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SINGULAR HOMOLOGY OVER Z ON TOPOLOGICAL MANIFOLDS

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0. Introduction

The differentiable case. On an arbitrary connected, differentiable manifold M_n of class C^∞, there always exists a real-valued nondegenerate (abbreviated ND) function f of class C^∞ with the following properties:

(a) *For each value c of f the subspace*

$$f_c = \{p \in M_n \mid f(p) \leq c\} \tag{0.1}$$

of M_n is compact.

(b) *The function f has different values a at different critical points.*

(c) *There is just one critical point of f of index* 0.

That such a function f exists on a manifold M_n of class C^∞ is established in the compact case in [12]. For the non-compact case see Theorem 23.5 and Lemma 22.4 of [1].

Singular homology groups on subspaces of M_n are understood in the sense of Eilenberg [2]. See also Part III of [1]. In this paper these groups are taken over Z, the ring of integers. With each critical point of a ND f we shall associate "relative numerical invariants"[1] such that the following is true:

Theorem 0.1. *There exists an inductive group-theoretic mechanism by virtue of which relative numerical invariants "associated"[2] with the critical points of f on f_c determine, up to an isomorphism, the singular homology groups over Z of the subspace f_c of M_n.*

The results in this paper were abstracted in part in Appendix III of [1]. In preparation for this paper a preliminary paper [3] has been written. Paper [3] is concerned with quotients A/W of a finitely generated abelian group A by a cyclic subgroup W of A. Given the *invariants* of A, namely the torsion coefficients of the torsion subgroup $\mathscr{T}$ of A, and the dimension[3] β of a free group $\mathscr{B}$ "complementary" in A to $\mathscr{T}$, paper [3] makes explicit a simple mechanism

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[1] Defined in §6.

[2] Associated as in Condition 7.1.

[3] β is termed the "Betti number" of A and $\mathscr{B}$ a "Betti subgroup" of A.

for calculating the corresponding invariants of A/W. The data include an integral linear representation of a generator w of W in terms of a "basis" for A.

The relevance of a triangulation of M_n. The most novel and one of the most important aspects of this paper is that in setting up the mechanism affirmed to exist in Theorem 0.1 no use is made of a global triangulation of M_n, although such a triangulation exists in the differentiable case. For the purposes of theorems such as Theorem 0.1 the existence of a triangulation of the underlying space is neither necessary nor relevant. Cf. [1].[4]

The nondifferentiable case. Although this case will not be studied in this paper for the sake of simplicity, one can state the following. If there exists on M_n a real-valued function f which is topologically nondegenerate (abbreviated TND) in the sense of [10] and satisfies the above conditions (a), (b), and (c), in a topological sense, then the mechanism affirmed to exist in Theorem 0.1 can still be set up. Some differences in proof are required. In particular the trajectories globally transverse to the f-level manifolds in the differentiable case must be replaced by trajectories whose definition is local and which in general cannot be globally extended. See [5].

The class of topological manifolds M_n which admit TND functions includes the class of combinatorial triangulated manifolds admitted by Eells and Kuiper, in [8] as shown by these authors. The experience gained in the study of deformations in [5] has led Morse to the conjecture that there exist compact topological manifolds which admit no triangulation but do admit TND functions.

It is hoped that the discussion of the relevance and generality of the methods used in this paper will not obscure the nature of the mechanism by which the singular homology groups of the sublevel sets f_c are determined.

For a more complete set of references see the book [1] by Morse and Cairns. The work of R. C. Kirby and L. Siebenmann on TND functions, as yet unpublished, is awaited with maximum interest.

1. Singular homology on a Hausdorff space χ

This section reviews some of the basic terms in singular homology theory on a Hausdorff space χ. As already indicated, this paper is concerned with homology groups over Z rather than over a field. However, much of the homology theory over a field presented in [1] carries over, with no change or minor changes, to homology theory over Z. These changes will be noted when necessary.

Homology theory over Z has its algebraic basis in abelian group theory. Singular homology theories over Z or over a field start with a common

[4] In the book [1] the singular homology groups are taken over an arbitrary field $\mathscr{K}$. The invariants there attached to a critical point p are its index and invariants characterizing p as of "linking" or "non-linking" type. These invariants uniquely determine the singular homology groups over $\mathscr{K}$ of the sublevel sets f_c, up to an isomorphism.

definition of a singular cell. As in [1] we make use of Eilenberg's definition of such cells. See [2].

Given a Hausdorff space χ and an integer $r \geq 0$, a singular r-cell σ^r is defined as in § 26 of [1]. The set of singular r-cells on χ is a "base" in the sense of Bourbaki [9, p. 42] of a Z-module $C_r(\chi, Z)$. The elements of $C_r(\chi, Z)$ are termed *integral r-chains*. For $r < 0$ we understand that $C_r(\chi, Z)$ is the Z-module 0. The "carrier" of a singular r-cell σ^r is denoted by $|\sigma^r|$.

The boundary operator ∂. Given a singular cell σ^q, $\partial\sigma^q$ is defined as in § 26 of [1]. One extends ∂ linearly over $C_q(\chi, Z)$ to define a homomorphism

$$\partial\colon C_q(\chi, Z) \to C_{q-1}(\chi, Z) \tag{1.1}$$

for each integer q. By virture of Lemma 24.4 of [1] the composite homomorphism $\partial\partial$ maps $C_q(\chi, Z)$ onto the null element in $C_{q-2}(\chi, Z)$.

The Z-module $Z_q(\chi, Z)$. An integral q-chain c^q is termed an *integral q-cycle* if $\partial c^q = 0$. The integral q-cycles of $C_q(\chi, Z)$ generate a sub-Z-module of $C_q(\chi, Z)$ denoted by $Z_q(\chi, Z)$, each element of which is an integral q-cycle.

The Z-module $B_q(\chi, Z)$. An integral q-cycle c^q is termed *bounding* over Z if $c^q = \partial c^{q+1}$ for some integral $(q + 1)$-chain c^{q+1}. The integrally bounding q-cycles of $Z_q(\chi, Z)$ generate a sub-Z-module of $Z_q(\chi, Z)$ denoted by $B_q(\chi, Z)$. Each element of $B_q(\chi, Z)$ is an integrally bounding q-cycle.

Homology groups over Z. The quotient group

$$H_q(\chi, Z) = Z_q(\chi, Z)/B_q(\chi, Z) \qquad (q = 0, 1, \cdots) \tag{1.2}$$

is called the q-th *homology group* of χ over Z. The cosets of B_q in Z_q are called *integral homology classes*. Two q-cycles c^q and e^q in the same integral homology class are termed *integrally* homologous, and one writes $c^q \sim e^q$ or $c^q - e^q \sim 0$ over Z.

The homology group $H_q(\chi, Z)$ is a Z-module, or simply an abelian group. If this group is finitely generated, a torsion subgroup $\mathscr{T}_q(\chi)$ and a free gubgroup $\mathscr{B}_q(\chi)$ complementary to $\mathscr{T}_q(\chi)$ exist, so that

$$H_q(\chi, Z) = \mathscr{B}_q(\chi) \oplus \mathscr{T}_q(\chi) \qquad \text{(cf. [4, p. 151]).} \tag{1.3}$$

One calls $\mathscr{B}_q(\chi)$ a *Betti-subgroup* of $H_q(\chi, Z)$ and terms $\dim \mathscr{B}_q(\chi)$ the q-th *Betti number*, $\beta_q(\chi)$ of χ.

We shall find the following concept useful.

Definition 1.1. *Prebases for Betti groups.* Let $\mathscr{B}_q(\chi)$ be a Betti group with a base

$$u_1, \cdots, u_r \qquad (r > 0)\ . \tag{1.4}$$

Each u_i is an integral homology class. If c_i is a cycle in u_i the set

$$c_1, \cdots, c_r \tag{1.5}$$

will be called a *prebase* for $\mathscr{B}_q(\chi)$. See Def. 24.7 of [1].

Although we shall be almost exclusively concerned with homology groups over Z, in Theorem 1.1 we shall recall the relation between connectivities over the field Q of rational numbers and Betti numbers.

Singular homology groups over Q. Homology groups $H_q(\chi, Q)$ are defined in § 26 of [1]. See Eilenberg [2]. Chains, cycles and homology classes over Q are called *rational* and are "singular".

The operator ∂ is defined over the *vector space* $C_q(\chi, Q)$, as in § 26 of [1]. This operator will be denoted by ∂_0 to distinguish it from the operator ∂ of (1.1). The operator ∂_0 acts as a homomorphism

$$\partial_0 \colon C_q(\chi, Q) \to C_{q-1}(\chi, Q) . \tag{1.6}$$

One should recall that Z is a subring of the ring Q and that

$$\partial = \partial_0 | C_q(\chi, Z) . \tag{1.7}$$

Thus $\partial z = \partial_0 z$ when z is an integral chain.

The following theorem will be useful in our study of homology groups over Z. See [6, Ch. V, § 2].

Theorem 1.1. *If $H_q(\chi, Z)$ is finitely generated, then the vector space $H_q(\chi, Q)$ has a finite dimension $R_q(\chi, Q)$ and*

$$\beta_q(\chi) = R_q(\chi, Q) . \tag{1.8}$$

Theorem 1.1 will follow from Lemmas 1.1, 1.2, 1.3.

Notation. If c^q is an integral q-cycle its integral homology class will be denoted by $\bar{c}^q$. If c^q is a rational q-cycle its rational homology class will be denoted by $\tilde{c}^q$. It will be convenient to set

$$(\bar{c}_1^q, \cdots, \bar{c}_\beta^q) = \bar{\boldsymbol{c}}_\beta^q , \tag{1.9}$$

$$(\tilde{c}_1^q, \cdots, \tilde{c}_\beta^q) = \tilde{\boldsymbol{c}}_\beta^q . \tag{1.10}$$

Lemma 1.1. *If c^q is a rational q-cycle, then $c^q \sim 0$ over Q, if and only if for some positive integer m, $mc^q \sim 0$ over Z.*

Proof. If $mc^q \sim 0$ over Z, then

$$mc^q = \partial c^{q+1} = \partial_0 c^{q+1} \tag{1.11}$$

for some integral chain c^{q+1}. Since $m > 0$ and Q is a field, $c^q \sim 0$ over Q.

If $c^q \sim 0$ over Q, then $c^q = \partial_0 c^{q+1}$ for some rational chain c^{q+1}. It follows that for some positive integer m, mc^{q+1} is an integral chain and that

$$mc^q = \partial_0(mc^{q+1}) = \partial mc^{q+1} .$$

Hence $mc^q \sim 0$ over Z.

Corollary 1.1. *Under the hypothesis of Theorem* 1.1 *two integral q-cycles on χ are homologous over Q, if and only if their integral homology classes are equal* mod $\mathscr{T}_q(\chi)$, *that is, differ by a homology class in* $\mathscr{T}_q(\chi)$.

Proof. Let c^q and e^q be integral q-cycles such that $c^q - e^q \sim 0$ over Q. Then by Lemma 1.1 for some integer $m > 0$

$$(1.12) \qquad m(c^q - e^q) \sim 0 \qquad (\text{over } Z) .$$

Hence

$$(1.13) \qquad \bar{c}^q - \bar{e}^q = 0 \qquad (\text{mod } \mathscr{T}_q(\chi)) .$$

Conversely, if (1.13) holds, (1.12) holds for some m. Hence by Lemma 1.1, $c^q - e^q \sim 0$ over Q, completing the proof of the corollary.

We distinguish between the cases $\beta_q(\chi) > 0$ and $\beta_q(\chi) = 0$.

Lemma 1.2. *If $\beta = \beta_q(\chi) > 0$, let $\bar{c}^q_\beta$ (see (1.9)) be a base of a Betti group of $H_q(\chi, Z)$. Then each integral q-cycle c^q satisfies a homology*

$$(1.14) \qquad c^q \sim m_1 c^q_1 + \cdots + m_\beta c^q_\beta \qquad (\text{over } Q) ,$$

where $m_1, \cdots, m_\beta$ are integers determined by c^q.

