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FLOER HOMOLOGY OF BRIESKORN HOMOLOGY SPHERES

NIKOLAI SAVELIEV

Abstract

Every Brieskorn homology sphere $\Sigma(p,q,r)$ is a double cover of the 3–sphere ramified over a Montesinos knot $k(p,q,r)$. We express the Floer homology of $\Sigma(p,q,r)$ in terms of certain invariants of the knot $k(p,q,r)$, among which are the knot signature and the Jones polynomial. We also define an integer valued invariant of integral homology 3–spheres which agrees with the $\bar{\mu}$–invariant of W. Neumann and L. Siebenmann for Seifert fibered homology spheres, and investigate its behavior with respect to homology 4–cobordism.

Let p, q, and r be pairwise coprime positive integers. A Brieskorn homology 3–sphere $\Sigma(p,q,r)$ is the link of the singularity of $f^{-1}(0)$, where $f : \mathbb{C}^3 \to \mathbb{C}$ is a map of the form $f(x,y,z) = x^p + y^q + z^r$. The complex conjugation in $\mathbb{C}^3$ acts on $\Sigma(p,q,r)$ turning it into a double branched cover of S^3 branched over a Montesinos knot $k(p,q,r)$.

The Floer homology groups $I_n(\Sigma)$, $0 \le n \le 7$, are abelian groups associated with an integral homology 3–sphere Σ; see [14]. The Floer homology of $\Sigma(p,q,r)$ was studied by R. Fintushel and R. Stern [13] who showed in particular that the groups $I_*(\Sigma(p,q,r))$ are free abelian, and if a_n denotes the rank of $I_n(\Sigma(p,q,r))$ then $a_n = 0$ for odd n. Therefore,

$$a_0 + a_2 + a_4 + a_6 = 2\,\lambda(\Sigma(p,q,r)), \tag{1}$$

where $\lambda(\Sigma(p,q,r))$ is the Casson invariant; see [36]. We add to this knowledge the following result:

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Theorem 1. *Let $\Sigma(p,q,r)$ be a Brieskorn homology sphere represented as a double branched cover of S^3 ramified over a Montesinos knot $k(p,q,r)$. Let a_n denote the rank of the free abelian group $I_n(\Sigma(p,q,r))$, and $\operatorname{sign} k(p,q,r)$ the signature of the knot $k(p,q,r)$. Then*

$$-a_0 + a_2 - a_4 + a_6 = \frac{1}{4}\operatorname{sign} k(p,q,r). \tag{2}$$

According to [15], the groups $I_n(\Sigma(p,q,r))$ are 4–periodic, that is, $I_n(\Sigma(p,q,r)) = I_{n+4}(\Sigma(p,q,r))$ for all n. Therefore, equations (1) and (2) give a closed form formula for the Floer homology of Brieskorn homology spheres, and answer Question 3.5 of [26] (which is attributed to M. Atiyah).

Theorem 1 establishes links between the Floer homology and the Jones polynomial. The following result is obtained by combining equation (2) with D. Mullins formula for the Casson invariant of two–fold branched covers; see [24].

Theorem 2. *Let $\Sigma(p,q,r)$ be a Brieskorn homology sphere represented as a double branched cover of S^3 with the branch set a Montesinos knot $k(p,q,r)$. Let a_n denote the rank of the free abelian group $I_n(\Sigma(p,q,r))$, and V_k the Jones polynomial of the knot $k = k(p,q,r)$. Then*

$$a_2 + a_6 = \frac{1}{12}\cdot\left.\frac{d}{dt}\right|_{t=-1} \ln V_k\,(t).$$

For an integral homology 3–sphere Σ, we defined in [31] an invariant ν by the formula

$$\nu(\Sigma) = \frac{1}{2}\sum_{n=0}^{7}(-1)^{\frac{(n+1)(n+2)}{2}}\operatorname{rank}_{\mathbb{Q}} I_n(\Sigma). \tag{3}$$

For a Brieskorn homology sphere $\Sigma(p,q,r)$ with the orientation of an algebraic link, the invariant ν is of the form $\nu(\Sigma(p,q,r)) = \frac{1}{2}(-a_0 + a_2 - a_4 + a_6)$ where $a_n = \operatorname{rank} I_n(\Sigma(p,q,r))$ as before. It should be also mentioned that the ν–invariant changes sign with the change of orientation. We conjectured in [31] that the ν–invariant equals the $\bar{\mu}$–invariant of W. Neumann [25] and L. Siebenmann [34] for all Seifert fibered homology spheres. Now we can prove that this conjecture is true.

Theorem 3. *For any Brieskorn homology 3–sphere $\Sigma(p,q,r)$, the ν–invariant defined by (3) and the $\bar\mu$–invariant coincide, $\nu(\Sigma(p,q,r)) = \bar\mu(\Sigma(p,q,r))$.*

Theorem 3 is equivalent to Theorem 1 due to the observation that $\bar\mu(\Sigma(p,q,r)) = \frac{1}{8}\operatorname{sign} k(p,q,r)$; see Section 7. The statement of the conjecture for a general Seifert fibered homology sphere $\Sigma(a_1,\dots,a_n)$ now follows from Theorem 3 by induction with the help of the so called splicing additivity. Namely, it has been proved in [12]; see also [31], that $\bar\mu(\Sigma(a_1,\dots,a_n)) = \bar\mu(\Sigma(a_1,\dots,a_j,p)) + \bar\mu(\Sigma(q,a_{j+1},\dots,a_n))$ where j is any integer between 2 and $n-2$, and the integers $q = a_1\cdots a_j$ and $p = a_{j+1}\cdots a_n$ are the products of the first j and the last $(n-j)$ Seifert invariants, respectively. The same additivity holds for the invariant ν; see [31].

Corollary 4. *For any Seifert fibered homology sphere $\Sigma(a_1,\dots,a_n)$, the ν– and the $\bar\mu$–invariants coincide,*

$$\nu(\Sigma(a_1,\dots,a_n)) = \bar\mu(\Sigma(a_1,\dots,a_n)).$$

Thus, the invariant ν is an extension of the $\bar\mu$–invariant from Seifert fibered homology spheres to arbitrary integral homology spheres. In fact, the invariant $\bar\mu$ is defined for a broader class of homology 3–spheres, including all integral graph homology spheres. The question whether $\bar\mu$ equals ν on this bigger class remains open.

W. Neumann conjectured in [25] that the $\bar\mu$–invariant is in fact a homology cobordism invariant. In [30] and [33] we proved this conjecture for certain classes of Seifert fibered homology spheres. Due to the identification $\bar\mu = \nu$, the corresponding results hold for the ν–invariant.

Theorem 5. *(1) Let $\Sigma(a_1,\dots,a_n)$ be a Seifert fibered homology sphere homology cobordant to zero. Then $\nu(\Sigma(a_1,\dots,a_n)) \ge 0$.*

(2) Let $\Sigma = \Sigma(p,q,pqm\pm 1)$ be a surgery on a (p,q)–torus knot, and suppose that Σ is homology cobordant to zero. Then $\nu(\Sigma) = 0$.

(3) For all Seifert fibered homology 3–spheres Σ which are known to be homology cobordant to zero, including the lists of Casson-Harer [7] and Stern [35], $\nu(\Sigma) = 0$.

Further progress in this direction is due to Fukumoto and Furuta [16], who proved the homology cobordism invariance of $\bar\mu$ (and therefore that of ν) for all Seifert fibered homology spheres $\Sigma(a_1,\dots,a_n)$ with a_1 even. It should be mentioned that the homology cobordism invariance of ν has also been checked on certain classes of integral homology 3–spheres which are not graph manifolds; see [32].

The idea of proof of Theorem 1 is shortly as follows. In [36], C. Taubes proved that, for any integral homology 3–sphere Σ,

$$\lambda(\Sigma) = 1/2 \cdot \chi(I_*(\Sigma)). \tag{4}$$

The Casson's λ–invariant on the left is defined using a Heegaard splitting of Σ and SU(2)–representation spaces. The number on the right is the Euler characteristic of Floer homology $I_*(\Sigma)$; it can be defined using SU(2) gauge theory as an infinite dimensional generalization of the classical Euler characteristic.

Let now $\Sigma = \Sigma(p,q,r)$ be endowed with an involution σ induced on $\Sigma \subset \mathbb{C}^3$ by the complex conjugation. We work out σ–invariant versions of both invariants in (4) for $\Sigma = \Sigma(p,q,r)$. We use a σ–invariant Heegaard splitting of Σ and the corresponding equivariant SU(2)–representation spaces to define an invariant $\lambda^\rho(\Sigma)$ in a manner similar to that of A. Casson. On the other hand, an equivariant gauge theory produces an equivariant Euler characteristic which we denote $\chi^\rho(\Sigma)$. Note that the latter can be defined without actually working out any Floer homology. Then, a σ–invariant version of the Taubes result (4) is that

$$\lambda^\rho(\Sigma(p,q,r)) = 1/2 \cdot \chi^\rho(\Sigma(p,q,r)). \tag{5}$$

Our next step is to show that $\lambda^\rho(\Sigma(p,q,r)) = \frac{1}{8}\,\operatorname{sign} k(p,q,r)$. We achieve this by pushing equivariant representations of the manifolds in the σ–invariant Heegaard splitting down to their quotients. We note that the push-down representations are only defined on the complements of the ramification sets, and they map all their meridians to trace-free matrices in SU(2). After that we identify $\lambda^\rho(\Sigma(p,q,r))$ with one forth of the Casson–Lin invariant $h(k(p,q,r))$ of the knot $k(p,q,r)$. The invariant $h(k)$ was defined in [21] for an arbitrary knot k by using trace–free representation spaces associated with the knot k complement. X.-S. Lin proved in [21] that the invariant $h(k)$ equals one half of the knot k signature, which in our case immediately implies that $\lambda^\rho(\Sigma(p,q,r)) = \frac{1}{8}\,\operatorname{sign} k(p,q,r)$. The crucial observation is that *all* the representations of $\pi_1\Sigma(p,q,r)$ are in fact equivariant, and the λ^ρ–invariant just counts them with signs different from those defined by Casson for his λ–invariant.

