. The constancy of the sign of the semidefinite second fundamental form of $\phi' : M' \rightarrow R^3$ implies that in a neighborhood of each $\phi(p)$, $p \in M'$, the surface $\phi(M')$ lies in one of the closed half-spaces determined by P_p. To verify this implication, observe that in a neighborhood of $\phi(p)$ the surface $\phi(M')$ is representable as the graph of a function, that by the constancy of the sign of the second fundamental form this function is convex, and that the graph of a convex function lies locally on one side of its tangent plane. Let H_p be the closed half-space determined by P_p which contains $\phi(M')$ in a neighborhood of $\phi(p)$. It was shown previously that if $p \in \textcircled{B}$, then $I \cup \phi(M)$ lies in H_p. Since $\textcircled{B}$ is not empty and H_p varies continuously with p, it follows that for every $p \in M'$, $I \cup \phi(M)$ lies in H_p in some neighborhood of $\phi(p)$. It remains to show that $I \cup \phi(M)$ has a local support plane at each of the $\phi(p_i)$, $i = 1, \cdots, n$. For a given i, there is a sequence $\{q_j \mid j = 1, 2, \cdots; q_j \in \textcircled{B}\}$ of points of $\textcircled{B}$ approaching p_i, since $\textcircled{B}$ is dense in the set of points of M' of positive curvature. There is a subsequence $\{q'_j\}$ such that $\{P_{q'_j}\}$ is a convergent sequence of planes in R^3. The limit P of $\{P_{q'_j}\}$ is a (local) support plane at $\phi(p_i)$, for if $I \cup \phi(M)$ did not lie in one of the closed half-spaces determined by P, then for q'_j sufficiently near p_i, $I \cup \phi(M)$ could not lie entirely in $H_{q'_j}$. (Note that this argument depends upon the fact that $I \cap \phi(M)$ lies in $H_{q'_j}$ altogether, not just in some neighborhood of q'_j; a limit of local support planes is not necessarily a local support plane.) q.e.d.

It was observed earlier that the continuous map ϕ is one-to-one except for possible identities among the $\phi(p_1), \cdots, \phi(p_n)$. Since $\phi(M)$ is the boundary of a compact convex body, it is homeomorphic to S^2 and $H_1(\phi(M); Z) = 0$. But it is easy to see by using the triangulations of M and $\phi(M)$ described in the proof of Lemma 2, that if identifications among the p_i occur under ϕ then $H_1(\phi(M); Z) \neq 0$. Hence $\phi\colon M \to \phi(M)$ must be one-to-one and, being then a one-to-one continuous map from one compact Hausdorff space to another, must be a homeomorphism.

Let D_ε be the disc of radius ε in M about a fixed point p, and E_ε be the set of all oriented unit normals to the planes of support of $\phi(M)$ at points of $\phi(D_\varepsilon)$. (*A* (global) *plane of support* P at a point of a surface is a plane through p such that the surface lies entirely in one of the closed half-spaces determined by P.) Define

$$\tilde{K}(\phi(p)) = \lim_{\varepsilon \to 0} \int_{E_\varepsilon} \Omega_S \Big/ \int_{D_\varepsilon} \Omega_M ,$$

where Ω_M is the volume element of M, and Ω_S is the volume element of the unit sphere S^2 of R^3. It will now be shown that $\tilde{K}(\phi(p)) = K(p)$, the Gaussian curvature of M at p. Since $I \cup \phi(M)$ is convex, a local support plane of $I \cup \phi(M)$ is necessarily a global support plane of $\phi(M)$. It was shown in the proof of Lemma 5 that for any $p \in M'$ the tangent plane to $\phi(M')$ at $\phi(p)$ was a local support plane of $I \cup \phi(M)$; clearly no plane other than this tangent plane can be a local support plane of $I \cup \phi(M)$ at $\phi(p)$. It now follows easily from the interpretation of the Gaussian curvature as the Jacobian of the Gauss map and the continuity of the Gaussian curvature of M that $\tilde{K}(\phi(p)) = K(p)$ if $p \in M'$. To see that $\tilde{K}(\phi(p)) = K(p)$ if $p \in M - M'$, note that the Gauss map is one-to-one on the set Ⓑ; this fact is a consequence of the previous observations that if $q \in$ Ⓑ, then the tangent plane to $\phi(M')$ at $\phi(q)$ intersects $I \cup \phi(M)$ only at the point $\phi(q)$ and that $I \cup \phi(M)$ lies everywhere on the same side (relative to the oriented normal) of the tangent planes of $\phi(M')$. Now $\eta(M') - \eta($Ⓑ$)$ has measure 0 in S^2, because $\eta(M') - \eta(M'_+) \subset$ Ⓒ has measure zero and $\eta(M'_+) - \eta($Ⓑ$) \subset$ Ⓒ $\cup -$ Ⓒ has measure 0, where M'_+ is the set of points of M' of positive curvature. Since