Proof. By definition of $\bar{c}^q_\beta$,

$$(1.15) \qquad \bar{c}^q = m_1 \bar{c}^q_1 + \cdots + m_\beta \bar{c}^q_\beta \qquad \text{mod } \mathscr{T}_q(\chi) \ (\text{over } Z)$$

for unique integers m_i. Lemma 1.2 follows from (1.15) and Corollary 1.1.

Lemma 1.3. *If $\beta_q(\chi) > 0$ and if $\bar{c}^q_\beta$ is a base of a Betti group of $H_q(\chi, Z)$, then $\tilde{c}^q_\beta$ is a base of $H_q(\chi, Q)$, where $\beta = \beta_q(\chi)$.*

Proof. Let c^q_0 be a rational q-cycle. Then there exists an integer $m \neq 0$ such that $c^q = mc^q_0$ is an integral q-cycle. By Lemma 1.2,

$$(1.16) \qquad c^q = mc^q_0 \sim m_1 c^q_1 + \cdots + m_\beta c^q_\beta \qquad (\text{over } Q)$$

for integers $m_1, \cdots, m_\beta$ determined by mc^q_0. Hence

$$(1.17) \qquad c^q_0 \sim \frac{m_1}{m} c^q_1 + \cdots + \frac{m_\beta}{m} c^q_\beta \qquad (\text{over } Q) .$$

It follows that *$\tilde{c}^q_\beta$ generates $H_q(\chi, Q)$.*

To verify that $\tilde{c}^q_\beta$ is a base of $H_q(\chi, Q)$, we must show the following:

(i) *The set $\tilde{c}^q_\beta$ is independent over Q; that is, if $r_1 c^q_1 + \cdots + r_\beta c^q_\beta \sim 0$ over Q, where r_i is rational, then each $r_i = 0$.*

Proof of (i). Let $m \neq 0$ be such that $m_i = mr_i$ is an integer ($i = 1, \cdots, \beta$). By Corollary 1.1,

$$(1.18) \qquad m_1 \bar{c}^q_1 + \cdots + m_\beta \bar{c}^q_\beta = 0 \qquad \text{mod } \mathscr{T}_q(\chi) \ (\text{over } Z) .$$

Since, by hypothesis, $\bar{c}_\beta^q$ is a base of a Betti group of $H_q(\chi, Z)$, it follows that each $m_i = mr_i = 0$. This, with $m \neq 0$, implies (i).

Theorem 1.1 follows for the case $\beta_q(\chi) > 0$. In the case $\beta_q(\chi) = 0$, each rational q-cycle is homologous to zero over Q, as a consequence of Lemma 1.1. Hence $R_q(\chi, Q) = 0$, completing the proof of Theorem 1.1.

Relative homologies on χ over Z. Relative homologies are necessary in critical point theory. By their use one characterizes the topological effect of removing a critical point p_a at which $f(p) = a$ from f_a. More precise statements will follow.

The introduction to relative homologies over a field $\mathscr{K}$, as given in § 28 of [1] is valid when Z replaces $\mathscr{K}$, if one replaces the cycles and homologies over $\mathscr{K}$ in § 28 by cycles and homologies over Z. We shall reformulate the fundamental Theorem 28.2 of [1] on "coset-contracting isomorphisms".

In Theorem 28.2 of [1], χ is a Hausdorff space and A a subspace of χ. If $A \neq \chi$ we term (χ, A) an *admissible set pair*, and A a *modulus* for χ. Cycles, homologies and homology classes are over Z.

Theorem 1.2. *Coset-contracting isomorphisms. Let (χ, A) and (χ', A') be two admissible "set pairs" with $\chi' \subset \chi$ and $A' \subset A$. Let U be an arbitrary rel.*[5] *homology class (possibly trivial) on χ, and $U' \subset U$ the sub-class of rel.*[6] *cycles on χ'. If, for each non-negative integer q,*

(a) *each rel.*[5] *q-cycle on is rel. homologous on χ to a rel.*[6] *q-cycle on χ', and if*

(b) *each rel.*[6] *q-cycle on χ' which is rel. bounding*[5] *on χ, is rel. bounding*[6] *on χ',*

then each set U' is a rel. homology class[6] *on χ', and the mapping*

$$U \to U' : H_q(\chi, A, Z) \to H_q(\chi', A', Z) \tag{1.19}$$

is a surjective isomorphism.

Note. The second arguments A and A' in H_q in (1.19) are moduli. The third argument is a ring or field, here a ring Z. The homology is over Z.

The isomorphism of Theorem 1.2 will be called *coset-contracting*. Its proof is formally similar to that of Theorem 28.2 of [1].

Excision theorem. The simplified Excision Theorem 28.3 of [1] affirms the existence of a coset-contracting isomorphism over $\mathscr{K}$. By the Excision Theorem over Z we shall mean a theorem similar to Theorem 28.3 but over Z.

Theorem 1.3. *Excision. Let χ be a metric space, A a proper subspace of χ, and A^* a subspace of A such that for some positive e*

$$(\chi - A)_e \subset \chi - A^* , \tag{1.20}$$

where $(\chi - A)_e$ is the open e-neighborhood of $\chi - A$ on χ. There then exists,

[5] That is mod A.
[6] That is mod A'.

for each integer q, a coset-contracting isomorphism

$$H_q(\chi, A, Z) \approx H_q(\chi - A^*, A - A^*, Z) . \tag{1.21}$$

A^* is "excised" from χ and A in the right member of (1.21). The proof of Theorem 1.3 is formally similar to the proof of Theorem 28.3 of [1]. One replaces $\mathscr{K}$ by Z. Cf. [7] Axiom 6.

Definition 1.2. *The induced homomorphism $\hat{\varphi}$.* As in (26.11) of [1] let there be given a continuous mapping φ: $\chi \to \chi'$ of a Hausdorff space χ into a Hausdorff space χ'. Corresponding to a singular q-cell σ^q on χ an image q-cell $\hat{\varphi}\sigma^q$ is defined on χ' by composing each of the "equivalent" continuous mappings τ into χ which define σ^q with φ. The mapping $\hat{\varphi}$ is extended linearly to define homomorphisms

$$\hat{\varphi}\colon C_q(\chi, Z) \to C_q(\chi', Z) \qquad (q = 0, 1, \cdots) . \tag{1.22}$$

Definition 1.3. *The induced homomorphism $\hat{\varphi}_*$.* One shows readily that $\hat{\varphi}$ is *∂-permutable*. Cf. Theorem 26.3b of [1]. It follows that $\hat{\varphi}$ defines homomorphisms

$$Z_q(\chi, Z) \to Z_q(\chi', Z); \qquad B_q(\chi, Z) \to B_q(\chi', Z)$$

for each q and so induces a "natural" homomorphism

$$\hat{\varphi}_*\colon H_q(\chi, Z) \to H_q(\chi', Z) . \tag{1.23}$$

Retracting deformations d are defined as in § 23 of [1]. Theorem 28.4 of [1] has Theorem 1.4 as an analogue.

Theorem 1.4. *Retraction $\to$ isomorphism. Let (χ, A) and (χ', A') be admissible set pairs with $\chi' \subset \chi$, $A' \subset A$ and d a deformation retracting χ onto χ' and A onto A'. There then exist coset-contracting isomorphisms*

$$H_q(\chi, A, Z) \approx H_q(\chi', A', Z) \qquad (q = 0, 1, \cdots) , \tag{1.24}$$

under which the rel. homology class on χ of a rel. q-cycle z on χ goes into the rel. homology class on χ' of $\hat{d}_1 z$, where d_1 is the "terminal" mapping of d.

We add the fundamental theorem giving the homological consequence of a homeomorphism of Hausdorff spaces χ' and χ''.

Theorem 1.5. *Suppose that a Hausdorff space χ' is topologically equivalent to a Hausdorff space χ'' under a homeomorphism Φ of χ' onto χ'' that maps a proper subspace A' of χ' onto a subspace A'' of χ''. There are then induced surjective isomorphisms*

$$\hat{\Phi}_*\colon H_q(\chi', A', Z) \approx H_q(\chi'', A'', Z) \qquad (q = 0, 1, \cdots) , \tag{1.25}$$

under which a rel. homology class on χ' of a rel. q-cycle z goes into the rel. homology class on χ'' of $\hat{\Phi}(z)$.

The proof is formally similar to the proof of Theorem 28.1 of [1], replacing the field $\mathscr{K}$ by the ring Z.

Definition 1.4. *$\sharp$-Mappings.* Consider an "inclusion" map φ of a Hausdorff space χ' into a Hausdorff space χ. The mapping φ induces homomorphisms

$$\widehat{\varphi}_*: H_q(\chi', Z) \to H_q(\chi, Z) \qquad (q = 0, 1, \cdots), \tag{1.26}$$

which we shall call *$\sharp$-mappings* ψ.

If z is a coset in $Z_q(\chi', Z)$ of $B_q(\chi', Z)$ the image $\psi(z)$ is that coset $\widehat{\varphi}_*(z)$ in $Z_q(\chi, Z)$ of $B_q(\chi, Z)$ which *includes* z. We shall find it convenient to set

$$\widehat{\varphi}_*(z) = \psi(z) = z^\sharp . \tag{1.27}$$

2. The manifold M_n

Let M_n be a connected differentiable manifold of class C^∞. On M_n there exists a ND function f of class C_∞ satisfying (a), (b), and (c) of § 0.

Program. Let c be a value of f. In § 5 we shall show that $H_q(f_c, Z)$ is finitely generated for each q without making use of any triangulation of M_n. In § 7 we shall show how to determine the fundamental "invariants" of each group $H_q(f_c, Z)$, that is, the Betti numbers of $H_q(f_c, Z)$ and its elementary divisors in terms of properties of *spherically carried* $(k - 1)$-cycles associated with the respective critical points of f on f_c.

The sphere S_k. The following facts concerning the singular homology groups of S_k are needed.

According to Theorem 1.1,

$$\beta_q(S_k) = R_q(S_k, Q) \qquad (q = 0, 1, \cdots), \tag{2.1}$$

provided the homology groups $H_q(S_k, Z)$ are "finitely generated". The right member of (2.1) is given by Table I, § 29 of [1]. Moreover, the classical theory shows that the torsion groups of S_k are trivial.

(*) These properties of S_k could be verified inductively by the methods of this paper, taking account of the fact that there exists on S_k, when $k > 0$, a ND function f with just two critical points of indices 0 and k respectively. However, for the sake of brevity we take over these classical theorems on S_k and turn to the analysis of the changes in the singular homology groups $H_q(f_\alpha, Z)$ as α increases through the critical values a of f.

On a topological n-sphere Γ_n there exist n-cycles which are "simply-carried" in a sense which we shall now define.

Simply-carried singular n-cells and n-chains. We shall recall terms introduced in Defs. 30.2 and 30.4 of [1].

A "singular n-simplex" on M_n which is defined by a *homeomorphism* of a vertex-ordered euclidean n-simplex into M_n will be said to be *simply-carried* by M_n, as will the corresponding singular n-cell σ^n. Let $|M_n|$ be the topological manifold carrying M_n. A singular n-chain

$$z^n = e_1\sigma_1^n + \cdots + e_m\sigma_m^n \qquad (m \geq 1;\, e_i = \pm 1) \tag{2.2}$$

on M_n will be said to be *simply-carried* on $|M_n|$, if each of the cells σ_i^n is simply-carried on $|M_n|$, and, for i and j unequal integers on the range $1, \cdots, m$, $|\sigma_i^n| \cap |\sigma_j^n|$ includes no open subset of $|M_n|$.

Lemma 30.3 of [1] is couched in these terms and together with Theorem 37.1 of [1] implies the following lemma.

Lemma 2.1. *If $n > 0$ there exist simply-carried n-cycles on a prescribed topological n-sphere Γ_n. If z^n is such an n-cycle*

$$z^n \not\sim 0 \quad \text{on} \quad \Gamma_n,\ |z^n| = \Gamma_n\ , \tag{2.3}$$

and z^n is a "prebase" of $H_n(\Gamma_n, Z)$ and hence of $H_n(\Gamma_n, Q)$.