Finally, we identify $1/2 \cdot \chi^\rho(\Sigma(p,q,r))$ with $\nu(\Sigma(p,q,r))$ by comparing the equivariant spectral flow used to define $\chi^\rho(\Sigma(p,q,r))$ with the "regular" spectral flow of [36] with the help of the G–index theorem of [11].

It should be pointed out that many results and definitions in the paper hold in more general situation. Recently, several papers have appeared that discuss a similar circle of ideas. C. Herald in [18] extended the Lin's result to SU(2)–representations of the knot complement whose trace along the meridian is fixed but not necessarily zero; the resulting invariant is an equivariant knot signature. This result was refined by O. Collin and B. Steer, who developed in [10] an instanton Floer homology for knots via orbifolds; the Euler characteristic of their Floer homology is an equivariant knot signature. This orbifold theory was related to an equivariant gauge theory on finite cyclic branched covers of a knot in [8] and [9]. W. Li in [20] constructed a symplectic Floer homology theory for knots in S^3 whose Euler characteristic equals the Casson–Lin invariant.

The paper is organized as follows. It begins with a short introduction to the relevant topology of Brieskorn homology spheres and Montesinos knots. In Section 2 we investigate equivariant SU(2)–representation spaces associated with a σ–invariant Heegaard splitting of $\Sigma(p,q,r)$ and define the λ^ρ–invariant. The equality $\lambda^\rho(\Sigma(p,q,r)) = \frac{1}{8}\operatorname{sign} k(p,q,r)$ is proved in Section 3. Section 4 is devoted to equivariant gauge theory and the definition of the χ^ρ–invariant. In Section 5 we express $\chi^\rho(\Sigma(p,q,r))$ in terms of the Floer homology as $\chi^\rho(\Sigma(p,q,r)) = 2\,\nu(\Sigma(p,q,r))$. The equality of λ^ρ and $\frac{1}{2}\chi^\rho$ for Brieskorn homology spheres is proved in Section 6. One should mention that all the equalities of the invariants above are proved up to a universal $(\pm)$–sign. The sign is fixed at the end of Section 6, which completes the proof of Theorem 1. Section 7 contains a detailed discussion on the $\bar\mu$–invariant of W. Neumann and L. Siebenmann.

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1. Topology of Brieskorn homology spheres

In this section we describe shortly topology of Brieskorn homology spheres and Montesinos knots. More detailed account will be given in Section 7.

Let p, q, r be relatively prime integers greater than or equal to 2. The Brieskorn homology sphere $\Sigma(p,q,r)$ is defined as the algebraic link

$$\Sigma(p,q,r) = \{\, (x,y,z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = 0 \,\} \cap S^5,$$

where S^5 is a sphere in $\mathbb{C}^3$. This is a smooth naturally oriented 3–manifold with $H_*(\Sigma(p,q,r)) = H_*(S^3)$. Moreover, $\Sigma(p,q,r)$ is Seifert fibered; see [27], with the Seifert invariants $\{\, b, (p,b_1), (q,b_2), (r,b_3) \,\}$ such that

$$b_1 qr + b_2 pr + b_3 pq = 1 + bpqr. \tag{6}$$

The complex conjugation on $\mathbb{C}^3$ obviously acts on $\Sigma(p,q,r)$ as

$$\sigma : \Sigma(p,q,r) \to \Sigma(p,q,r), \quad (x,y,z) \mapsto (\bar{x},\bar{y},\bar{z}). \tag{7}$$

The fixed point set of this action is never empty. The quotient of $\Sigma(p,q,r)$ by the involution σ is S^3, with the branch set the so called Montesinos knot $k(p,q,r)$; see [6], [23], or [34]. The knot $k(p,q,r)$ can be described by the diagram in Figure 1, where a box with α, β in it stands for a rational (α,β)–tangle; see [6], Fig. 12.9.

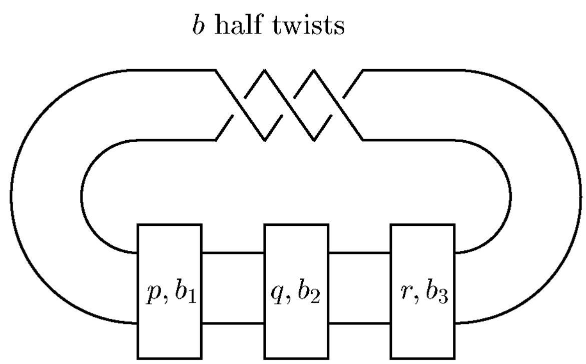

FIGURE 1

The parameters $b, (p,b_1), (q,b_2), (r,b_3)$ are the Seifert invariants of the corresponding $\Sigma(p,q,r)$. According to [6], Theorem 12.28, these parameters together with (6) determine the knot $k(p,q,r)$ uniquely up to isotopy.

2. The invariant λ^ρ

In this section we first shortly recall the definition of Casson's λ–invariant following the exposition [1]. Our λ^ρ–invariant for $\Sigma(p,q,r)$ will be an equivariant version of λ. To define λ^ρ, we introduce a σ–invariant Heegaard splitting of $\Sigma(p,q,r)$. Then we define the corresponding equivariant representation spaces and investigate closely the representation space of $\pi_1\Sigma(p,q,r)$. It turns out that all the representations in the latter space are σ–invariant. After computing the necessary dimensions and checking the transversality condition, we define the λ^ρ–invariant as an intersection number of equivariant representation spaces.

2.1 Casson invariant

Let M be an oriented integral homology 3–sphere with a Heegaard splitting $M = M_1 \cup M_2$ where M_1 and M_2 are handlebodies of genus $g \geq 2$ glued along their common boundary, a Riemann surface M_0. Let

$$\mathcal{R}(M_k) = \mathrm{Hom}^*(\pi_1 M_k, \mathrm{SU}(2))/\,\mathrm{SU}(2),\ \ k = 0,1,2,\emptyset,$$

be the set of conjugacy classes of irreducible representations of $\pi_1 M_k$ is $\mathrm{SU}(2)$. Each $\mathcal{R}(M_k)$, $k = 0,1,2$, is naturally an oriented manifold. The dimension of $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ is $3g-3$, and that of $\mathcal{R}(M_0)$ is $6g-6$. The inclusions $M_0 \subset M_k$, $k = 1,2$, induce embeddings $\mathcal{R}(M_k) \subset \mathcal{R}(M_0)$. The points of intersection of $\mathcal{R}(M_1)$ with $\mathcal{R}(M_2)$ are in one-to-one correspondence with $\mathcal{R}(M)$. If the intersection is transversal we define the *Casson invariant* as the integer

$$\lambda(M) = \frac{1}{2} \sum_{\alpha \in \mathcal{R}(M)} \varepsilon_\alpha \tag{8}$$

where $\varepsilon_\alpha = \pm 1$ is a sign obtained by comparing the orientations on $T_\alpha\mathcal{R}(M_1) \oplus T_\alpha\mathcal{R}(M_2)$ and $T_\alpha\mathcal{R}(M_0)$. Note that the intersection *is* transversal if $\Sigma = \Sigma(p,q,r)$; see R. Fintushel and R. Stern [13, Proposition 2.5].

The formula (8) defines the invariant λ up to a (± 1)–sign, which only depends on the orientation of Σ, and changes by (-1) if the orientation of Σ changed. One can show that $|\lambda(\Sigma(2,3,5))| = 1$; the invariant λ is then normalized by the requirement that $\lambda(\Sigma(2,3,5)) = +1$. An explicit formula for $\lambda(\Sigma(p,q,r))$ and more generally, for $\lambda(\Sigma(a_1,\ldots,a_n))$, is given in [26, Lemma 1.5].

2.2 σ–invariant Heegaard splitting

We are going to construct a σ–invariant Heegaard splitting of $\Sigma(p,q,r)$. Let us first fix notation. By Σ we denote a Brieskorn homology sphere $\Sigma(p,q,r)$, and by $\sigma : \Sigma \to \Sigma$ the involution defined by (7). The projection on the quotient space will be denoted by π, so $\pi : \Sigma \to \Sigma/\sigma = S^3$. The projection π maps the fixed point set $\mathrm{Fix}(\sigma) \subset \Sigma$ onto a Montesinos knot $k = k(p,q,r) \subset S^3$.

The knot $k \subset S^3$ can be represented as the closure of a braid β on n strings. We think about k as consisting of the braid β and n untangled arcs in S^3 forming its closure; see Figure 2. Let $S \subset S^3$ be an embedded 2–sphere in S^3 splitting S^3 in two 3–balls, B_1 and B_2, with common boundary S, and such that the entire braid β belongs to $\mathrm{int}\, B_1$. The intersection $k \cap S$ consists of $2n$ points, $P_1, \dots, P_{2n}$.

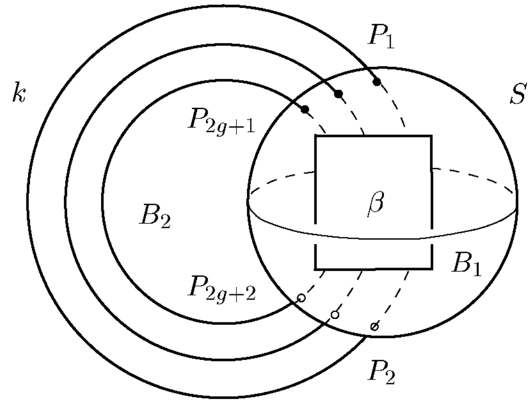

FIGURE 2

Now, we define a Heegaard splitting $\Sigma = M_1 \cup_{M_0} M_2$ as follows:

$$M_1 = \pi^{-1}(B_1), \quad M_2 = \pi^{-1}(B_2), \quad M_0 = \pi^{-1}(S). \tag{9}$$

Obviously, M_1 and M_2 are handlebodies branched over the braid β and the n untangled arcs, respectively. Their common boundary is M_0, which is a closed surface branched over the points $P_1, \dots, P_{2n} \in S$. The genus g of M_0 can be found by comparing Euler characteristics,

$$2 \cdot \chi(S) = \chi(M_0) + \chi(\,\mathrm{Fix}(\sigma) \cap S),$$

where $\chi(S) = 2$, $\chi(\,\mathrm{Fix}(\sigma) \cap S) = 2n$, and $\chi(M_0) = 2 - 2g$. Therefore, $g = n - 1$. The Heegaard splitting we just defined is σ–invariant in the sense that $\sigma(M_k) = M_k$ for $k = 0, 1, 2$.