$$\int_M K\Omega_M = \int_{M'} K\Omega_{M'} = \int_{M'} \eta^*(\Omega_S) = 4\pi \ ,$$

the fact that $\eta(M') - \eta($Ⓑ$)$ has measure 0 implies that

$$4\pi = \int_B \eta^*(\Omega_S) \ .$$

Thus, since $\eta\,|$ Ⓑ is one-to-one, the measure of $\eta($Ⓑ$)$ in S^2 is 4π. Hence

$$\int_{E_\varepsilon} \Omega_S = \int_{\eta(Ⓑ) \cap E_\varepsilon} \Omega_S = \int_{Ⓑ \cap D_\varepsilon} \eta^* \Omega_S = \int_{D_\varepsilon} K\Omega_M \ ,$$

$$\tilde{K}(\phi(p)) = \lim_{\varepsilon \to 0} \left(\int_{E_\varepsilon} \Omega_S \Big/ \int_{D_\varepsilon} \Omega_M \right) = \lim_{\varepsilon \to 0} \left(\int_{D_\varepsilon} K\Omega_M \Big/ \int_{D_\varepsilon} \Omega_M \right) = K(p) \ .$$

Thus $\tilde{K}(\phi(p))$ is equal to $K(p)$ for all $p \in M$, and $\tilde{K}$ is consequently bounded everywhere from above, and in a neighborhood of each p_i is bounded away from zero as well. By a theorem of Alexandrov [1](see also [12, p. 27]), $\phi(M)$ is a C^1 submanifold of R^3. ϕ: $M \to \phi(M)$ is an isometry except at (possibly) a finite number of points. Thus ϕ is a distance preserving map in the sense that both ϕ and ϕ^{-1} map rectifiable curves to rectifiable curves and that the length of rectifiable curves is preserved under ϕ and ϕ^{-1}. The following regularity theorem [13] of Pogorelov implies that M is a C^4 submanifold of R^3.

Theorem (*Pogorelov*). *Let N be a convex surface of class C^1 in R^3. Suppose that in a neighborhood of some point p in N the generalized Gaussian curvature is positive and finite and that moreover there is a distance-preserving homeomophism of some neighborhood of p in N onto a Riemannian manifold whose metric is of class C^k ($k \geq 2$). Then N is necessarily a C^{k-1} submanifold of R^3 in some neighborhood of p.*

Pogorelov considers explicitly only the case in which the generalized Gaussian curvature of N is everywhere positive and finite, and in which a distance-preserving homeomorphism onto a Riemannian manifold with C^k metric is defined on all of N. However, his proof that the surface N is C^{k-1} in a neighborhood of an arbitrary point $p \in N$ uses these assumptions only locally, in a neighborhood of p, and it is easy to verify that the same arguments yield the local result stated here.

Since $\phi(M)$ is a C^4 submanifold of R^3, the Riemannian metric induced on $\phi(M)$ is of class C^3. The Myers-Steenrod theorem (cf. Palais [11]) implies that the distance-preserving map $\phi\colon M \to \phi(M)$ is an isometry of class C^2 at least. Thus $\phi\colon M \to R^3$ is a C^2 isometric embedding. By the rigidity theorem of Sacksteder [15] and Voss [18], the map ϕ is rigid in the class of C^2 isometric embeddings of M. Hence ϕ' is necessarily rigid in the class of C^3 isometric embeddings of M'.

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