That z^n is a prebase of $H_n(\Gamma_n, Z)$ can be proved by the methods of paragraph (*) or by classical methods.

Notation. Corresponding to each critical point p_a of f, with critical value a, we shall introduce the compact subspace f_a of M_n and the subspace

$$\dot{f}_a = f_a - p_a\ . \tag{2.4}$$

If a_0 is the absolute minimum of f on M_n, $\dot{f}_{a_0}$ is empty. If $a > a_0$, $\dot{f}_a$ is not empty and will serve as a *modulus associated* with f_a. Singular cycles on $f_a \bmod \dot{f}_a$ are well-defined and play a fundamental role.

The basic Theorem 0.1 presupposes that "numerical relative invariants" are associated with the respective critical points p_a on f_c. These invariants will be defined in terms of the algebraic boundaries of the universal k-caps which we now introduce.

Universal k-caps. In Def. 2.2 we shall associate special relative k-cycles κ_a^k on $f_a \bmod \dot{f}_a$, with each critical point p_a of positive index k, and for reasons which will be made clear, will term each such relative k-cycle a *universal k-cap belonging* to p_a. The paragraphs preceding Def. 2.2 will motivate that definition. We begin by recalling the nature of the k-caps employed in [1].

The k-caps over $\mathscr{K}$. The k-caps ζ^k defined in § 29 of [1] will be here called k-caps *over* the associated field $\mathscr{K}$. Recall that a k-cap, ζ^k *over* $\mathscr{K}$, associated with the critical point p_a is, by definition, a rel. k-cycle on $f_a \bmod \dot{f}_a$, which is non-bounding on $f_a \bmod \dot{f}_a$. Such k-caps over $\mathscr{K}$ were shown to exist in [1]. Any such k-cap of p_a is a homology prebase over $\mathscr{K}$ on $f_a \bmod \dot{f}_a$ for rel. k-cycles on $f_a \bmod \dot{f}_a$.

A definition of a *k-cap over Z* associated with p_a must take account of the great difference between a field $\mathscr{K}$ and the ring Z, as well as the complexity introduced by the possible presence of torsion groups. It is possible to define a *k-cap over Z, associated with p_a* so that each such k-cap over Z is a "*k-cap* *over $\mathscr{K}$*" for arbitrary field $\mathscr{K}$. However, k-caps over a field $\mathscr{K}$ are not in general k-caps over other fields or over Z.

The definition of a k-cap over Z calls for a restriction of *f-saddles* as we have defined them in § 36 of [1].

Definition 2.1. *An f-saddle L_k of M_n at p_a.* A C^∞-manifold L_k, $0 < k \leq n$, which is the C^∞-diffeomorph in M_n of an open euclidean k-ball and is C^∞-embedded in M_n so as to meet a critical point p_a of index k, has been called an *f-saddle of M_n at p_a*, if together with $|\dot{L}_k| = |L_k| - p_a$, it has the following properties:

I. *The point p_a is a* ND *critical point of $f|L_k$ of index k.*

II. *$|\dot{L}_k|$ is included in $\dot{f}_a$.*

Restricted f-saddles. The following has been shown in §36 of [1]. If $k > 0$, and $\mathscr{L}_k$ is a prescribed f-saddle of M_n at p_a, then a "subsaddle" L_k of $\mathscr{L}_k$, whose carrier $|L_k|$ is included in a sufficiently small open neighborhood of p_a relative to $|\mathscr{L}_k|$, will have the following property: a coset-contracting isomorphism of form

$$H_q(f_a, \dot{f}_a, \mathscr{K}) \approx H_q(|L_k|, |\dot{L}_k|, \mathscr{K}) \tag{2.5}$$

is valid for each $q \geq 0$. See (36.19) of [1].

A review of the proof of (36.19) shows that if $|L_k|$ is sufficiently small, there will similarly exist a coset-contracting isomorphism

$$H_q(f_a, \dot{f}_a, Z) \approx H_q(|L_k|, |\dot{L}_k|, Z)(0 < k \leq n) . \tag{2.6}$$

Such an f-saddle L_k will be termed a *Z-restricted f-saddle* L_k of M_n at p_a.

The crucial definition can now be given.

Definition 2.2. *Universal k-caps κ_a^k.* A singular k-cell σ^k which is simply-carried on a Z-restricted f-saddle L_k at p_a, with p_a on the open interior of $|\sigma^k|$ relative to $|L_k|$, will be called a *k-cap of p_a over Z*. It will be denoted by κ_a^k and termed a *universal k-cap of p_a* because it follows from the Carrier Theorem 36.2 of [1] that it is a k-cap of p_a "over" each field $\mathscr{K}$.

Theorem 2.1 below relates the homological structure of a universal k-cap of p_a to the homological structure of $f_a \bmod \dot{f}_a$. For each q the fundamental invariants of the isomorphic groups in (2.8) are then determined as in Theorem 2.2 below. Finally Theorem 2.3 shows how *two* universal k-caps of p_a are related.

Given a universal k-cap κ_a^k we shall set

$$|\kappa_a^k| - p_a = |\dot{\kappa}_a^k| . \tag{2.7}$$

Theorem 2.1. *If κ_a^k is a universal k-cap, then for each $q \geq 0$ there exists a coset-contracting isomorphism*

$$H_q(f_a, \dot{f}_a, Z) \approx H_q(|\kappa_a^k|, |\dot{\kappa}_a^k|, Z) . \tag{2.8}$$

Proof. It follows from Excision Theorem 1.3 that a coset-contracting isomorphism

$$H_q(|L_k|, |\dot{L}_k|, Z) \approx H_q(|\kappa_a^k|, |\dot{\kappa}_a^k|, Z) \tag{2.9}$$

is valid for each $q \geq 0$. To apply Theorem 1.3 to prove (2.9) one sets

$$\chi = |L_k|, A = |\dot{L}_k|, A^* = |L_k| - |\kappa_a^k| \tag{2.10}$$

and notes that $\chi - A = p_a$. Theorem 2.1 follows from (2.9) and (2.6).

Two lemmas are needed to establish Theorem 2.2.

Introduction to Lemma 2.2. Let Δ_k be a closed euclidean k-disc, $k > 0$, and $\dot{\Delta}_k$ be this disc with its center removed. The importance for us of Δ_k arises from the fact that there exists a homeomorphism

$$\theta\colon |\kappa_a^k| \to \Delta_k \qquad (k > 0) \tag{2.11}$$

of the carrier $|\kappa_a^k|$ of a prescribed universal k-cap κ_a^k onto Δ_k in which p_a corresponds to the center of Δ_k. Thus θ maps $|\dot{\kappa}_a^k|$ onto $\dot{\Delta}_k$.

In the following lemma, as in the remainder of this paper, unless otherwise stated, all cycles are integral and all homologies are over Z.

Lemma 2.2. *If y^q is a rel. cycle on Δ_k* mod *$\dot{\Delta}_k$, $k > 0$, then $\partial y^q \sim 0$ on $\dot{\Delta}_k$ if and only if $y^q \sim 0$ on Δ_k* mod *$\dot{\Delta}_k$.*

The proof is formally the same as the proof of Lemma 29.0 of [1], on replacing $\mathscr{K}$ by Z.

We continue with a lemma on $\dot{\Delta}_k$.

Lemma 2.3. *For $k > 0$ the torsion subgroup of $H_q(\dot{\Delta}_k, Z)$ vanishes for each q and*

$$\beta_q(\dot{\Delta}_k) = R_q(\dot{\Delta}_k, Q) , \tag{2.12}$$

where $R_q(\dot{\Delta}_k, Q)$ is given by Table II, § 29 *of* [1].

Proof. For $k > 0$, $\dot{\Delta}_k$ admits a radial deformation d retracting $\dot{\Delta}_k$ onto the outer boundary S_{k-1} of $\dot{\Delta}_k$, so that by Theorem 1.4, with the moduli A and A' taken as empty sets,

$$H_q(\dot{\Delta}_k, Z) \approx H_q(S_{k-1}, Z) . \tag{2.13}$$

Hence the torsion subgroup of $H_q(\dot{\Delta}_k, Z)$ vanishes for each q. The relation (2.12) follows from Theorem 1.1.

Theorem 2.2 below gives the structure of the right, and hence the left members of (2.8).

Theorem 2.2. (i) *If κ_a^k is a universal k-cap of p_a, $k > 0$, then for each q the group*

$$H_q(|\kappa_a^k|, |\dot{\kappa}_a^k|, Z) \tag{2.14}$$

is a finitely generated free abelian group whose dimension is δ_q^k.

(ii) *The homology class* $\bar{\kappa}_a^k$ *of* κ_a^k *on* $|\kappa_a^k|$ mod $|\dot{\kappa}_a^k|$ *is a base for the free abelian group* (2.14) *when* $q = k$.

Because of the existence of the homeomorphism θ of (2.11), Theorem 2.2 is equivalent to the following lemma.

Lemma 2.4. (i) *For* $k > 0$

$$H_q(\Delta_k, \dot{\Delta}_k, Z) \tag{2.15}$$

is a finitely generated free abelian group whose dimension is δ_q^k.

(ii) *If* η^k *is a simply-carried k-cell with carrier* Δ_k, *then* η^k *is a k-cycle on* Δ_k mod $\dot{\Delta}_k$ *whose homology class* $\bar{\eta}^k$, *on* Δ_k mod $\dot{\Delta}_k$, *is a base for the group* (2.15), *when* $q = k$.

Proof of (i). We distinguish the case $k > 1$ from the case $k = 1$.

The case $k > 1$. In this case when $q > 1$ the group (2.15) is isomorphic, as we shall see, to the group $H_{q-1}(\dot{\Delta}_k, Z)$ under a mapping φ such that the homology class of a q-cycle c^q on Δ_k mod $\dot{\Delta}_k$ goes into the homology class on $\dot{\Delta}_k$ of ∂c^q.

It is clear that φ is a homomorphism *onto* $H_{q-1}(\dot{\Delta}_k, Z)$ when $q > 1$; to a cycle e^{q-1} on $\dot{\Delta}_k$ corresponds a q-cycle c^q on Δ_k mod $\dot{\Delta}_k$ such that $\partial c^q = e^{q-1}$. The mapping φ is *biunique* since its kernel is zero in accord with Lemma 2.2. The mapping φ is thus an isomorphism of the group (2.15) onto the group $H_{q-1}(\dot{\Delta}_k, Z)$. According to Lemma 2.3, $H_{q-1}(\dot{\Delta}_k, Z)$ is free with dimension δ_q^k.

Thus (i) of Lemma 2.4 is true when $q > 1$. When $q = 1$ or 0 and $1 < k$, (i) is trivial.

Proof of (i). $1 = k$. This case is left to the reader.

Proof of (ii). *The case* $1 < k$. Let η^k be given as in (ii). The cycle $\partial\eta^k$ is simply-carried, with $|\partial\eta^k|$ the $(k-1)$-sphere S_{k-1} which is the geometric boundary of Δ_k. According to Lemma 2.1, $\partial\eta^k$ is a prebase for $H_{k-1}(S_{k-1}, Z)$. The coset-contracting isomorphism (2.13) implies that $\partial\eta^k$ is then a prebase for $H_{k-1}(\dot{\Delta}_k, Z)$. It follows from Lemma 2.2 that η^k is a prebase for the group (2.15) when $q = k$ and $1 < k$.

Proof of (ii), $1 = k$. This case is left to the reader.

Any two universal k-caps associated with the same critical point p_a are related as follows.