2.3 Representations of Brieskorn homology spheres

Let $\Sigma(p,q,r)$ be a Brieskorn homology 3–sphere, and

$$\mathcal{R}(\Sigma(p,q,r)) = \operatorname{Hom}^*(\pi_1\Sigma(p,q,r), \mathrm{SU}(2))/\,\mathrm{SU}(2) \tag{10}$$

the space of the conjugacy classes of irreducible representations of its fundamental group in SU(2). In this subsection we are concerned with describing the space $\mathcal{R}(\Sigma(p,q,r))$ and the involution σ^* induced on the representation space by σ.

We will follow R. Fintushel and R. Stern [13] and first identify the group SU(2) with the group S^3 of unit quaternions in the usual way, so that

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Under this identification, the trace $\operatorname{tr}(A)$ of an element $A \in \mathrm{SU}(2)$ coincides with the number $2\operatorname{Re}A$, $A \in S^3$, and the mapping $r : S^3 \to [0,\pi]$, $r(A) = \arccos(\operatorname{Re}A)$, is a complete invariant of the conjugacy class of the element A. This conjugacy class is a copy of S^2 in S^3 unless $A = \pm 1$, in which case the conjugacy class consists of just one point.

The fundamental group $\pi_1\Sigma(p,q,r)$ has the following presentation; see [6] or [13],

$$\begin{aligned}\pi_1\Sigma(p,q,r) = \langle x, y, z, h \mid h \text{ central}\,, x^p =& h^{-b_1}, y^q = h^{-b_2}, \\ z^r =& h^{-b_3}, xyz = h^{-b}\rangle,\end{aligned}$$

where b_1, b_2, b_3, and b satisfy (6). Specifying an irreducible representation

$$\alpha : \pi_1\Sigma(p,q,r) \to \mathrm{SU}(2)$$

amounts to specifying a set $\{\alpha(h), \alpha(x), \alpha(y), \alpha(z)\}$ of unit quaternions. In fact, we only need to specify the first three quaternions in this set because $\alpha(z)$ will then be expressed in their terms as

$$\alpha(z) = (\alpha(x)\alpha(y))^{-1}\alpha(h)^{-b}.$$

Since h is central and the representation α is irreducible, $\alpha(h) = \pm 1$. Let us denote $\varepsilon_i = \alpha(h)^{b_i} = \pm 1$, $i = 1,2$, and $\varepsilon_3 = \alpha(h)^{b_3 - rb} = \pm 1$. Then the relations $x^p = h^{-b_1}, y^q = h^{-b_2}$ and $(xy)^r = h^{b_3 - rb}$ imply the following restrictions on $\alpha(x)$ and $\alpha(y)$:

$$r(\alpha(x)) = \pi\ell_1/p, \quad r(\alpha(y)) = \pi\ell_2/q, \quad r(\alpha(x)\alpha(y)) = \pi\ell_3/r, \tag{11}$$

where ℓ_i is even if $\varepsilon_i = 1$, ℓ_i is odd if $\varepsilon_i = -1$, and

$$0 < \ell_1 < p, \; 0 < \ell_2 < q, \; 0 < \ell_3 < r.$$

After conjugation, we may assume that $\alpha(x) = e^{\pi i \ell_1/p}$. The quaternions $\alpha(y)$ and $\alpha(x)\alpha(y)$ should lie in their respective conjugacy classes, $S_2 = r^{-1}(\pi\ell_2/q)$ and $S_3 = r^{-1}(\pi\ell_3/r)$. On the other hand, $\alpha(x)\alpha(y)$ lies in $\alpha(x)\cdot S_2$, therefore, in order for $\alpha(x)$ and $\alpha(y)$ to define a representation, the intersection $\alpha(x)\cdot S_2 \cap S_3$ must be non-empty. Since $\alpha(x)\cdot S_2$ is a 2-sphere centered at $\alpha(x)$, the intersection $\alpha(x)\cdot S_2 \cap S_3$ in S^3 (if non-empty) is a circle. This circle parametrizes a whole collection of representations α' coming together with α such that $r(\alpha'(x)) = r(\alpha(x))$, $r(\alpha'(y)) = r(\alpha(y))$, and $r(\alpha'(x)\alpha'(y)) = r(\alpha(x)\alpha(y))$. In fact, all these representations are conjugate to each other by simultaneous conjugation of $\alpha(x)$ and $\alpha(y)$ by the complex circle $S^1 \subset S^3$. This can be seen from the following elementary technical lemma.

Lemma 6. *Let α and β be irreducible representations of the group $\pi_1\Sigma(p,q,r)$ in $\mathrm{SU}(2)$ such that*

(1) $\alpha(h) = \beta(h)$ and $\alpha(x) = \beta(x) \in \mathbb{C}$,

(2) $r(\alpha(y)) = r(\beta(y))$, and

(3) $r(\alpha(x)\alpha(y)) = r(\beta(x)\beta(y))$.

Then the representations α and β are conjugate to each other, that is there exists a unit quaternion c such that

$$\beta(t) = c\cdot\alpha(t)\cdot c^{-1} \quad \text{for all} \quad t \in \pi_1\Sigma(p,q,r).$$

Moreover, the quaternion c may be chosen to be a complex number, $c \in \mathbb{C}$.

Corollary 7 (Compare [13]). *The representation space $\mathcal{R}(\Sigma(p,q,r))$ is finite.*

The involution $\sigma : \Sigma(p,q,r) \to \Sigma(p,q,r)$ induces the involution on the fundamental group,

$$\sigma_* : \pi_1\Sigma(p,q,r) \to \pi_1\Sigma(p,q,r),$$
$$h \mapsto h^{-1}, \quad x \mapsto x^{-1}, \quad y \mapsto xy^{-1}x^{-1}, \quad z \mapsto xyz^{-1}y^{-1}x^{-1};$$

see [6, Proposition 12.30], which in turn induces an involution on the corresponding representation space ($[\cdot]$ stands for conjugacy class),

$$\sigma^* : \mathcal{R}(\Sigma(p,q,r)) \to \mathcal{R}(\Sigma(p,q,r)), \quad [\alpha] \mapsto [\alpha'], \tag{12}$$

where

$$\alpha'(t) = \alpha(\sigma_*(t)), \quad t \in \pi_1\Sigma(p,q,r). \tag{13}$$

Proposition 8. *If $\alpha' : \pi_1\Sigma(p,q,r) \to \mathrm{SU}(2)$ is a representation defined by the formula (13), then there exists an element $\rho \in \mathrm{SU}(2)$ such that $\rho^2 = -1$ and*

$$\alpha'(t) = \rho \cdot \alpha(t) \cdot \rho^{-1} \quad \textit{for all} \quad t \in \pi_1\Sigma(p,q,r).$$

The element ρ is defined uniquely up to multiplication by ± 1, and the elements ρ corresponding to different representations α are conjugate to each other. In particular, the action (12) on the space $\mathcal{R}(\Sigma(p,q,r))$ of the conjugacy classes of irreducible representations of $\pi_1\Sigma(p,q,r)$ in $\mathrm{SU}(2)$ is trivial.

Proof. After conjugation we may assume that $\alpha(x)$ is a unit complex number. Then $\alpha'(x) = \alpha(\sigma_*(x)) = \alpha(x^{-1}) = \alpha(x)^{-1}$ is a complex number as well. Let us consider another representation, β, defined as a conjugate of α by the unit quaternion j,

$$\beta(t) = j^{-1} \cdot \alpha'(t) \cdot j \quad \text{for all} \quad t \in \pi_1\Sigma(p,q,r).$$

Next we want to verify that the representations α and β satisfy the conditions of Lemma 6. We have

$$\begin{aligned}
\beta(x) &= j^{-1} \cdot \alpha'(x) \cdot j \\
&= j^{-1} \cdot \alpha(x)^{-1} \cdot j \\
&= j^{-1} \cdot \overline{\alpha(x)} \cdot j \\
&= j^{-1} \cdot j \cdot \alpha(x), \qquad \text{since} \quad \alpha(x) \in \mathbb{C}, \\
&= \alpha(x).
\end{aligned}$$

Note that, for any unit quaternion a, we have $r(a) = r(\bar{a}) = r(a^{-1})$. Therefore,

$$\begin{aligned}
r(\beta(y)) &= r(j^{-1} \cdot \alpha'(y) \cdot j) = r(\alpha'(y)) \\
&= r(\alpha(xy^{-1}x^{-1})) = r(\alpha(x)\alpha(y)^{-1}\alpha(x)^{-1}) \\
&= r(\alpha(y)^{-1}) = r(\alpha(y)),
\end{aligned}$$

and similarly,

$$\begin{aligned} r(\beta(x)\beta(y)) &= r(j^{-1}\cdot\alpha'(x)\alpha'(y)\cdot j) = r(\alpha'(x)\alpha'(y)) \\ &= r(\alpha(x)^{-1}\alpha(x)\alpha(y)^{-1}\alpha(x)^{-1}) = r((\alpha(x)\alpha(y))^{-1}) \\ &= r(\alpha(x)\alpha(y)). \end{aligned}$$

Since $r(\alpha(h))$ is equal to $r(\beta(h))$ we can apply Lemma 6 to the representations α and β to find a complex number $c \in \mathbb{C}$ such that $\beta(t) = c\cdot\alpha(t)\cdot c^{-1}$ for all $t \in \pi_1\Sigma(p,q,r)$. Since $\beta(t) = j^{-1}\cdot\alpha'(t)\cdot j = c\cdot\alpha(t)\cdot c^{-1}$, we get that $\alpha'(t) = \rho\cdot\alpha(t)\cdot\rho^{-1}$ with $\rho = jc$.