Theorem 2.3. *If* $\kappa_a^k(1)$ *and* $\kappa_a^k(2)$ *are two universal k-caps,* $k > 0$ *of the same critical point* p_a, *then for some integer* $e = \pm 1$

$$\kappa_a^k(1) \sim e\kappa_a^k(2) \qquad (\text{on } f_a \bmod \dot{f}_a)\ , \tag{2.16}$$

and consequently

$$\partial\kappa_a^k(1) \sim e\partial\kappa_a^k(2) \qquad (\text{on } \dot{f}_a)\ . \tag{2.17}$$

Proof of (2.16). For μ on the range 1, 2 Theorem 2.2 (ii) implies that

$\kappa_a^k(\mu)$ is a prebase of the free abelian group

$$H_k(|\kappa_a^k(\mu)|, |\dot{\kappa}_a^k(\mu)|, Z) . \tag{2.18}$$

We infer from the coset-contracting isomorphism (2.8) that both $\kappa_a^k(1)$ and $\kappa_a^k(2)$ are prebases of $H_k(f_a, \dot{f}_a, Z)$. Relation (2.16) follows.

Proof of (2.17). The homology (2.16) implies that

$$\kappa_a^k(1) - e\kappa_a^k(2) = \partial c_+^{k+1} + c_-^k , \tag{2.19}$$

where c_+^{k+1} and c_-^k are integral chains on f_a and $\dot{f}_a$ respectively. The application of ∂ to both members of (2.19) yields (2.17).

Permanent notation. We shall set

$$H_q(f_a, Z) = H_q^a \tag{2.20}$$

for each integer q and critical value a. If a_0 is the minimum critical value, and $a > a_0$, then we shall set

$$H_q(\dot{f}_a, Z) = \dot{H}_q^a . \tag{2.21}$$

In § 5 we shall show that for each integer q and critical value $a > a_0$, the groups $\dot{H}_q^a$ and H_q^a are *finitely generated* (FG).

Granting that $\dot{H}_q^a$ is FG we shall denote the torsion subgroup of $\dot{H}_q^a$ by $\dot{\mathscr{T}}_q^a$ and a complementary Betti subgroup by $\dot{\mathscr{B}}_q^a$. Similarly we shall denote the torsion subgroup of H_q^a by $\mathscr{T}_q^a$ and a complementary Betti group by $\mathscr{B}_q^a$. One then has

$$\dot{H}_q^a = \dot{\mathscr{B}}_q^a \oplus \dot{\mathscr{T}}_q^a , \tag{2.22}$$

$$H_q^a = \mathscr{B}_q^a \oplus \mathscr{T}_q^a . \tag{2.23}$$

3. Some terms in abelian group theory

We begin by recalling the definitions of the torsion coefficients and elementary divisors of the torsion subgroup $\mathscr{T}$ of a finitely generated abelian group A.

The torsion coefficients of $\mathscr{T}$. It is a classical theorem that a finite nontrivial abelian group $\mathscr{T}$ is a direct sum of a finite set of cyclic subgroups of $\mathscr{T}$, which if *canonically* arranged have orders

$$q_1, q_2, \cdots, q_\rho \tag{3.1}$$

exceeding 1 each of which, except q_ρ, is divisible by its successor. The integers of the sequence (3.1) are uniquely determined by $\mathscr{T}$ and are termed its *torsion coefficients*. It is convenient for our purposes to order the torsion coefficients as above and not in the inverse order employed by some writers.

Elementary divisors of $\mathscr{T}$. It is known that a finite, nontrivial, abelian group $\mathscr{T}$ is a direct sum $g_1 \oplus \cdots \oplus g_r$ of cyclic groups g_i such that the order of g_i is a power $p_i^{e_i} > 1$ of a prime p_i and g_i is a subgroup of no cyclic subgroup of $\mathscr{T}$ whose order is a higher power of p_i. Such a direct sum is called a "*cyclic primary decomposition*" (abbrev. CPD) of $\mathscr{T}$. The prime powers

$$p_1^{e_1}, \cdots, p_r^{e_r} \qquad (e_i > 0; i = 1, \cdots, r)\,, \tag{3.2}$$

which are the orders of respective summands in a CPD of $\mathscr{T}$, are called *elementary divisors* of $\mathscr{T}$. The ED's of $\mathscr{T}$ are said to be *normally arranged* if $p_1 \geq p_2 \geq \cdots \geq p_r$, and if, when $p_i = p_{i+1}$, then $e_i \geq e_{i+1}$. $\mathscr{T}$ uniquely determines a set of normally arranged ED's.

We state a classical theorem:

Theorem 3.1. *Canonically ordered torsion coefficients of a finite non-trivial abelian group $\mathscr{T}$ determine and are uniquely determined by normally ordered elementary divisors of $\mathscr{T}$.* See [11, p. 147].

By the *multiplicity* of an ED λ of $\mathscr{T}$ is meant the number of ED's in a normally ordered list of ED's of $\mathscr{T}$ which are numerically equal to λ.

Definition 3.1. *A "basis" of a FG A.* Suppose that A has a nontrivial torsion subgroup $\mathscr{T}$ and that[7], with i on the range $1, 2, \cdots, \rho$,

$$\mathscr{T} = \{x_1\} \oplus \cdots \oplus \{x_\rho\} \qquad (x_i \in \mathscr{T}) \tag{3.3}$$

is a CPD of $\mathscr{T}$. Let $\mathscr{B}$ be a Betti subgroup of A with a non-empty base $(u_1, \cdots, u_\beta)$. The set of elements

$$u_1, \cdots, u_\beta; x_1, \cdots, x_\rho \tag{3.4}$$

of A is called a "*basis*" for A. If $\mathscr{B}$ is trivial there are no elements u_i, and if $\mathscr{T}$ is trivial, no elements x_j.

A "basis" for A is to be distinguished from a *base* for $\mathscr{B}$ which is free.

A basis for A is unique if and only if A is a cyclic group of order 2.

Let w be a prescribed element in A. Then

$$w = \mu_1 u_1 + \cdots + \mu_\beta u_\beta + m_1 x_1 + \cdots + m_\rho x_\rho\,, \tag{3.5}$$

where μ_i is an integer uniquely determined by w and the choice of the "basis" (3.4), while m_j is an integer uniquely determined by w and the choice of the basis (3.4), provided m_j is restricted to integral values such that

$$0 \leq m_j < \text{order } x_j \qquad (j = 1, 2, \cdots, \rho)\,. \tag{3.5$'$}$$

When $\beta = 0$ there are no integers μ_i, and when $\rho = 0$ no integers m_j.

Definition 3.2. The set of integral coefficients

$$\mu_1, \cdots, \mu_\beta; m_1, \cdots, m_\rho \tag{3.6}$$

[7] $\{x_i\}$ denotes the cyclic subgroup of $\mathscr{T}$ generated by x_i.

in the right member of (3.5), subject to (3.5)′, will be termed a *profile of* w relative to the basis (3.4) of A. When $\beta = \rho = 0$ the profile (3.6) is an empty set.

We come to a group-theoretic definition of two indices to be attached to a prescribed element w of A. In the application of this section to critical point theory, A will be taken as $\dot{H}^{a}_{k-1}$, where k is the index of the critical point p_a, and w will be the homology class on $\dot{f}_a$ of the algebraic boundary of a prescribed "universal" k-cap κ^k_a.

The free index s and torsion index t of $w \in A$.

A is assumed FG. We assign to each element $w \in A$ an integer $s \geq 0$ termed the *free index* of w. When $w \notin \mathscr{T}$, the integer s is characterized in Lemma 3.1 below. When $w \in \mathscr{T}$, s shall be 0.

Notation. In formulating Lemma 3.1 we write $x = y \bmod \mathscr{T}$ whenever x and y are elements in A such that $x - y$ is in $\mathscr{T}$. Lemma 3.1 is established in § 3 of [3].

Lemma 3.1. (i) *Corresponding to an element $w \in A$ of infinite order there exists an integer $s > 0$ such that a subgroup $\mathscr{B}$ of A, prescribed among the Betti subgroups of A, has a base with a first element*[8] u_B *such that*

$$w = su_B \bmod \mathscr{T} . \tag{3.7}$$

(ii) *If there is given a second Betti group $\mathscr{B}'$ of A and a positive integer s' such that for a first element*[8] $u_{B'}$ *in a base for $\mathscr{B}'$*

$$w = s'u_{B'} \bmod \mathscr{T} , \tag{3.8}$$

then $s = s'$.

We recapitulate the definition of the index s of an element $w \in A$.

Definition 3.3. *The free index s of w. If $w \in \mathscr{T}$, set $s = 0$. If $w \notin \mathscr{T}$ let s be a positive integer affirmed to exist in Lemma* 3.1.

By virtue of this definition of s,

$$w = su_B + \tau_B \qquad (\tau_B \in \mathscr{T}) , \tag{3.9}$$

where $u_B = 0$ or is the first element in a base for $\mathscr{B}$, according as order w is finite or infinite.

Definition 3.4. *The torsion index t of w.* In the notation of (3.9) set

$$\text{order } \tau_B = t_B , \qquad \min_B t_B = t , \tag{3.10}$$

where the group $\mathscr{B}$ represented by B ranges over all Betti subgroups of A complementary to $\mathscr{T}$. We term t the *torsion index* of w. When $s = 0$, $t = t_B$ for every choice of $\mathscr{B}$.

[8] The subscript B represents $\mathscr{B}$, the subscript B' represents $\mathscr{B}'$. Script letters are not available as subscripts.

In § 4 of [3] we have proved the following theorem.

Theorem 3.2. *A profile*

$$\mu_1, \cdots, \mu_\beta, m_1, \cdots, m_\rho \qquad \textit{(possibly empty)} \tag{3.11}$$

of an element $w \in A$ relative to a basis (3.4) *of a* FG *abelian group A uniquely determines the free index s of w and, when $s = 0$, the torsion index t of w. These values of s and t are independent of the choice of the basis* (3.4) *relative to which a profile of w is taken.*

When the basis (3.4) is given it is understood that the orders of each element in the basis are known.

4. The critical cyclic subgroup W^a_{k-1} of $\dot{H}^a_{k-1}$

We are supposing that $a > a_0$, the minimum critical value of f, and that $k =$ index p_a.

Definition 4.1. *The group W^a_{k-1} and its critical generators.* According to Theorem 2.3 the algebraic boundaries $\partial\kappa^k_a$ of universal k-caps κ^k_a have homology classes on $\dot{f}_a$ of form $\pm w^{k-1}_a$, where w^{k-1}_a is any one such homology class. These homology classes generate a unique cyclic subgroup

$$\{\pm w^{k-1}_a\} = W^a_{k-1} \tag{4.1}$$

of $\dot{H}^a_{k-1}$. We term W^a_{k-1} the *critical cyclic subgroup* of $\dot{H}^a_{k-1}$, and $\pm w^{k-1}_a$ its *critical generators*.

There is just one "critical cyclic group" W^a_{k-1} associated with each critical point p_a of f of positive index k. The order of W^a_{k-1} may be finite or infinite. For $q \neq k - 1$, W^a_q is undefined.

Definition 4.2. *The $\sharp$-mapping ϕ^a_q.* Let φ_a be the inclusion map of $\dot{f}_a$ into f_a. The mapping φ_a induces homomorphisms

$$\widehat{\varphi}_a : C_q(\dot{f}_a, Z) \to C_q(f_a, Z) \qquad (q = 0, 1, 2, \cdots) \tag{4.2}$$

as in Def. 1.2. Let

$$\phi^a_q = (\widehat{\varphi}_a)_* : \dot{H}^a_q \to H^a_q \qquad (a > a_0) \tag{4.3}$$

be the natural homomorphism of $\dot{H}^a_q$ into H^a_q induced by $\widehat{\varphi}_a$. Cf. (1.23). We term ϕ^a_q a *$\sharp$-mapping* induced by the inclusion mapping φ_a of $\dot{f}_a$ into f_a.

Notation. Symbols such as c^q_+ and c^q_- shall denote chains or cycles on f_a and $\dot{f}_a$ respectively. As previously $a > a_0$ and $k =$ index p_a.

Theorem 4.1. *Concerning the $\sharp$-mapping ϕ^a_q of* (4.3) *the following is true*:

(i) *The kernel of ϕ^a_q is 0 when $q \neq k - 1$.*

(ii) *The kernel of* ψ_q^a *is* W_{k-1}^a *when* $q = k - 1$.

(iii) ψ_q^a *is onto when* $q \neq k$.

(iv) ψ_q^a *is onto when* $q = k$ *if*[9] *and only if* W_{k-1}^a *is of infinite order.*

We shall now prove (i), (ii), (iii) of this theorem. Statement (iv) follows from Lemma 5.1 and Lemma 4.1 (ii).

Proof of Theorem 4.1 (i). If c_-^q is a q-cycle on $\dot{f}_a$ such that $c_-^q \not\sim 0$ on $\dot{f}_a$, we shall show that $c_-^q \not\sim 0$ on f_a when $q \neq k - 1$, implying thereby that $\ker \psi_q^a = 0$ when $q \neq k - 1$. See last paragraph of § 1.

Suppose on the contrary that there exists a $(q+1)$-chain c_+^{q+1} on f_a such that

$$c_-^q = \partial c_+^{q+1} \,. \tag{4.4}$$

The chain c_+^{q+1} is a rel. cycle on $f_a \bmod \dot{f}_a$. It follows from Theorem 2.1 that if κ_a^k is a prescribed universal k-cap then

$$c_+^{q+1} \sim e_+^{q+1} \qquad (\text{on } f_a \bmod \dot{f}_a) \tag{4.5}$$

for some $(q+1)$-cycle e_+^{q+1} on $|\kappa_a^k| \bmod |\dot{\kappa}_a^k|$. Since $q + 1 \neq k$ by hypothesis, it follows from Theorem 2.2 (i) that

$$e_+^{q+1} \sim 0 \qquad (\text{on } |\kappa_a^k| \bmod |\dot{\kappa}_a^k|) \,. \tag{4.6}$$

Hence $c_+^{q+1} \sim 0$ on $f_a \bmod \dot{f}_a$, or equivalently

$$c_+^{q+1} = \partial c_+^{q+2} + c_-^{q+1} \tag{4.7}$$

for suitable chains c_+^{q+2} and c_-^{q+1}. The application of ∂ to both members of (4.7) implies the equality

$$c_-^q = \partial c_-^{q+1} \tag{4.8}$$

contrary to the nature of c_-^q.

Hence Theorem 4.1 (i) is true.

Proof of Theorem 4.1 (ii). It suffices to prove (a) and (b).

$$\text{(a)} \qquad W_{k-1}^a \subset \ker \psi_{k-1}^a \qquad (k = \text{index } p_a) \,.$$

To verify (a) it is sufficient to show that a "critical generator" w_a^{k-1} of W_{k-1}^a is in $\ker \psi_{k-1}^a$. If κ_a^k is a universal k-cap then $\partial \kappa_a^k$ is in the homology class on $\dot{f}_a$ of a generator w_a^{k-1} of W_{k-1}^a (by Def. 4.1). Since $|\kappa_a^k| \subset f_a$, $\partial \kappa_a^k \sim 0$ on f_a. Hence

$$\psi_{k-1}^a(w_a^{k-1}) = 0 \qquad (\text{by Def. 4.2}) \,. \tag{4.9}$$

Thus (a) holds.

$$\text{(b)} \qquad \ker \psi_{k-1}^a \subset W_{k-1}^a \qquad (k = \text{index } p_a) \,.$$

[9] Equivalently if and only if the free index s^a of p_a is positive.

To verify (b) it is sufficient to show that if a $(k-1)$-cycle a_-^{k-1} on $\dot{f}_a$ bounds a chain e_+^k on f_a, and $\bar{a}_-^{k-1}$ is the homology class of a_-^{k-1} on $\dot{f}_a$, then

$$\bar{a}_-^{k-1} \in W_{k-1}^a . \tag{4.10}$$

To verify (4.10) note that e_+^k is a k-cycle on $f_a \bmod \dot{f}_a$. It follows from Theorems 2.1 and 2.2 (ii) that if κ_a^k is a universal k-cap of p_a there exists an integer μ such that $e_+^k \sim \mu\kappa_a^k$ on $f_a \bmod \dot{f}_a$, or equivalently

$$e_+^k = \mu\kappa_a^k + \partial e_+^{k+1} + e_-^k \tag{4.11}$$

for suitable chains e_+^{k+1} and e_-^k. The application of ∂ to both members of (4.11) shows that on $\dot{f}_a$

$$a_-^{k-1} = \mu\partial\kappa_a^k + \partial e_-^k . \tag{4.12}$$

The homology class of $\partial\kappa_a^k$ on $\dot{f}_a$ is a generator w_a^{k-1} of W_{k-1}^a, and it follows from (4.12) that

$$\bar{a}_-^{k-1} = \mu w_a^{k-1} \in W_{k-1}^a . \tag{4.13}$$

Hence (b) is true and (ii) follows.

Proof of Theorem 4.1 (iii). It is sufficient to show that if c_+^q is a q-cycle on f_a and if $q \neq k$, then for some cycle c_-^q on $\dot{f}_a$

$$c_+^q \sim c_-^q \qquad (\text{on } f_a) . \tag{4.14}$$

We shall verify (4.14). It follows from Theorems 2.1 and 2.2 (i) when $q \neq k$ that

$$c_+^q = \partial e_+^{q+1} + e_-^q \qquad (\text{on } f_a) \tag{4.15}$$

for suitable chains e_+^{q+1} and e_-^q. An application of ∂ to both members of (4.15) shows that e_-^q is a cycle on $\dot{f}_a$, and (4.14) follows on setting $c_-^q = e_-^q$.

We continue with the critical value $a > a_0$.

Theorem 4.1 has the following corollary.

Corollary 4.1. (α) *When* $k = \text{index}\, p_a > 0$ *and* q *is neither* k *nor* $k-1$, *the* $\sharp$*-mapping* ϕ_q^a *is an isomorphism of* $\dot{H}_q^a$ *onto* H_q^a.

(β) *When* $k > 0$, ϕ_{k-1}^a *induces a surjective isomorphism*

$$\dot{H}_{k-1}^a / W_{k-1}^a \approx H_{k-1}^a . \tag{4.16}$$

(γ) *When* $k > 0$, ϕ_k^a *is an isomorphism of* $\dot{H}_k^a$ *onto* H_k^a *if and only if* W_{k-1}^a *is an infinite cyclic group.*

Statement (α) follows from (i) and (iii) of Theorem 4.1. Statement (β) follows from (ii) and (iii) of Theorem 4.1. Statement (γ) follows from Theorem 4.1 (iv), as yet unverified, and from Theorem 4.1 (i).

Let w_{k-1}^a be a "critical generator" of W_{k-1}^a.

Definition 4.3. *The free and torsion indices of p_a, $a > a_0$.* Under the assumption that $\dot{H}_{k-1}^a$ is FG (to be verified in § 5) we can assign "free" and "torsion" indices $s^a \geq 0$ and $t^a \geq 1$ to $w = ew_a^{k-1}$, $e = \pm 1$, as an element in $A = \dot{H}_{k-1}^a$ for each critical value $a > a_0$ in accord with the abstract definition of such indices given in § 3. These indices are independent of the choice of e as ± 1 as we shall see. They are uniquely determined by $\pm w_a^{k-1}$ and $\dot{H}_{k-1}^a$ and will be termed "*free*" and "*torsion indices*" respectively of p_a. The torsion indices t^a are not to be confused with the classical "torsion coefficients".

Definition of s^a. Corresponding to any free subgroup[10] $\dot{\mathscr{B}}_{k-1}^a$ of $\dot{H}_{k-1}^a$ complementary to the torsion subgroup $\dot{\mathscr{T}}_{k-1}^a$ of $\dot{H}_{k-1}^a$ there exists a unique integer $s^a \geq 0$ such that

$$w_a^{k-1} = s^a u_B + \tau_B^a \qquad (\tau_B^a \in \dot{\mathscr{T}}_{k-1}^a) , \tag{4.17}$$

where u_B is the null element in $\dot{H}_{k-1}^a$, or the first element in a suitably chosen base of $\dot{\mathscr{B}}_{k-1}^a$ according as the order of w_a^{k-1} in $\dot{H}_{k-1}^a$ is finite or infinite. In the first case $s^a = 0$, in the second $s^a > 0$. The element τ_B^a is uniquely determined by w_a^{k-1} when $s^a = 0$, and when $s^a > 0$, by w_a^{k-1} and the choice of $\dot{\mathscr{B}}_{k-1}^a$ among the subgroups of $\dot{H}_{k-1}^a$ complementary to $\dot{\mathscr{T}}_{k-1}^a$. If one replaces w_a^{k-1} by $-w_a^{k-1}$ in (4.17), (4.17) remains valid if one keeps s^a and multiplies both u_B and τ_B^a by -1.

Definition of t^a. As in § 3 we denote the order of $\pm\tau_B^a$ by t_B^a and define the *torsion index* t^a of $\pm w_a^{k-1}$ by setting

$$t^a = \min_B t_B^a , \tag{4.18}$$

where B ranges over the free groups $\dot{\mathscr{B}}_{k-1}^a$ complementary to $\dot{\mathscr{T}}_{k-1}^a$.

If $s^a = 0$, $t^a = t_B^a$ regardless of the choice of $\dot{\mathscr{B}}_{k-1}^a$.

New cycles on f_a. When the critical point p_a has an index $k > 0$ there may be k-cycles λ^k on f_a whose homology classes on f_a contain no k-cycles on $\dot{f}_a$. Such a k-cycle λ^k on f_a will be called a *new* k-cycle on f_a. For dimensions q other than k there are, in a similar sense, no "new" q-cycles on f_a if $q \neq k$, as Theorem 4.1 (iii) implies. We shall see that there are "new" k-cycles on f_a if and only if $s^a = 0$.

In § 5 we shall show that the singular homology groups of f_c are FG for each value c of f. In § 7 the mechanism affirmed to exist in Theorem 0.1 will be inductively defined. However, in both § 5 and § 7 one needs to know that there are "new" k-cycles on f_a when $s^a = 0$. In the next paragraphs we shall define a special homology class of "new" k-cycles on f_a when $s^a = 0$.

In anticipation of § 5, suppose that $a > a_0$ and that $\dot{H}_{k-1}^a$ is FG. Then the

[10] As a subscript $\dot{\mathscr{B}}_{k-1}^a$ is represented by B in (4.17).

indices s^a and t^a are *well-defined*. Let κ_a^k be a universal k-cap. Then

$$t^a \partial \kappa_a^k \sim 0 \qquad (\text{on } \dot{f}_a, \text{ when } s^a = 0) \tag{4.19}$$

in accord with (4.17) and the definition of $t^a (t^a \geq 1)$.

Definition 4.4. *A t^a-fold linking k-cycle λ_a^k.* By virture of (4.19) there exists a k-chain c_-^k on $\dot{f}_a$ such that

$$\partial t^a \kappa_a^k = \partial c_-^k \qquad (\text{when } s^a = 0) , \tag{4.20}$$

and hence a k-cycle

$$\lambda_a^k = t^a \kappa_a^k - c_-^k \qquad (\text{on } f_a) . \tag{4.21}$$

We term λ_a^k a *t^a-fold linking k-cycle* on f_a belonging to p_a and associated with κ_a^k.

The following lemma is essential.

Lemma 4.1. (i) *Any two t^a-fold linking k-cycles $\lambda_a^k(1)$ and $\lambda_a^k(2)$ on f_a satisfy a rel. homology*

$$\lambda_a^k(1) \sim e\lambda_a^k(2) \qquad (\text{on } f_a \bmod \dot{f}_a) , \tag{4.22}$$

where e has one of the values $e = \pm 1$.

(ii) *If λ_a^k is a t^a-fold linking k-cycle on f_a, then $m\,\lambda_a^k \sim 0$ on f_a* mod $\dot{f}_a$ *for no positive integer m.*

Proof of (i). The rel. homology (4.22) follows from the relative homology (2.16).

Proof of (ii). If λ_a^k is a "t^a-fold linking k-cycle", then λ_a^k is "linking" over the rational field Q in the sense of Def. 29.2 of [1], as we now verify.