If we apply σ^* twice to the representation α, we will get α again because σ^* is an involution. Therefore, α is conjugate to itself by the element ρ^2. But the representation α is irreducible, therefore, $\rho^2 = \pm 1$. The following easy computation shows that ρ^2 is in fact -1,

$$\rho^2 = jcjc = jj\bar{c}c = -|c|^2 = -1.$$

The uniqueness of ρ up to ± 1 follows from the irreducibility of α. The fact that the elements ρ defined by different representations are conjugate follows from the fact that $\operatorname{tr}(jc) = 0$ for any complex number c. q.e.d.

2.4 Heegaard splitting and representation spaces

Let $\Sigma = \Sigma(p,q,r)$ be a Brieskorn homology sphere with a σ–invariant Heegaard splitting of genus g,

$$\Sigma = M_1 \cup_{M_0} M_2,$$

constructed in (9). According to Proposition 8, for any $\alpha : \pi_1\Sigma(p,q,r) \to \mathrm{SU}(2)$, there exists an element $\rho \in \mathrm{SU}(2)$ such that $\sigma^*\alpha = \rho\alpha\rho^{-1}$ and $\rho^2 = -1$. Generally speaking, ρ depends on α but it follows from Proposition 8 that the elements ρ corresponding to different α's are conjugate to each other. Therefore, the following definition makes sense.

For each manifold M_k, $k = 0,1,2$, in the splitting (9), we define *the equivariant representation space*,

$$(14)\quad \begin{aligned} \mathcal{R}^\rho(M_k) = \{\, \alpha : \pi_1 M_k \to \mathrm{SU}(2) \mid \alpha \ &\text{ is irreducible and} \\ &\alpha(\sigma_*(t)) = \rho\alpha(t)\rho^{-1},\ t \in \pi_1 M_k \,\}/S^1_\rho, \end{aligned}$$

as the space of all irreducible ρ–invariant representations modulo the adjoint action of the 1–dimensional Lie group $S^1_\rho \subset \mathrm{SU}(2)$ consisting of all $g \in \mathrm{SU}(2)$ such that $\rho g = g\rho$. Note that whenever $\rho' = h\rho h^{-1}$, $h \in \mathrm{SU}(2)$, we have that $\mathcal{R}^{\rho}(M_k) = \mathcal{R}^{\rho'}(M_k)$, hence each of the spaces (14) only depends on the conjugacy class of ρ and not on ρ itself.

We first describe the space $\mathcal{R}^{\rho}(M_0)$. The classical result of Lüroth and Hurwitz; see [4, Theorem 3.4], implies that a double branched cover of S^2 is uniquely determined by the cardinality of its branch set. Thus we are free to visualize the covering $M_0 \to S^2$ by using the model shown in Figure 3, the action of σ being the rotation through 180° about the horizontal symmetry axis.

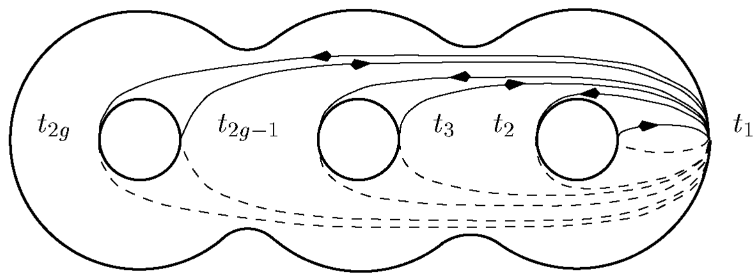

FIGURE 3

One can easily see that

$$\pi_1 M_0 = \langle\, t_1, \dots , t_{2g} \,|\, t_1 \cdots t_{2g} = t_{2g} \cdots t_1 \,\rangle,$$

where the generators $t_1, \dots , t_{2g}$ are represented by the respective curves in Figure 3. The induced involution $\sigma_* : \pi_1 M_0 \to \pi_1 M_0$ acts on the generators t_i by inverting them, $\sigma_*(t_i) = t_i^{-1}$.

Let us introduce the surface $M_0^* = M_0 \setminus \{\, D, \sigma(D) \,\}$ where D and $\sigma(D)$ are a pair of disjoint open discs as shown in Figure 4.

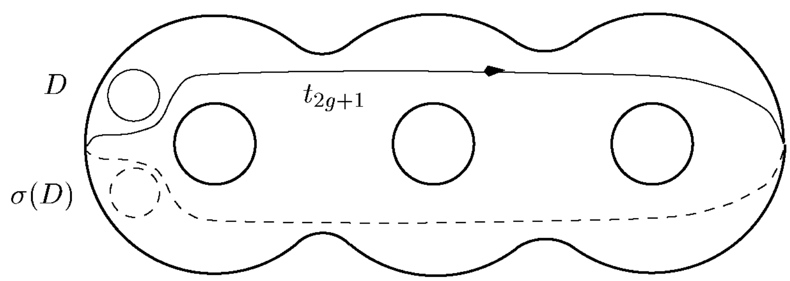

FIGURE 4

The group $\pi_1 M_0^*$ is a free group on $2g+1$ generators, and obviously

$$\mathcal{R}^\rho(M_0) \subset \mathrm{Hom}^\rho(\pi_1 M_0^*, \mathrm{SU}(2))/S^1_\rho,$$

where $\mathrm{Hom}^\rho(\pi_1 M_0^*, \mathrm{SU}(2))$ consists of all the representations $\alpha : \pi_1 M_0^* \to \mathrm{SU}(2)$ such that $\sigma^*\alpha = \mathrm{Ad}_\rho\, \alpha$. Let t_{2g+1} be the curve shown in Figure 4. Then $\sigma_*(t_{2g+1}) = t_{2g+1}^{-1}$, and the curves $t_1, \dots, t_{2g+1}$ together give a basis of the free group $\pi_1 M_0^*$. This choice of generators allows us to make the following identification,

$$\mathrm{Hom}^\rho(\pi_1 M_0^*, \mathrm{SU}(2)) = \{\, (T_1, \dots, T_{2g+1}) \in \mathrm{SU}(2)^{2g+1} \mid T_i^{-1} = \rho T_i \rho^{-1},\ i = 1, \dots, 2g+1 \,\}.$$

Lemma 9. *Let $\rho \in \mathrm{SU}(2)$ be such that $\rho^2 = -1$. Then the subset S^2_ρ of $\mathrm{SU}(2)$ consisting of all $a \in \mathrm{SU}(2)$ such that $a^{-1} = \rho a \rho^{-1}$ is a two-dimensional sphere.*

Proof. Any element ρ of $\mathrm{SU}(2)$ with $\rho^2 = -1$ has zero trace. Therefore, there exists $x \in \mathrm{SU}(2)$ such that $\rho = xjx^{-1}$ (remember that we identify $\mathrm{SU}(2)$ with the group of unit quaternions), and then the map $a \mapsto x^{-1}ax$ establishes a diffeomorphism between S^2_j and S^2_ρ. Now, S^2_j consists of all $b \in \mathrm{SU}(2)$ such that $\bar{b}j = jb$ and $|b|^2 = 1$, hence $b = u + vi + wk$ for some real u, v, w with $u^2 + v^2 + w^2 = 1$. This of course determines a 2–sphere. q.e.d.

Therefore, $\mathrm{Hom}^\rho(\pi_1 M_0^*, \mathrm{SU}(2)) = (S^2)^{2g+1}$; in particular, $\mathrm{Hom}^\rho(\pi_1 M_0^*, \mathrm{SU}(2))$ gets a smooth manifold structure from $(S^2)^{2g+1}$. A $(2g+1)$–tuple $(T_1, \dots, T_{2g+1}) \in (S^2)^{2g+1}$ defines an equivariant representation of $\pi_1 M_0$ in $\mathrm{SU}(2)$ if $T_1 \cdots T_{2g+1} = 1$. We call a point $(T_1, \dots, T_{2g+1}) \in (S^2)^{2g+1}$ *reducible* if there is a matrix $A \in \mathrm{SU}(2)$ such that AT_iA^{-1} are all diagonal matrices for $i = 1, \dots, 2g+1$, and call it *irreducible* otherwise.

Lemma 10. *Let $S^2_\rho \subset \mathrm{SU}(2)$ be the 2–sphere of unit quaternions T such that $T^{-1} = \rho T \rho^{-1}$, and θ the product map,*

$$\theta : (S^2_\rho)^{2g+1} \to \mathrm{SU}(2),$$
$$\theta(T_1, \dots, T_{2g+1}) = T_1 \cdots T_{2g+1}.$$

Then the differential $d\theta$ is onto at any irreducible point $(T_1, \dots, T_{2g+1})$ such that $T_1 \cdots T_{2g+1} = 1$.

Proof. Let us assume that $\rho = j$ and suppose that $(T_1, \dots, T_{2g+1})$ is an irreducible point such that $\theta(T_1, \dots, T_{2g+1}) = 1 \in \mathrm{SU}(2)$. Without loss of generality, we may assume that

$$T_1 = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \quad \text{and} \quad T_{2g+1} = \begin{pmatrix} A & iB \\ iB & \bar{A} \end{pmatrix}$$

with A a complex number and B a non–zero real number.

The tangent space to S^2_j at X_1 is the image under the left multiplication by T_1 of the linear subspace of $\mathfrak{su}(2)$ consisting of the matrices

$$u_1 = \begin{pmatrix} i\alpha & i\beta e^{-i\phi} \\ i\beta e^{i\phi} & -i\alpha \end{pmatrix}$$

with real α and β. For such a u_1,

$$\begin{aligned} (d\theta)(u_1, 0, \dots, 0) &= T_1 u_1 T_2 \dots T_{2g+1} = T_1 u_1 T_1^{-1} \\ &= \begin{pmatrix} i\alpha & i\beta e^{i\phi} \\ i\beta e^{-i\phi} & -i\alpha \end{pmatrix} \\ &= \alpha \cdot i - \beta \sin\phi \cdot j + \beta \cos\phi \cdot k, \quad \alpha, \beta \in \mathbb{R}. \end{aligned} \tag{15}$$

Thus, the image of $d\theta$ restricted to the tangent space to S^2_j at T_1 is 2–dimensional.

The tangent space to S^2_j at T_{2g+1} is the image under the left multiplication by T_{2g+1} of the linear subspace of $\mathfrak{su}(2)$ consisting of the matrices

$$u_{2g+1} = \begin{pmatrix} i\alpha & b \\ -\bar{b} & -i\alpha \end{pmatrix}$$

such that $\alpha \in \mathbb{R}$ and $\mathrm{Re}(Ab + \alpha B) = 0$. For such a u_{2g+1},

$$(d\theta)(0, \dots, 0, u_{2g+1}) = T_1 \cdots T_{2g+1} u_{2g+1} = u_{2g+1}.$$

Let us choose $b = \beta\cos\phi + i\beta\sin\phi$ with ϕ as in (15), and $\beta \in \mathbb{R}$. The equation $\mathrm{Re}(Ab + \alpha B) = 0$ can be solved for $\alpha \in \mathbb{R}$ to get $\alpha = \mathrm{Re}(Ab)/B$, and then the vector

$$\mathrm{Re}(Ab)/B \cdot i + \beta\cos\phi \cdot j + \beta\sin\phi \cdot k \tag{16}$$

will lie in the image of $d\theta$. The vectors (15) and (16) span the entire Lie algebra $\mathfrak{su}(2)$, therefore, $d\theta$ is onto. q.e.d.

This lemma implies that $\mathcal{R}^\rho(M_0)$ is a smooth (open) manifold of dimension $4g-2$. A choice of basis $t_1,\dots,t_g$ in $\pi_1 M_1$ with the property that $\sigma_*(t_i)=t_i^{-1}$ identifies $\mathcal{R}^\rho(M_1)$ with a smooth submanifold in $(S^2_\rho)^g/S^1_\rho$ of dimension $2g-1$. The construction for $\mathcal{R}^\rho(M_2)$ is completely analogous.

Due to Seifert–Van Kampen Theorem, we have the following commutative diagrams of the fundamental groups

$$\begin{array}{ccc} \pi_1\Sigma(p,q,r) & \longleftarrow & \pi_1 M_1 \\ \uparrow & & \uparrow \\ \pi_1 M_2 & \longleftarrow & \pi_1 M_0 \end{array}$$

and of the equivariant representation spaces

$$\begin{array}{ccc} \mathcal{R}^\rho(\Sigma) & \longrightarrow & \mathcal{R}^\rho(M_1) \\ \downarrow & & \downarrow \\ \mathcal{R}^\rho(M_2) & \longrightarrow & \mathcal{R}^\rho(M_0) \end{array}$$

where

$$\mathcal{R}^\rho(\Sigma)=\{\,\alpha:\pi_1\Sigma\to \mathrm{SU}(2)\mid \alpha \text{ is irreducible, } \sigma^*\alpha=\rho\alpha\rho^{-1}\,\}/S^1_\rho.$$

The maps in the latter diagram are injective, so we can think of $\mathcal{R}^\rho(\Sigma)$ as the intersection of $\mathcal{R}^\rho(M_1)$ and $\mathcal{R}^\rho(M_2)$ inside $\mathcal{R}^\rho(M_0)$.

Lemma 11. *The space $\mathcal{R}^\rho(\Sigma)$ is a (regular) double cover of $\mathcal{R}(\Sigma(p,q,r))$. The manifolds $\mathcal{R}^\rho(M_1)$ and $\mathcal{R}^\rho(M_2)$ intersect transversally inside the manifold $\mathcal{R}^\rho(M_0)$ in a finite number of points.*

Proof. Let $\alpha,\beta:\pi_1\Sigma\to\mathrm{SU}(2)$ be irreducible representations such that $\sigma^*\alpha=\rho\alpha\rho^{-1}$ and $\sigma^*\beta=\rho\beta\rho^{-1}$. If they are conjugated, $\beta=h\alpha h^{-1}$, $h\in\mathrm{SU}(2)$, then $\sigma^*(h\alpha h^{-1})=\rho\cdot h\alpha h^{-1}\cdot\rho^{-1}$, and therefore,

$$\sigma^*\alpha=(h^{-1}\rho h)\cdot\alpha\cdot(h^{-1}\rho h)^{-1}=\rho\alpha\rho^{-1}.$$

Since α is irreducible, $\rho=\pm h\rho h^{-1}$, or $\rho h=\pm h\rho$. Hence,

$$\mathcal{R}(\Sigma(p,q,r))=\{\,\alpha:\pi_1\Sigma\to\mathrm{SU}(2)\mid \alpha\text{ is irreducible, } \sigma^*\alpha=\rho\alpha\rho^{-1}\,\}/\bar{S}^1_\rho,$$

where $\bar{S}^1_\rho$ is a subgroup of SU(2) consisting of all h such that $\rho h = \pm h\rho$. The group S^1_ρ is a subgroup of $\bar{S}^1_\rho$ of index 2. This proves the first part of the lemma.

Each of the manifolds $\mathcal{R}^\rho(M_k)$, $k = 0, 1, 2$, is the fixed point submanifold of the involution $\alpha \mapsto \mathrm{Ad}_\rho \, \sigma^* \alpha$ on the corresponding manifold $\mathcal{R}(M_k)$. The intersection of $\mathcal{R}(M_1)$ with $\mathcal{R}(M_2)$ in $\mathcal{R}(M_0)$ is transversal; see e.g. [13, Proposition 2.5], and $\mathcal{R}^\rho(M_1) \cap \mathcal{R}^\rho(M_2) = \mathcal{R}^\rho(\Sigma(p,q,r))$ is finite. q.e.d.

2.5 The definition of λ^ρ

Let $\Sigma = \Sigma(p,q,r)$ be a Brieskorn homology sphere. Choose a σ–invariant Heegaard splitting $\Sigma = M_1 \cup M_2$ along a Riemann surface $M_0 = M_1 \cap M_2$ as in (9). We define the *λ^ρ–invariant* of Σ as one-forth of the algebraic intersection number of $\mathcal{R}^\rho(M_1)$ and $\mathcal{R}^\rho(M_2)$ inside $\mathcal{R}^\rho(M_0)$, so that

$$\lambda^\rho(\Sigma) = \frac{1}{4} \cdot \sum_{\alpha \in \mathcal{R}^\rho(\Sigma)} \varepsilon_\alpha, \tag{17}$$

where ε_α equals ± 1 depending on whether the orientations on the spaces $T_\alpha \mathcal{R}^\rho(M_1) \oplus T_\alpha \mathcal{R}^\rho(M_2)$ and $T_\alpha \mathcal{R}^\rho(M_0)$ agree; see below. Due to (8) and the fact that $\mathcal{R}^\rho(\Sigma)$ double covers $\mathcal{R}(\Sigma)$, the number $\lambda^\rho(\Sigma)$ is an integer.

The manifolds $\mathcal{R}^\rho(M_0)$ and $\mathcal{R}^\rho(M_i)$, $i = 1, 2$, are oriented as follows. First, we fix an orientation on SU(2). There are two transversal submanifolds in SU(2),

$$S^1_\rho = \{\, g \in \mathrm{SU}(2) \mid g\rho = \rho g \,\} \quad \text{and} \quad S^2_\rho = \{\, g \in \mathrm{SU}(2) \mid g^{-1} = \rho g \rho^{-1} \,\}.$$

At $1 \in \mathrm{SU}(2)$, we obviously have the decomposition

$$T_1 \,\mathrm{SU}(2) = T_1 S^1_\rho \oplus T_1 S^2_\rho \tag{18}$$

of $T_1 \,\mathrm{SU}(2) = \mathfrak{su}(2)$ into the (± 1)–eigenspaces of the operator $\mathrm{Ad}_\rho : \mathfrak{su}(2) \to \mathfrak{su}(2)$. We orient S^1_ρ and S^2_ρ so that the orientations are consistent with the decomposition (18).

Let us now fix an orientation on M_0. This orientation orients the two boundary circles of M_0^*, γ and $\sigma(\gamma)$. As we have seen, a choice of basis in $\pi_1 M_0^*$ identifies $\mathrm{Hom}^\rho(\pi_1 M_0^*, \mathrm{SU}(2))$ with $(S^2_\rho)^{2g+1}$, which allows to orient $\mathrm{Hom}^\rho(\pi_1 M_0^*, \mathrm{SU}(2))$ as a product. This orientation is independent of the choice of a basis due to the fact that S^2_ρ is even–dimensional.

The orientations on γ and on $S^1_\rho \in \mathrm{SU}(2)$ orient the manifold $\mathcal{R}^\rho(M_0)$. Note that this orientation is independent of the initial choice of orientations on SU(2) and S^1_ρ, as soon as the orientation of S^2_ρ is consistent with those via (18). The orientation of $\mathcal{R}^\rho(M_0)$, however, depends on the orientation of M_0 through the induced orientation of γ; change the orientation of M_0, and the orientation of $\mathcal{R}^\rho(M_0)$ changes by (-1).

The manifolds $\mathcal{R}^\rho(M_1)$ and $\mathcal{R}^\rho(M_2)$ are oriented as submanifolds of $(S^2_\rho)^g/S^1_\rho$. Their orientations depend on the choices of SU(2) and S^1_ρ orientations, but the orientation of $T\mathcal{R}^\rho(M_1) \oplus T\mathcal{R}^\rho(M_2)$ at a point of intersection in $\mathcal{R}^\rho(M_0)$ is insensitive to these choices. Switching the roles of M_1 and M_2 in the sum $T\mathcal{R}^\rho(M_1) \oplus T\mathcal{R}^\rho(M_2)$ changes its orientation by (-1).