A universal k-cap κ_a^k is also a "k-cap over Q" in the sense of Def. 29.1 of [1], as we have already seen.[11] Hence λ_a^k is a k-cap *over Q*. Since λ_a^k is an integral cycle it is also a rational cycle. As such λ_a^k is "linking" in the sense of [1].

It follows from Theorem 29.3 (ii) of [1] that $\lambda_a^k \not\sim 0$ on $f_a \bmod \dot{f}_a$ over Q. Hence $m\,\lambda_a^k \sim 0$ on $f_a \bmod \dot{f}_a$ over Z for no positive integer m.

Statement (ii) follows.

5. The finite generation of groups $H_q(f_c, Z)$

A priori, c is any value of f. When c is the minimum value a_0 of f, f_c reduces to the critical point p_{a_0}. The group $H_q(f_c, Z)$ is then trivially FG.

We suppose that $c > a_0$.

The critical values of f at most c form a sequence

$$a_0 < a_1 < a_2 < \cdots < a_m \leq c . \tag{5.1}$$

[11] As a consequence of "Carrier Theorem" 36.2 of [1].

We shall prove inductively that the homology groups in the sequences

$$\boxed{H_q^{a_0};\ \dot{H}_q^{a_1}, H_q^{a_1};\ \dot{H}_q^{a_2}, H_q^{a_2};\ \cdots;\ \dot{H}_q^{a_m}, H_q^{a_m};\ H_q(f_c, Z)} \tag{5.2q}$$

are finitely generated for each integer q.

Since $H_q^{a_0}$ is FG for each q, it suffices to prove Theorems 5.1, 5.2 and 5.3 below.

Theorem 5.1. *If* $0 < r \leq m$, *q is an integer, and* $H_q^{a_{r-1}}$ *is* FG, *then* $\dot{H}_q^{a_r}$ *is* FG.

Proof. Corollary 23.1 of [1] implies that $\dot{f}_{a_r}$ admits a deformation retracting $\dot{f}_{a_r}$ onto $f_{a_{r-1}}$. There then exists a coset-contracting isomorphism

$$H_q(\dot{f}_{a_r}, Z) \approx H_q(f_{a_{r-1}}, Z) \tag{5.3}$$

in accord with Theorem 1.4, on taking A and A' in Theorem 1.4 as empty sets. Theorem 5.1 is a consequence of the isomorphism (5.3).

Theorem 5.2. *If* $a_m < c$ *in the sequence* (5.1), *q is an integer and* $H_q^{a_m}$ *is* FG, *then* $H_q(f_c, Z)$ *is* FG.

Proof. In case there exists a critical value a_{m+1} such that $c < a_{m+1}$, Corollary 23.1 of [1] implies that there exists a deformation D retracting $\dot{f}_{a_{m+1}}$ onto f_{a_m}. The restriction of the deformation D to $f_c \times [0, 1]$ will be a deformation retracting f_c onto f_{a_m}. That $H_q(f_c, Z)$ is FG follows with the aid of Theorem 1.4.

In case there is no critical value of f exceeding c, we infer from Corollary 23.2 of [1] that there exists a deformation D retracting all of M_n onto f_{a_m}. The restriction of D to $f_c \times [0, 1]$ will be a deformation retracting f_c onto f_{a_m}. Theorem 5.2 follows from Theorem 1.4.

Theorem[12] 5.3. *If* $0 < r \leq m$, *and* $\dot{H}_q^{a_r}$ *is* FG *for each integer q, then* $H_q^{a_r}$ *is* FG *for each q.*

We set $k =$ index p_{a_r} and distinquish three cases, $q \neq k$ or $k - 1$, $q = k - 1$, $q = k$.

The case $q \neq k$ or $k - 1$. $H_q^{a_r}$ is FG in this case, since $\dot{H}_q^{a_r}$ is then isomorphic to $H_q^{a_r}$, in accord with Corollary 4.1 (α).

The case $q = k - 1$. By hypothesis there is a finite set $z_1^{k-1}, \cdots, z_\mu^{k-1}$ of generators of $\dot{H}_{k-1}^{a_r}$. If θ is the natural homomorphism of $\dot{H}_{k-1}^{a_r}$ onto the group quotient Q_r of $\dot{H}_{k-1}^{a_r}$ by $W_{k-1}^{a_r}$, then

$$\theta(z_1^{k-1}), \cdots, \theta(z_\mu^{k-1}) \tag{5.4}$$

is a set of generators of Q_r. By Corollary 4.1 (β) the quotient Q_r is isomorphic to $H_{k-1}^{a_r}$, so that $H_{k-1}^{a_r}$ is FG.

[12] When $q = k =$ index p_{a_r} the proof that $H_k^{a_r}$ is FG makes use of the hypothesis that *both* $\dot{H}_{k-1}^{a_r}$ and $\dot{H}_k^{a_r}$ are FG. When $q \neq$ index p_{a_r}, $H_q^{a_r}$ is FG if $\dot{H}_q^{a_r}$ is FG. The proof of Theorem 5.3 shows this to be true.

We come to the most difficult case.

The case $q = k > 0$. We shall refer to the index k, the free index s^a and torsion index t^a of p_a of Def. 4.3 and to the t^a-fold linking k-cycle λ_a^k on f_a of Def. 4.4. Given the $\#$-homomorphism

$$\phi_q^a : \dot{H}_q^a \to H_q^a \tag{5.5}$$

of Def. 4.2, we shall term the images $\phi_q^a(z^q)$ of elements $z^q \in \dot{H}_q^a$ $\#$-images, $z^{q\#}$, and verify the following lemma.

Lemma 5.1. (i) *If* $s^a > 0$, *the* $\#$-*images of a set of generators of* $\dot{H}_k^a$ *form a set of generators of* H_k^a.

(ii) *If* $s^a = 0$, *then the* $\#$-*images of a set of generators of* $\dot{H}_k^a$, *supplemented by the homology class* Λ_a^k *of a "*t^a*-fold linking k-cycle"* λ_a^k, *form a set of generators of* H_k^a.

Lemma 5.1 will follow once we have established Prop. 5.1 below. The trivial case $k = 0$ is excluded.

Proposition 5.1. *If* e_+^k *is an arbitrary k-cycle on* f_a, *then*

$$e_+^k \sim m\lambda_a^k \qquad (\textit{on } f_a \bmod \dot{f}_a \textit{ when } s^a = 0)\ , \tag{5.6}$$

where λ_a^k *is a* t^a*-fold linking k-cycle on* f_a, *m is an integer, and*

$$e_+^k \sim 0 \qquad (\textit{on } f_a \bmod \dot{f}_a \textit{ when } s^a > 0)\ . \tag{5.6'}$$

Proof. It follows from Theorems 2.1 and 2.2 that for some integer μ and prescribed universal k-cap κ_a^k

$$e_+^k = \mu\kappa_a^k + \partial e_+^{k+1} - e_-^k \tag{5.7}$$

for a suitably chosen chain e_+^{k+1} on f_a and a chain e_-^k on $\dot{f}_a$. It follows from (5.7) that the homology class of $\mu\partial\kappa_a^k$ on $\dot{f}_a$ vanishes since

$$\partial e_-^k = \mu\partial\kappa_a^k\ . \tag{5.8}$$

The case $s^a > 0$. In this case (5.8) is valid only if $\mu = 0$. To verify this, recall that the homology class of $\partial\kappa_a^k$ on $\dot{f}_a$ is an element w_a^{k-1} by Def. 4.1. Moreover (4.17) shows that order $w_a^{k-1} = \infty$ in $\dot{H}_{k-1}^a$ when $s^a > 0$. Hence, when $s^a > 0$, (5.8) can hold only if $\mu = 0$. When $s^a > 0$, (5.6)′ accordingly holds.

The case $s^a = 0$. In this case order $w_a^{k-1} = t^a$, as (4.17) shows. By virtue of (5.8), $\mu w_a^{k-1} = 0$. Thus μ annihilates the element w_a^{k-1} in $\dot{H}_{k-1}^a$. We infer that μ is a multiple mt^a of the order t^a of w_a^{k-1}. From (5.7) we conclude that

$$e_+^k \sim mt^a\kappa_a^k \qquad (\text{on } f_a \bmod \dot{f}_a)\ . \tag{5.9}$$

According to Def. 4.4, when $s^a = 0$ there is associated with κ_a^k a t^a-fold linking

k-cycle λ_a^k such that

$$t^a \kappa_a^k \sim \lambda_a^k \qquad (\text{on } f_a \bmod \dot{f}_a) \, . \tag{5,10}$$

A rel. homology of form (5.6) follows from (5.9) and (5.10).

This completes the proof of Prop. 5.1.

Proof of Lemma 5.1 *completed.* Let the $\#$-image $\psi_k^a(z^k)$ of an element $z^k \in \dot{H}_k^a$ be denoted by $z^{k\#}$, and denote the group $\psi_k^a(\dot{H}_k^a)$ by $(\dot{H}_k^a)^\#$.

By hypothesis there exists a finite set $(z_1^k, \cdots, z_p^k)$ of generators of $\dot{H}_k^a$, or equivalently we write

$$\dot{H}_k^a = \{z_1^k, \cdots, z_p^k\} \, . \tag{5.11}$$

Since a $\#$-mapping is a homomorphism, (5.11) implies that

$$(\dot{H}_k^a)^\# = \{z_1^{k\#}, \cdots, z_p^{k\#}\} \, . \tag{5.12}$$

When $s^a > 0$, (5.6)$'$ holds so that in this case $(\dot{H}_k^a)^\# = H_k^a$. Hence

$$H_k^a = \{z_1^{k\#}, \cdots, z_p^{k\#}\} \, , \tag{5.13}$$

establishing Lemma 5.1 (i), when $s^a > 0$.

When $s^a = 0$, (5.6) holds and implies that e_+^k is homologous on $f_a \bmod \dot{f}_a$ to a k-cycle $m\lambda_a^k$ on f_a. If Λ_a^k is the homology class on f_a of λ_a^k one concludes that when $s^a = 0$

$$H_k^a = \{\Lambda_a^k, z_1^{k\#}, \cdots, z_p^{k\#}\} \tag{5.14}$$

thereby establishing Lemma 5.1 (ii).

Thus Lemma 5.1 is true.

This completes the proof of Theorem 5.3.

Theorems 5.1, 5.2 and 5.3 together show that each homology group in the sequences (5.2q) is FG. We have thus proved the following:

Theorem 5.4. *For each value c of f and each integer q, $H_q(f_c, Z)$ is finitely generated.*

6. Relative invariants

The term "relative numerical invariants" in Theorem 0.1 requires definition.

The diff Θ. Let there be given an arbitrary C^∞-diff

$$x \to \Theta(x) \colon M_n \to M'_n \tag{6.1}$$

of M_n onto a second differentiable manifold M'_n. Corresponding to the ND f given on M_n there exists a ND function f' on M'_n such that $f(x) = f'(x')$, where $x \in M_n$ and $x' = \Theta(x)$. To a critical point p_a of f corresponds a ND critical point p'_a such that $p'_a = \Theta(p_a)$ and index $p_a =$ index p'_a. If f' is defined in terms of f as above, then the index of a critical point p_a of f is *invariant* under Θ. See Theorem 5.5 and § 13 of [1].

The Θ-induced isomorphism Φ^a_q. The diff Θ induces an isomorphism Φ^a_q of the homology group $\dot{H}^a_q$ of $\dot{f}_a$ onto the corresponding homology group $\dot{H}'^a_q$ of $\dot{f}'_a$. Cf. Def. 1.3 and Theorem 1.5. If $\dot{H}^a_q$ is FG, and

$$\dot{H}^a_q = \dot{\mathscr{B}}^a_q \oplus \dot{\mathscr{T}}^a_q \qquad \text{(cf. (2.22))}\ , \tag{6.2}$$

then under Φ^a_q the groups in (6.2) are mapped isomorphically onto the corresponding groups of a direct sum

$$\dot{H}'^a_q = \dot{\mathscr{B}}'^a_q \oplus \dot{\mathscr{T}}'^a_q\ . \tag{6.3}$$

The invariance of universal k-caps of p_a. The definition in § 2 of a "universal k-cap" κ^k_a of p_a involves prior definition of a "restricted f-saddle" on M_n at p_a. If L_k is a restricted[13] f-saddle on M_n at p_a, given as the C^∞-diffeomorph in M_n of an open euclidean k-ball B_k, then $\Theta(L_k)$ is a "restricted f'-saddle" L'_k of M'_n at p'_a.