Thus, the signs ε_α in (17) are well–defined given the orientation of M_0 and the order of the handlebodies. Changing both choices leaves $\lambda^\rho(\Sigma)$ invariant, so it is only the induced orientation of Σ that must be specified to avoid an ambiguity. The sign of $\lambda^\rho(\Sigma)$ changes with the change of orientation of Σ.

The invariant λ^ρ may *a priori* depend on the choice of Heegaard splitting. In fact, it does not, and this will follow from Corollary 16, which expresses λ^ρ in terms of the knot signature.

3. The invariant λ^ρ and the knot signature

The signature of a knot in S^3 has been interpreted by X.-S. Lin in [21] as an intersection number of trace–free SU(2)–representation spaces associated with the knot complement. This number is usually referred to as the Casson–Lin invariant. In this section we use this interpretation to show that, under proper normalizations, the invariant λ^ρ of a Brieskorn homology sphere $\Sigma(p,q,r)$ equals the signature of the Montesinos knot $k(p,q,r)$.

3.1 Definition of the Casson–Lin invariant

Let B_n be the braid group of rank n with the standard generators $\beta_1, \dots, \beta_{n-1}$ represented in a free group F_n on symbols $x_1, \dots, x_n$ as follows:

$$\begin{aligned} \beta_i: \quad x_i &\mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} &\mapsto x_i, \\ x_j &\mapsto x_j, \quad \text{if} \quad j \neq i, i+1. \end{aligned}$$

If $\beta \in B_n$, then the automorphism of F_n representing β maps each x_i to a conjugate of some x_j and preserves the product $x_1 \cdots x_n$.

Let $k \subset S^3$ be a knot represented as the closure of a braid $\beta \in B_n$. Let us fix an embedding of k into S^3 as shown in Figure 2, the sphere S separating S^3 in two 3–balls, B_1 and B_2, with the braid β inside B_1 and n untangled arcs inside B_2. The fundamental group $\pi_1 K$ of the knot k complement $K = S^3 \setminus k$, has the presentation

$$\pi_1 K = \langle\, x_1, \dots, x_n \mid x_i = \beta(x_i),\ i = 1, \dots, n \,\rangle,$$

the generators $x_1, \dots, x_n$ being represented by the meridians of β.

The knot complement K can be now decomposed as $K = M_1' \cup_{M_0'} M_2'$ where $M_0' = S \cap K$ and $M_k' = B_k \cap K$, $k = 1, 2$. The manifolds M_1' and M_2' are handlebodies of genus $n = g + 1$, and M_0' is a 2–sphere with $2g + 2$ punctures at the points $P_1, \dots, P_{2g+2}$; see Figure 2. Due to Seifert–Van Kampen Theorem, we get the following commutative diagram of fundamental groups

$$\begin{array}{ccc}
\pi_1 K & \longleftarrow & \pi_1 M_1' \\
\uparrow & & \uparrow \\
\pi_1 M_2' & \longleftarrow & \pi_1 M_0'
\end{array}$$

where

$$\pi_1 M_0' = \langle\, x_1, \dots, x_{2g+2} \,|\, x_1 \cdots x_{2g+2} = 1 \,\rangle,$$
$$\pi_1 M_1' = \langle\, x_1, \dots, x_{g+1} \,|\quad \rangle, \quad \pi_1 M_2' = \langle\, x_{g+2}, \dots, x_{2g+2} \,|\quad \rangle$$

are all isomorphic to free groups, and the generators $x_1, \dots, x_{2g+2}$ of $\pi_1 M_0'$ are represented by the loops in Figure 5.

In complete analogy with the definition of the Casson invariant, we will define SU(2)–representation spaces of the groups $\pi_1 K$ and $\pi_1 M_k'$, $k = 0, 1, 2$, and compute the corresponding intersection number. To make this program work, we impose the extra condition on the representations that all the meridians $x_1, \dots, x_{2g+2}$ go to trace-free matrices in SU(2). Thus the representation spaces in question are

$$\mathcal{R}_0(K) = \{\, \alpha : \pi_1 K \to \mathrm{SU}(2) \quad |\ \alpha \text{ is irreducible, } \operatorname{tr} \alpha(x_i) = 0 \,\} / \,\mathrm{SU}(2),$$
$$\mathcal{R}_0(M_k') = \{\, \alpha : \pi_1 M_k' \to \mathrm{SU}(2) \ |\ \alpha \text{ is irreducible, } \operatorname{tr} \alpha(x_i) = 0 \,\} / \,\mathrm{SU}(2),$$

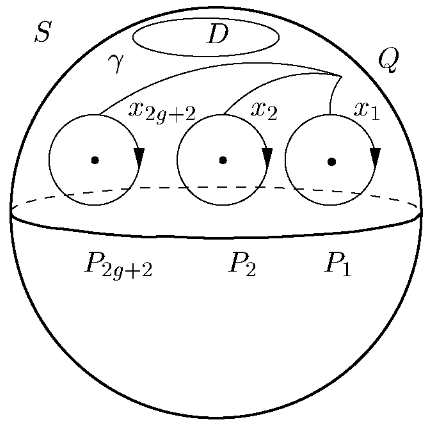

FIGURE 5

where $k = 0, 1, 2$. The commutative diagram of the fundamental groups above induces the following commutative diagram of inclusions of representation spaces,

$$\begin{array}{ccc} \mathcal{R}_0(K) & \longrightarrow & \mathcal{R}_0(M_1') \\ \downarrow & & \downarrow \\ \mathcal{R}_0(M_2') & \longrightarrow & \mathcal{R}_0(M_0') \end{array}$$

In particular, the irreducible trace-free representations of $\pi_1 K$ in SU(2) are in one-to-one correspondence with the intersection points of $\mathcal{R}_0(M_1')$ with $\mathcal{R}_0(M_2')$ in $\mathcal{R}_0(M_0')$. X.-S. Lin showed in [21] that $\mathcal{R}_0(M_k')$, $k = 0, 1, 2$, are smooth (open) manifolds of dimensions $\dim \mathcal{R}_0(M_0') = 4g - 2$ and $\dim \mathcal{R}_0(M_1') = \dim \mathcal{R}_0(M_2') = 2g - 1$, and that the intersection $\mathcal{R}_0(K) = \mathcal{R}_0(M_1') \cap \mathcal{R}_0(M_2')$ in $\mathcal{R}_0(M_0')$ is compact. As we will see later, this intersection is finite if the knot k is a Montesinos knot. The *Casson–Lin invariant* for $k = k(p, q, r)$ is now defined as

$$h(k) = \sum_{\alpha \in \mathcal{R}_0(K)} \varepsilon'_\alpha, \tag{19}$$

where $\varepsilon'_\alpha = \pm 1$ is a sign obtained by comparing the orientations (see below) on $T_\alpha \mathcal{R}_0(M_1') \oplus T_\alpha \mathcal{R}_0(M_2')$ and $T_\alpha \mathcal{R}_0(M_0')$. For a general knot, the number (19) is well defined after some compact perturbations.

The orientations of $\mathcal{R}(M_k')$, $k = 0, 1, 2$, are defined as follows. Let SU(2) be oriented by the standard basis i, j, k in its Lie algebra $\mathfrak{su}(2)$. The submanifold of SU(2) consisting of the trace-free matrices is a 2–dimensional sphere, which we denote by $S_0^2 \subset \mathrm{SU}(2)$. It is naturally oriented as $S_0^2 = \exp(S(0, \frac{\pi}{2}))$ where $S(0, \frac{\pi}{2})$ is the 2–sphere in $\mathfrak{su}(2)$ centered at 0 and of radius $\frac{\pi}{2}$ in the metric

$$\langle u, v\rangle = \frac{1}{2}\operatorname{tr}(u\bar{v}^t), \quad u, v \in \mathfrak{su}(2).$$

Let us introduce the manifold $M_0' \setminus D$ where D is an open disc in M_0' away from the points $P_1, \dots, P_{2g+2}$; see Figure 5. An orientation of M_0' orients the boundary circle, γ, of $M_0' \setminus D$. The choice of generators $x_1, \dots, x_{2g+2}$ in $\pi_1 M_0'$ identifies the representation space of $M_0' \setminus D$, $\{\, \alpha : \pi_1 M_0' \to \mathrm{SU}(2) \mid \operatorname{tr}\alpha(x_i) = 0 \,\}$, with the product $(S_0^2)^{2g+2}$. The manifold $\mathcal{R}_0(M_0')$ is then identified with an SU(2)–quotient of the open set of irreducible representations in $\theta^{-1}(1)$ where

$$\theta : (S_0^2)^{2g+2} \to \mathrm{SU}(2)$$

is defined by parallel transport around γ. The differential $d\theta$ of the map θ is onto at all irreducible representations α such that $\theta(\alpha) = 1$; see [21, Lemma 1.5]. This orients $\mathcal{R}_0(M_0')$ so that the natural isomorphism

$$T_{X_1} S_0^2 \oplus \dots \oplus T_{X_{2g+2}} S_0^2 = T_1\,\mathrm{SU}(2) \oplus T_{(X_1, \dots, X_{2g+2})}\mathcal{R}_0(M_0') \oplus T_1\,\mathrm{SU}(2)$$

(with the second $T_1\,\mathrm{SU}(2)$–factor corresponding to the SU(2)–action) is orientation preserving. Note that the orientation of $\mathcal{R}_0(M_0')$ changes by (-1) with the change of orientation on M_0' (through the orientation of γ).

The manifolds $\mathcal{R}_0(M_1')$ and $\mathcal{R}_0(M_2')$ are oriented as SU(2)–quotients of open subsets in $(S_0^2)^{g+1}$, and their orientations are well defined as soon as the orientations of SU(2) and S_0^2 are fixed as above.