Let κ^k_a be a universal k-cap on M_n at p_a with carrier on $|L_k|$. If τ is a homeomorphism of a vertex-ordered euclidean k-simplex into $|L_k|$ whose singular "equivalence class" is κ^k_a, then $\Theta \mathbin{\bar{\circ}} \tau$ defines[14] a singular simplex on $|L'_k|$ whose "equivalence class" is a universal k-cap κ'^k_a on M'_n at p'_a. In this sense *universal k-caps are invariant under Θ.*

The relative invariance of "profiles" of elements of A. Let w be an element in a homology group A in a sequence (5.2q). If

$$u_1, \cdots, u_\beta;\ x_1, \cdots, x_\rho \tag{6.4}$$

is a basis of A, then the *profile* of w (Def. 3.2), relative to the basis (6.4) of A, is a set of integers

$$\mu_1, \cdots, \mu_\beta;\ m_1, \cdots, m_\rho \tag{6.5}$$

such that

$$0 \leq m_j < \text{order}\, x_j \qquad (j = 1, \cdots, \rho) \tag{6.6}$$

and such that

$$w = \mu_1 u_1 + \cdots + \mu_\beta u_\beta + m_1 x_1 + \cdots + m_\rho x_\rho\ . \tag{6.7}$$

Under $\dot{\Phi}^a_q$, the groups A, element w and the basis (6.5) go respectively into a group A', element w' and basis

$$u'_1, \cdots, u'_\beta;\ x'_1, \cdots, x'_\rho \tag{6.8}$$

---

[13] A saddle L_k with $|L_k|$ so small that (2.6) holds.

[14] The symbol $\bar{\circ}$ indicates an "extended composition" as defined in Appendix I of [1].

of A' such that (6.5) is a profile of w' relative to the basis (6.8) of A'. It is in this sense that a profile of w is *invariant* under Θ, *relative to a basis* (6.4) of A.

The principal application for us of the concepts of the preceding paragraph is to the case in which

$$A = \dot{H}^a_{k-1} , \quad w = w^{k-1}_a , \quad k = \text{index } p_a , \tag{6.9}$$

where $a > a_0$ is a critical value in the sequence (5.1). For each such a, a "profile" of w^{k-1}_a of form (6.5) "relative" to a basis of $\dot{H}^a_{k-1}$ of form (6.4) will be given and admitted at the appropriate step (see Condition 7.1) in the inductive proof of Theorem 0.1.

The invariance of s^a, and of t^a when $s^a = 0$. We have seen in Theorem 3.2 that if s^a is the free index of a critical point p_a and t^a the torsion index, then t^a, when $s^a = 0$, and s^a are uniquely determined by a profile of a critical generator w^{k-1}_a of W^a_{k-1}. Let M'_n, f', p'_a be defined as above. Let s'^a be the free index of p'_a, and t'^a its torsion index when $s'^a = 0$. Then $s^a = s'^a$ and $t^a = t'^a$; for the "relative invariance" of a profile of a critical generator, w^{k-1}_a associated with p_a, means that the *same* profile can be used to determine s^a and s'^a, and when s^a and $s'^a = 0$ to determine t^a and t'^a.

7. Interpretation and proof of Theorem 0.1

The homology groups A in a sequence (5.2q) are finitely generated, as shown in § 5, and each accordingly has a well-defined "basis", in general many such. A group A in a sequence (5.2q) is a homology group $H_q(\chi, Z)$ in which χ is one of the subspaces

$$\dot{f}_{a_r}, f_{a_r}, f_c \qquad (0 \leq r \leq m) \tag{7.0}$$

of M. We term χ the *space* of A.

The object of this section is to give an inductive proof that the Betti number and ED's of each of the above groups A are uniquely determined by "relative numerical invariants" which we shall associate with the respective critical points of f in the "space" of A.

Definition 7.0. *Groups A of type AA.* A homology group in a sequence (5.2q) whose Betti numbers and ED's are determined by the relative numerical invariants associated with the respective critical points *on its "space"* will be said to be of *type AA*.

Condition 7.1. *As relative numerical invariants* of a critical point p_a of f we shall *admit* the index k of p_a, and if $a > a_0$, the *profile* (Def. 3.2)

$$\mu_1, \cdots, \mu_\beta, m_1, \cdots, m_\rho \tag{7.1}$$

of a critical generator w^{k-1}_a of the group W^a_{k-1} of § 4 relative to a basis of a homology group $\dot{H}^a_{k-1}$ known to be of type AA.

The first group in a sequence (5.2q) is of type AA.

We shall continue the proof of Theorem 0.1, admitting data subject to Condition 7.1. In *Paragraphs* P_1 and P_2 below we single out the homology groups A in the sequence (5.2q) which are isomorphic to their successors.

Paragraph P_1. If $a > a_0$, one sets $k = \text{index } p_a$, and the dimension q of the homology group $\dot{H}_q^a$ is neither k nor $k-1$, then $\dot{H}_q^a$ and H_q^a are isomorphic by virtue of Corollary 4.1 (α). If then $\dot{H}_q^a$ is of type AA, H_q^a is of type AA.

Paragraph P_2. It was seen in the proof of Theorem 5.1 that $H_q^{a_{r-1}} \approx \dot{H}_q^{a_r}$ for each r such that $0 < r \leq m$, and in the proof of Theorem 5.2 that if $a_m < c$, then $H_q^{a_m} \approx H_q(f_c, Z_m)$. In both cases the second group is of type AA if the first group is of type AA.

There remain the cases of a group $\dot{H}_q^a$ followed by a group H_q^a in the sequence (5.2q). Two cases are to be distinguished.

$$\begin{array}{lll} \text{Case I:} & q = k - 1 \;, & \\ & & (k = \text{index } p_a)\;. \\ \text{Case II:} & q = k \;, & \end{array}$$

Lemma 7.1 below concerns Case I, and Lemma 7.2 concerns Case II.

Lemma 7.1. *If the index k of p_a is positive, and $\dot{H}_{k-1}^a$ is of type AA, then H_{k-1}^a is of type AA.*

Proof. It is sufficient to show that the Betti number β of H_{k-1}^a and the[15] ED's of H_{k-1}^a are uniquely determined by the Betti number $\dot{\beta}$ of $\dot{H}_{k-1}^a$ together with the ED's

$$n_1, \cdots, n_\rho \qquad (\rho \geq 0) \tag{7.2}$$

of $\dot{H}_{k-1}^a$ and a profile

$$\mu_1, \cdots, \mu_{\dot{\beta}}; m_1, \cdots, m_\rho \tag{7.3}$$

of a critical generator w_a^{k-1} of W_{k-1}^a relative to a basis of $\dot{H}_{k-1}^a$.

We recall the surjective isomorphism

$$\dot{H}_{k-1}^a / W_{k-1}^a \approx H_{k-1}^a \tag{7.4}$$

introduced in Corollary 4.1 (β), and determine the Betti number and ED's of the quotient in (7.4) as follows.

The determination of β. If $\dot{\beta} = 0$, the free index s^a of p_a, as defined in § 4, is zero. If $\dot{\beta} \neq 0$, s^a is the GCD of the integers $\mu_1, \cdots, \mu_{\dot{\beta}}$. The following proposition is implied by Theorem[16] 3.2 of [3] and Lemma 4.1 of [3]. As previously $k = \text{index } p_a > 0$.

Proposition 7.0. *If $\dot{\beta}$ is the Betti number of $\dot{H}_{k-1}^a$, then the Betti number β of H_{k-1}^a equals $\dot{\beta}$ or $\dot{\beta} - 1$ according as $s = 0$ or $s^a > 0$.*

[15] Or equivalently the torsion quotients of H_{k-1}^a.

[16] Theorem 3.2 as supplemented by (3.48) of [3] when $\mathscr{T} = 0$.

The determination of the ED's *of* H^a_{k-1}. We distinguish between the cases in which $\dot{H}^a_{k-1}$ is torsion-free and not torsion-free. When $\dot{H}^a_{k-1}$ is torsion-free an application of Theorem 3.3 of [3] to $\dot{H}^a_{k-1}/\dot{W}^a_{k-1}$ gives the following.

Proposition 7.1. *When $\dot{H}^a_{k-1}$ is torsion-free, H^a_{k-1} is torsion-free unless $s^a > 1$. If $s^a > 1$, H^a_{k-1} then has a unique torsion coefficient s^a.*

Thus when $\dot{H}^a_{k-1}$ is torsion-free and of type AA, H^a_{k-1} is of type AA.

When $\dot{H}^a_{k-1}$ is not torsion-free, we shall make use of the ED's of $\dot{H}^a_{k-1}$ given in (7.2) and the integers $m_1, \cdots, m_\rho$ in the "profile" (7.3), and verify the following.

Proposition 7.2. *When $\dot{H}^a_{k-1}$ is not torsion-free, the torsion coefficients of H^a_{k-1} are the invariant factors exceeding* 1 *of the $\rho + 1$ square matrix*

$$\left\| \begin{matrix} n_1 & & & \\ & \ddots & & \\ & & n_\rho & \\ m_1 & \cdots & m_\rho & s^a \end{matrix} \right\| = \|a_{ij}\| , \tag{7.5}$$

where each element in $\|a_{ij}\|$ not in the diagonal or last row is zero.

Proof. When $s^a > 0$, Prop. 7.2 follows from the isomorphism (7.4) of this paper and Cor. 3.1 of [3]. When $s^a = 0$, Prop. 7.2 follows from (7.4) and Cor. 4.1 of [3].

It follows from Prop. 7.2 that if $\dot{H}^a_{k-1}$ of Prop. 7.2 is of type AA, then H^a_{k-1} is of type AA.

Lemma 7.1 *follows from* Props 7.0, 7.1 *and* 7.2.

Lemma 7.2 covers Case II.

Lemma 7.2. *If the index k of p_a is positive, and $\dot{H}^a_{k-1}$ and $\dot{H}^a_k$ are of type AA, then H^a_k is of type AA.*

Lemma 7.2 will be shown to follow from Prop. 7.3 and Prop. 7.4. Prop. 7.3 evaluates the Betti number of H^a_k and Prop. 4.4 the ED's of H^a_k.

Proposition 7.3. *If $\dot{\beta}$ is the Betti number of $\dot{H}^a_k$, then the Betti number β of H^a_k equals $\dot{\beta}$ or $\dot{\beta} + 1$ according as $s^a > 0$ or $s^a = 0$.*

Proof. The following three statements[17] are *equivalent*. Cf. (4.17).

(a) The free index s^a of p_a is zero.

(b) The "critical generators" of $\dot{W}^a_{k-1}$ have a finite order. Cf. Def. 4.1.

(c) Some non-null integral multiple of the algebraic boundary $\partial\kappa^k_a$ of each universal k-cap of p_a bounds on $\dot{f}_a$.

The determination of β of Prop. 7.3. As affirmed in § 2, a "universal k-cap" of p_a is a "k-cap" in the sense of § 29 of [1] over each field $\mathscr{K}$. It follows from the equivalence of the preceding statements, (a) and (c), that the critical point p_a is of "linking" or "non-linking" type in the sense of § 29 of [1] "over $\mathscr{K}$",

[17] True or false.

according as $s^a = 0$ or $s^a > 0$. According to Theorem 1.1 the k-th connectivity of f_a or $\dot{f}_a$ over Q equals the k-th Betti number of f_a or $\dot{f}_a$ respectively. Theorem 29.2 of [1] implies that the k-th connectivity of f_a equals the k-th connectivity R of $\dot{f}_a$ or equals $R + 1$, according as p_a is of "non-linking" or "linking" type. Proposition 7.3 follows.

Note. The data used in the proof of Prop. 7.3 are admissible under Condition 7.1. In particular the Betti number β of $\dot{H}_k^a$ is admissible since $\dot{H}_k^a$ is of type AA by hypotheses of Lemma 7.2. The free index s^a of p_a is admissible; for $\dot{H}_{k-1}^a$ is of type AA by hypotheses of Lemma 7.2, and s^a is determined by a profile of W_a^{k-1} relative to a basis of $\dot{H}_{k-1}^a$.

The following proposition implies that the ED's of the group $\dot{H}_k^a$ of Lemma 7.2 are the ED's of H_k^a.

Proposition 7.4. *The torsion subgroup $\dot{\mathscr{T}}_k^a$ of $\dot{H}_k^a$ is mapped isomorphically onto the torsion subgroup $\mathscr{T}_k^a$ of H_k^a by the inclusion induced $\sharp$-mapping ϕ_k^a of* (5.5).

Proof in case $s^a > 0$. ϕ_k^a maps $\dot{H}_k^a$ *onto* H_k^a when $s^a > 0$ by Lemma 5.1 (i), and is an isomorphism by virtue of Theorem 4.1 (i).

Proof in case $s^a = 0$. According to Lemma 5.1 (ii), H_k^a is generated by the groups $\dot{H}_k^{a*}$ and $\{\Lambda_a^k\}$, or equivalently[18]

$$H_k^a = \{\dot{\mathscr{B}}_k^{a*}, \dot{\mathscr{T}}_k^{a*}, \{\Lambda_a^k\}\} . \tag{7.6}$$

We shall show that

$$H_k^a = (\dot{\mathscr{B}}_k^{a*} \oplus \{\Lambda_a^k\}) \oplus \dot{\mathscr{T}}_k^{a*} . \tag{7.7}$$

Since ϕ_k^a maps $\dot{\mathscr{B}}_k^a$ onto $\dot{\mathscr{B}}_k^{a*}$, it follows from Theorem 4.1 (i) that $\dot{\mathscr{B}}_k^{a*}$ is isomorphic to $\dot{\mathscr{B}}_k^a$ and hence free. Let $\mathscr{B}_k^a$ be a Betti group of H_k^a. Prop. 7.3 implies that when $s^a = 0$

$$\dim \mathscr{B}_k^a = 1 + \dim \dot{\mathscr{B}}_k^a . \tag{7.8}$$

Since $\dot{\mathscr{T}}_k^{a*}$ is a finite group and order $\Lambda_a^k = \infty$ by Lemma 4.1 (ii), the relation (7.8) is compatible with (7.6) only if (7.7) holds, or equivalently, if $\dot{\mathscr{T}}_k^{a*}$ is the torsion subgroup of H_k^a, and $\dot{\mathscr{B}}_k^{a*} \oplus \{\Lambda_a^k\}$ is a complementary Betti subgroup of H_k^a.

Prop. 7.4 is thereby established.

Lemma 7.2 *follows from Prop.* 7.3 *and Prop.*7.4.

Proof of Theorem 0.1 *reviewed.* The "relative numerical invariants" admitted in Theorem 0.1 have been specified in Condition 7.1. The proof of Theorem 0.1 is inductive. The first group in a sequence (5.2q) of homology

[18] The outer brace in (7.6) denotes the group generated by the three subgroups of H_k^a which are enclosed.

groups is trivially of type AA for each integer q. Let $n(c)$ by the number of elements in the sequence (5.2q). The results of *Paragraph* P_1 and P_2 of this section and of Lemmas 7.1 and 7.2 imply the following.

Let r be an integer such that $1 < r < n(c)$. If for each integer q the homology group A in the r-th place in the sequence (5.2q) *is of type AA, then the homology group in the $(r+1)$-th place in the sequence* (5.2q) *will also be of type AA.*

Theorem 0.1 follows.

The following theorem is a by-product of this section. It summarizes how the free indices s^{a_r} of the critical points p_{a_r} determine the q-th Betti number of f_c. See Cor. 4.1(α), Prop 7.0 and Prop 7.3.

Theorem 7.1. *For $q > 0$ let A and A' be two successive groups in a sequence* (5.2q). *Then the Betti number of A fails to equal the Betti number of A' if and only if one of the following two cases occurs.*

Case I. *For some critical value $a > a_0$ of f in* (5.1), $A = \dot{H}^a_q$, *index* $p_a = q + 1$, *and* $s^a > 0$.

Case II. *For some critical value $\alpha > a_0$ of f in* (5.1), $A = \dot{H}^a_q$, *index* $p_a = q$, $s^a = 0$.

In Case I *the Betti number of H^a_q is one less than the Betti number of $\dot{H}^a_q$.*

In Case the Betti number of H^a_q is one more than the Betti number of $\dot{H}^a_q$.

Our results on Betti numbers are summarized in still another way in the following theorem.

Theorem 7.2. *Let a be the critical value of a critical point p_a of positive index k. Then*

$$(7.9) \qquad \beta_{k-1}(f_a) - \beta_{k-1}(\dot{f}_a) = 0 \quad or \quad -1 \, ,$$

$$(7.10) \qquad \beta_k(f_a) - \beta_k(\dot{f}_a) = 1 \quad or \quad 0$$

according as $s^a = 0$ or $s^a > 0$. Moreover,

$$(7.11) \qquad \beta_r(f_a) = \beta_r(\dot{f}_a) \qquad (r \neq k \text{ or } k-1) \, .$$

Theorem 7.2 follows from Propositions 7.0 and 7.3 and Corollary 4.1 (α).

A corollary of Theorem 7.1 concerns the following.

Sublevel sets f_c of lacunary type. Given a value $c > a_0$ of f, let N_c be the set of all indices of critical points on f_c. We say that f_c is of *lacunary type* if there are no two positive integers in N_c, which differ by 1. If f is a Milnor function of a complex projective space, each f_c is of lacunary type. See § 35 of [1].

Corollary 7.1. (i). *Each critical point of positive index on a sublevel set f_c of M_n of lacunary type has a vanishing free index $s = 0$.*

(ii) *As a consequence the q-th Betti number of f_c equals the number of critical points on f_c with index q.*

Proof of (i). Given $q > 0$ if the Betti number $\beta_q(f_c)$ of f_c is positive, there must be a first group A' in the sequence (5.2q) whose Betti number is positive. Since there are no negative Betti numbers, it follows from Theorem 7.1 that $A' = H_q^a$ for some critical point p_α on f_c, and that $A = \dot{H}_q^a$ must come under Case II with index $p_\alpha = q$ and $s^\alpha = 0$.

We see that when q is the index of no critical point on f_c, then $\beta_q(f_c) = 0$.

If f_c is of lacunary type then for fixed $q > 0$, Case I of Theorem 7.1 can never occur. Cf. Theorem 7.2.

Statement (i) follows.

Theorem 7.1 and (i) imply (ii).

Since Corollary 7.1 (i) is true, it follows from Prop 7.1, Prop 7.4 and the isomorphisms of Paragraphs P_1 and P_2 that the singular homology groups of sublevel sets f_c of "lacunary type" are *torsion free*.

The following theorem gives a summary of our results on the determination of torsion subgroups of the homology groups H_r^a.

Theorem 7.3. *Let a be the critical value of a critical point p_a of positive index k.* (i) *For each integer $r \neq k - 1$ the torsion subgroup of $\dot{H}_r^a$ admits a $\#$-isomorphism onto the torsion subgroup of H_r^a.* (ii) *The torsion coefficients of H_{k-1}^a can be determined with the aid of Propositions* 7.1 *and* 7.2.

This theorem is an immediate consequence of Corollary 4.1 (α) and Propositions 7.1, 7.2, and 7.4.

A compact M_n. In the case in which M_n is both compact and connected special results concerning $H_n(M_n, Z)$ are well known. These results have been classically proved with the aid of a triangulation of M_n. The extension of these results, formulated in Theorem 7.4, in reality depends upon no triangulation of M_n.

In Theorem 7.4 we refer to the "geometric orientability" of M_n as defined in § 39 of [1]. A criterion for this orientability of M_n is presented[19] in § 39 of [1], namely that $R_n(M_n, Q) = 1$. According to Theorem 1.1 of this paper, when M_n is compact and connected

$$\beta_n(M_n) = R_n(M_n, Q) . \tag{7.12}$$

We shall make use of the fact, established by Morse in [12], that there exists a polar ND function f on M_n, which is a ND function on M_n of class C^∞ whose set of critical points includes just one critical point of index 0 and one of index n.

We are led to the following theorem.

Theorem 7.4. *Concerning a compact, connected C^∞-manifold M_n the following is true.*

[19] To be verified in a later paper.

(i) *The singular homology group* $H_n(M_n, Z)$ *is torsion-free.*

(ii) *The manifold* M_n *is "geometrically orientable" if and only if* $\beta_n(M_n) = 1$.

(iii) *The Betti number* $\beta_n(M_n) = 1$ *if and only if, for some polar* ND *function* f *on* M_n *and the critical point* p *of* f *of index* n, *the "free index" of* p *is* 0.

Proof of (i). Suppose that in the sequence (5.1) of critical values the terminal value $c = a_m$, and that c is the maximum value on M_n of a polar ND function f on M_n. The index n of p_c then exceeds the indices of each of the other critical points of f. In the sequence

$$H_n^{a_0};\ \dot{H}_n^{a_1},\ H_n^{a_1};\ \cdots;\ \dot{H}_n^{a_{m-1}},\ H_n^{a_{m-1}};\ \dot{H}_n^{a_m} \tag{7.13}$$

of homology groups each "dotted" group is isomorphic to its predecessor by virtue of the isomorphism (5.3), while each dotted group except the last is isomorphic to its successor by virtue of Corollary 4.1 (α) with $q = n > k$ therein. Hence $\dot{H}_n^{a_m}$ is torsion-free. Finally the torsion subgroup of $\dot{H}_n^{a_m}$ is isomorphic to the torsion subgroup of $H_n^{a_m}$ by virtue of Theorem 7.3 (i) with $a = c$, $k = n = r$ therein.

Hence H_n^c is torsion-free and (i) is true.

Proof of (ii). Statement (ii) follows from (7.12) and the criterion for orientability of M_n, given in § 39 of [1].

Proof of (iii). Statement (iii) follows from Theorem 7.1 and the fact that the indices of critical points of f other than p_c have values $k < n$.

The equivalence (ii) *of geometrical orientability and homologically defined orientability.*

In formulating a proof of this equivalence without any global use of a triangulation of M_n certain discoveries were made, one of which will be outlined in brief. We suppose $n > 2$.

Let a critical value a of f be assigned an index equal to index p_a. Let the ND function f on M_n be so chosen (as is possible) that the critical values of f of index $n - 1$ are greater than the critical values with smaller indices and, dually, the critical values of f of index 1 are less than the critical values of f with larger indices. Such an f will be termed of *biordered* type.

Corresponding to any open interval (c, e) of values of f set

$$f_{(c,e)} = \{x \in |M_n| \ c < f(x) < e\}\ , \tag{7.14}$$

and let $\mathbf{f}_{(c,e)}$ be the submanifold of M_n with carrier $f_{(c,e)}$ and differentiable structure induced by that of M_n. Let M and m be respectively the maximum and minimum of the values of f on M_n.

Definition. *Inverting critical values.* A critical value a of f with index k such that $0 < k < n$ will be called *orientation inverting* if M_n is nonorientable and if a is a largest critical value for which $f_{(m,a)}$ is orientable, or is a smallest critical value for which $f_{(a,M)}$ is orientable.

We state a fundamental theorem.

Theorem 7.5. *With $n > 2$ suppose that f is of biordered type, and that M_n is compact and connected. Then each level set f^c of M_n is connected.*

If M_n is non-orientable, there are just two orientation inverting critical values, one a' of index 1, and the other a'' of index $n-1$. Of the differentiable submanifolds

$$f_{(m,a')}, f_{(a',a'')}, f_{(a'',M)} \tag{7.15}$$

of M_n the first and third are geometrically orientable and the second geometrically non-orientable.

Detailed proofs of Theorems 7.4 and 7.5 will follow in a later paper.

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