The switch of M_1 and M_2 changes the orientation of $T_\alpha\mathcal{R}_0(M_1') \oplus T_\alpha\mathcal{R}_0(M_2')$. Thus the sign ε_α' in (19) is well defined given the orientation of M_0' and the specification of the handlebody to call M_1'. Changing both choices does not change $h(k)$, therefore, we only need to fix an orientation of K to avoid an ambiguity. The sign of $h(k)$ changes by (-1) if the orientation of K is changed.

The next result follows from Corollary 2.10 of [21].

Proposition 12. *For a knot $k = k(p, q, r)$, $h(k) = \frac{\varepsilon}{2}\operatorname{sign} k$, where $\varepsilon = \pm 1$ is a universal constant independent of k.*

3.2 Some equivariant gauge theory

Let $\Sigma = \Sigma(p,q,r)$ be a Brieskorn homology sphere endowed with the involution (7) and a σ–invariant Heegaard splitting $\Sigma = M_1 \cup_{M_0} M_2$.

Let $E \to \Sigma$ be a (necessarily trivial) SU(2)–vector bundle over Σ with a fixed trivialization. Let us fix a σ–invariant Riemannian metric on Σ and consider the space $\mathcal{A}$ of smooth connections on E. The choice of trivialization gives an isomorphism of $\mathcal{A}$ with $\Omega^1(\Sigma, \mathfrak{su}(2))$, the linear space of smooth differential 1–forms on Σ with $\mathfrak{su}(2)$–coefficients. We use the L^2_1–inner product on $\Omega^1(\Sigma, \mathfrak{su}(2))$ to define $\mathcal{A}$ as a smooth manifold modeled on a pre–Hilbert space; see [29].

The bundle E restricts to trivial SU(2)–bundles E_k over M_k, $k = 0,1,2$. The metric on Σ induces σ–invariant Riemannian metrics on M_k, and we denote by $\mathcal{A}_k$ the corresponding space of smooth connections on E_k, $k = 0,1,2$. The spaces $\mathcal{A}_k$ are defined as smooth manifolds modeled on pre–Hilbert spaces $\Omega^1(M_k, \mathfrak{su}(2))$ with the L^2_1–inner product.

The group $\mathcal{G} = C^\infty(\Sigma, \mathrm{SU}(2))$ is called the gauge group; it acts on $\mathcal{A}$ as $(g, A) \mapsto g \cdot dg^{-1} + g \cdot A \cdot g^{-1}$. Let $\mathcal{B} = \mathcal{A}/\mathcal{G}$ be the quotient space. Set

$$\mathcal{B}^* = \mathcal{A}^*/\mathcal{G},$$

where $\mathcal{A}^* \subset \mathcal{A}$ is the subspace of irreducible connections A on which $\mathcal{G}$ acts with the stabilizer ± 1. It is the complement of a space of infinite codimension in $\mathcal{A}$. Then $\mathcal{B}^*$ is an infinite dimensional manifold modeled on a pre–Hilbert space using the L^2_1–theory.

The projection $\mathcal{A}^* \to \mathcal{B}^*$ can be shown to be a principal $\mathcal{G}$–bundle, so that $\pi_1(\mathcal{B}^*) = \pi_0(\mathcal{G}) = \pi_0\, C^\infty(\Sigma, \mathrm{SU}(2)) = [\,\Sigma, \mathrm{SU}(2)\,] = \mathbb{Z}$, the last isomorphism given by the mapping degree.

For every $k = 0,1,2$, the gauge group $\mathcal{G}_k = C^\infty(M_k, \mathrm{SU}(2))$ acts on $\mathcal{A}_k$, and we define $\mathcal{B}^*_k$ as the $\mathcal{G}_k$–quotient of the subspace $\mathcal{A}^*_k \subset \mathcal{A}_k$ of irreducible connections. Each of the spaces $\mathcal{B}^*_k$ can be endowed with the structure of an infinite dimensional simply–connected manifold.

It is a classical result in differential geometry that the holonomy map establishes a one–to–one correspondence between the gauge equivalence classes of irreducible flat connections in a bundle and the conjugacy classes of irreducible representations of the fundamental group of its base.

For every $k = \emptyset, 0, 1, 2$, the involution σ can be lifted to a bundle endomorphism of E_k. Any endomorphism of E_k clearly induces an action on $\mathcal{A}^*_k$ by pull-back, and an action on $\mathcal{B}^*_k$ as well. Since any two liftings of σ differ by a gauge transformation, we have a well-defined action $\sigma^*:\mathcal{B}^*_k\to\mathcal{B}^*_k$. Denote by $\bar{\mathcal{B}}^\sigma_k$ the manifold consisting of connections invariant with respect to σ^*.

Let $\rho\in\mathrm{SU}(2)$ be such that $\rho^2=-1$. The formula

$$(x,\xi)\mapsto(\sigma(x),\rho\cdot\xi) \tag{20}$$

defines a lifting of $\sigma:\Sigma\to\Sigma$ on E_k, which will again be denoted by $\rho:E\to E$. Let $\bar{\mathcal{B}}^\rho_k\subset\bar{\mathcal{B}}^\sigma_k$ be the manifold consisting of the gauge equivalence classes of irreducible connections A in E_k such that $\rho^*A=A$. For instance, Proposition 8 implies that all irreducible flat connections on Σ belong to $\bar{\mathcal{B}}^\rho$. The following lemma can be easily checked; compare with Proposition 1 of [37].

Lemma 13. *Let $k=\emptyset,0,1,2$. For any $\rho,\rho'\in\mathrm{SU}(2)$ such that $\rho^2=\rho'^2=-1$, $\bar{\mathcal{B}}^\rho_k=\bar{\mathcal{B}}^{\rho'}_k$. Furthermore, $\bar{\mathcal{B}}^\rho_k$ is bijective to $\mathcal{A}^\rho_k/\bar{\mathcal{G}}^\rho_k$ where $\bar{\mathcal{G}}_k=\mathcal{G}_k/\pm1$ and $\mathcal{A}^\rho_k=\{\,A\in\mathcal{A}^*_k\mid\rho^*A=A\,\}$.*

With the inner L^2_1–product, $\mathcal{A}^\rho_k$ is a smooth manifold whose tangent space at any point A is identified with the pre–Hilbert space of smooth 1–forms $\omega\in\Omega^1(\Sigma,\mathfrak{su}(2))$, respectively, $\omega\in\Omega^1(M_k,\mathfrak{su}(2))$ if $k=0,1,2$, such that $\omega(\sigma(x))=\mathrm{Ad}_\rho\,\omega(x)$ $(=\rho\omega(x)\rho^{-1})$ for all $x\in\Sigma$, respectively, $x\in M_k$. The gauge group $\bar{\mathcal{G}}^\rho_k$ consists of all smooth maps $g:\Sigma\to\mathrm{SU}(2)$ such that $g(\sigma(x))\rho=\pm\rho g(x)$, the maps g and $-g$ being identified.

It is more convenient for us to work with the double covers

$$\mathcal{B}^\rho_k=\mathcal{A}^\rho_k/\mathcal{G}^\rho_k \tag{21}$$

of the manifolds $\bar{\mathcal{B}}^\rho_k$, $k=\emptyset,0,1,2$, where the gauge groups $\mathcal{G}^\rho_k$ consist of all the smooth maps $g:\Sigma\to\mathrm{SU}(2)$, respectively, $g:M_k\to\mathrm{SU}(2)$, such that $g\rho=\rho g$. Again, $\mathcal{A}^\rho_k\to\mathcal{B}^\rho_k$, $k=\emptyset,0,1,2$, is a principal $\mathcal{G}^\rho_k$–bundle, hence,

$$\pi_1\mathcal{B}^\rho=\pi_0\mathcal{G}^\rho=[\,\Sigma,\mathrm{SU}(2)\,]_{\mathbb{Z}/2},$$
$$\pi_1\mathcal{B}^\rho_k=\pi_0\mathcal{G}^\rho_k=[\,M_k,\mathrm{SU}(2)\,]_{\mathbb{Z}/2},$$

where $[\ ,\ ]_{\mathbb{Z}/2}$ stands for the group of $\mathbb{Z}/2$–equivariant homotopy classes. The equivariant obstruction theory of [5] can be used to show that $\pi_1\mathcal{B}^\rho=\mathbb{Z}\oplus\mathbb{Z}$, and that the manifolds $\mathcal{B}^\rho_k$ are simply–connected for $k=0,1,2$.

3.3 Representations of the knot $k(p,q,r)$ complement and the λ^ρ–invariant

Let $\Sigma = \Sigma(p,q,r)$ be a Brieskorn homology sphere with a σ–invariant Heegaard splitting $\Sigma = M_1 \cup_{M_0} M_2$ as in (9). Let K be the knot $k(p,q,r)$ complement in S^3; then $K = M_1' \cup_{M_0'} M_2'$ where

$$M_1' = B_1 \cap K, \quad M_2' = B_2 \cap K, \quad M_0' = S \cap K. \tag{22}$$

Let us fix orientations on Σ, K and M_0, M_0' so that the projection π is orientation preserving.

Let $E \to \Sigma$ be a trivialized SU(2)–bundle over Σ. The quotient of Σ by the involution σ is S^3. Since $\rho \neq \pm 1$, it is impossible to define the quotient bundle of E over S^3. However, one can define it away from the knot $k \subset S^3$, that is, on the knot complement K. Given an irreducible equivariant flat connection $A \in \mathcal{A}^\rho$ in E, its push–down A' is an irreducible SU(2)–connection over K. In other words, A' is an irreducible flat SU(2)–connection singular along k in the sense of P. Kronheimer and T. Mrowka [19]. The next result follows from Proposition 17 of [37].

Lemma 14. *The flat SU(2)–connection A' has holonomy $1/4$ around k, i.e., the holonomy is trace free.*

Let E_k be the restriction of the bundle E to the submanifold M_k, $k = 0,1,2$. Given an irreducible flat connection in E_k, its push–down is a singular irreducible flat connection over M_k' with trace–free holonomy.

Proposition 15. *The push–down of flat connections induces orientation preserving diffeomorphisms $\mathcal{R}^\rho(M_k) \to \mathcal{R}_0(M_k')$, $k = 0,1,2$, and $\mathcal{R}^\rho(\Sigma) \to \mathcal{R}_0(K)$, which commute with the embeddings of representation spaces.*

Corollary 16. *There exists a universal constant $\varepsilon = \pm 1$ such that, for any Brieskorn homology sphere $\Sigma(p,q,r)$ and the corresponding Montesinos knot $k(p,q,r)$,*

$$\lambda^\rho(\Sigma(p,q,r)) = \frac{1}{4} h(k(p,q,r)) = \frac{\varepsilon}{8} \operatorname{sign} k(p,q,r).$$

Proof of Proposition 15. The push–down of flat connections defines a map of representation spaces, $\phi_0 : \mathcal{R}^\rho(M_0) \to \mathcal{R}_0(M_0')$, as follows. Let $\alpha : \pi_1 M_0 \to \mathrm{SU}(2)$ be the holonomy representation of a flat connection A over M_0, $\alpha = \mathrm{hol}_A$. For every loop x in M_0', we define $\phi_0(\alpha)(x) =$

$\mathrm{hol}_A(\tilde{x})$ where $\tilde{x}$ is a lift of the loop x to M_0. Note that the curve $\tilde{x}$ is not necessarily closed.

With respect to the generators $x_1, \dots, x_{2g+2}$ and $t_1, \dots, t_{2g+1}$ of the groups $\pi_1 M_0'$ and $\pi_1 M_0$, respectively; see Figures 3, 4 and 5, the map ϕ_0 can be given by the following formula (remember that we used the left multiplication by ρ in (20) to identify the fibers over the pairs of points P and $\sigma(P)$)

$$\begin{aligned}
\phi_0(\alpha)(x_1) &= \rho, \\
\phi_0(\alpha)(x_2) &= \alpha(t_1)\rho^{-1}, \\
\phi_0(\alpha)(x_3) &= \rho\alpha(t_2), \\
&\dots\dots \\
\phi_0(\alpha)(x_{2g+2}) &= \alpha(t_{2g+1})\rho^{-1}.
\end{aligned} \tag{23}$$

If we think about $\mathcal{R}^\rho(M_0)$ and $\mathcal{R}_0(M_0')$ as the following smooth manifolds,

$$\mathcal{R}^\rho(M_0) = \{\, (T_1, \dots, T_{2g+1}) \mid (T_1, \dots, T_{2g+1}) \text{ is irreducible}, \\ T_i \in S^2_\rho,\ T_1 \cdots T_{2g+1} = 1 \,\} / \, S^1_\rho \quad \text{and}$$

$$\mathcal{R}_0(M_0') = \{\, (X_1, \dots, X_{2g+2}) \mid (X_1, \dots, X_{2g+2}) \text{ is irreducible}, \\ X_i \in S^2_0,\ X_1 \cdots X_{2g+2} = 1 \,\} / \ \mathrm{SU}(2),$$

the map ϕ_0 is given by the formula

$$\phi_0(T_1, \dots, T_{2g+1}) = (\rho, T_1\rho^{-1}, \rho T_2, \dots, T_{2g+1}\rho^{-1}).$$

The map ϕ_0 is obviously smooth. To check its injectivity we suppose that there exists $h \in \mathrm{SU}(2)$ such that

$$\begin{aligned}
\rho &= h\rho h^{-1}, \\
T_1'\rho^{-1} &= hT_1\rho^{-1}h^{-1}, \\
\rho T_2' &= h\rho T_2 h^{-1}, \\
&\dots\dots \\
T_{2g+1}'\rho^{-1} &= hT_{2g+1}\rho^{-1}h^{-1}
\end{aligned}$$

for some $(T_1, \dots, T_{2g+1})$ and $(T_1', \dots, T_{2g+1}')$ in $\mathcal{R}^\rho(M_0)$. Then the first equation implies that $h\rho = \rho h$, and the remaining equations assure that $T_i' = hT_ih^{-1},\ i = 1, \dots, 2g+1$, with $h \in S^1_\rho$.

To show that ϕ_0 is surjective we take an arbitrary $(2g+2)$–tuple in $\mathcal{R}_0(M_0')$, conjugate it to a $(2g+2)$–tuple of the form $(\rho, X_2, \dots, X_{2g+2})$, and define

$$(T_1, T_2, \dots, T_{2g+1}) = (X_2\rho, \rho^{-1}X_3, \dots, X_{2g+2}\rho).$$

Obviously, $\phi_0(T_1, \dots, T_{2g+1}) = (\rho, X_2, \dots, X_{2g+2})$.

Next we need to check that the map ϕ_0 is orientation preserving. We orient $\mathrm{SU}(2)$ and S_0^2 as in the definition of the Casson–Lin invariant. As we have seen in Section 2.5, the orientation of $\mathcal{R}^\rho(M_0)$ is independent of the orientation choices for S_ρ^2 and $\mathrm{SU}(2)$, as soon as the orientation on S_ρ^1 is consistent in the sense that the natural isomorphism

$$T_1\,\mathrm{SU}(2) = T_1 S_\rho^1 \oplus T_1 S_\rho^2 \tag{24}$$

is orientation preserving. We use the already fixed orientation on $\mathrm{SU}(2)$, and orient S_ρ^1 in such a way that the left multiplication by ρ,

$$S_\rho^2 \to S_0^2, \quad g \mapsto \rho g,$$

is orientation preserving. Then the right multiplication by ρ^{-1} is also orientation preserving.

In our construction of the map ϕ_0 we used the identification $S_0^2 = \mathrm{SU}(2)/S_\rho^1$ where S_ρ^2 is the orbit space of the adjoint $\mathrm{SU}(2)$–action on itself with S_ρ^1 the stabilizer at ρ. The natural isomorphism

$$T_\rho\,\mathrm{SU}(2) = T_1 S_\rho^1 \oplus T_\rho S_0^2 \tag{25}$$

orients S_ρ^1, and this orientation is consistent with the one obtained from (24) because the left multiplication by ρ is orientation preserving on both $\mathrm{SU}(2)$ and S_ρ^2.

The orientations of both $\mathcal{R}_0(M_0')$ and $\mathcal{R}^\rho(M_0)$ included a choice of orientations of manifolds M_0' and M_0, respectively. The latter two are oriented consistently by the requirement that the projection π be orientation preserving. With all the choices made, the map ϕ_0 is an orientation preserving diffeomorphism.

The maps $\phi_k : \mathcal{R}^\rho(M_k) \to \mathcal{R}_0(M_k')$, $k = 0, 1, 2$, are defined after choosing suitable bases in the free groups $\pi_1 M_k'$ and $\pi_1 M_k$, by the formulas similar to (23). Both ϕ_1 and ϕ_2 are orientation preserving diffeomorphisms with the above choices of the orientations.

The maps $\phi_k,\ k = 0, 1, 2$, commute with the embeddings of representation spaces due to the naturality of the construction. In particular, the sets of intersection points,

$$\mathcal{R}^\rho(\Sigma) = \mathcal{R}^\rho(M_1) \cap \mathcal{R}^\rho(M_2) \quad \text{and} \quad \mathcal{R}_0(K) = \mathcal{R}_0(M_1') \cap \mathcal{R}_0(M_2')$$

are in one–to-one correspondence. With the order of handlebodies (M_1, M_2) and (M_1', M_2') fixed by (9) and (22), the conclusions of both Proposition 15 and Corollary 16 follow. q.e.d.

4. The invariant χ^ρ

We use gauge theory to define an invariant χ^ρ for Brieskorn homology spheres as an infinite dimensional equivariant Euler characteristic. In Section 5 we will express χ^ρ in terms of the Floer homology, and in Section 6 will prove that, under proper normalizations, $\chi^\rho = 2 \cdot \lambda^\rho$.

4.1 Equivariant gauge theory on $\Sigma(p, q, r)$

Let $\mathcal{B}^\rho = \mathcal{A}^\rho/\mathcal{G}^\rho$ be the manifold defined in (21). Remember that Σ is endowed with a σ–invariant Riemannian metric. The assignment to an $\mathfrak{su}(2)$–valued smooth differential p–form ω of the form $\rho^*\omega = \mathrm{Ad}_\rho\,\sigma^*\omega$ defines an involution on the space $\Omega^p(\Sigma, \mathfrak{su}(2))$. This involution splits $\Omega^p(\Sigma, \mathfrak{su}(2))$ in the direct sum of its (± 1)–eigenspaces. Denote the $(+1)$–eigenspace by $\Omega^p_\rho(\Sigma, \mathfrak{su}(2))$ so that

$$\Omega^p_\rho(\Sigma, \mathfrak{su}(2)) = \{\, \omega \in \Omega^p(\Sigma, \mathfrak{su}(2)) \mid \sigma^*\omega = \rho\omega\rho^{-1} \,\}. \tag{26}$$

If $A \in \mathcal{A}^\rho$ is a ρ–invariant flat connection, the covariant differentiation d_A commutes with the splitting (26). Therefore, we have a splitting in de Rham cohomology. In particular, for any p,

$$H^p_\rho(\Sigma, d_A) \ \subset \ H^p(\Sigma, d_A),$$

where $H^p_\rho(\Sigma, d_A)$ is the p–th equivariant cohomology group whose elements are canonically represented by ρ–invariant harmonic forms $\omega \in \Omega^p_\rho(\Sigma, \mathfrak{su}(2))$.

A 1-form on $\mathcal{A}^\rho$ assigns to a connection A a homomorphism from $T_A\mathcal{A}^\rho = \Omega^1_\rho(\Sigma, \mathfrak{su}(2))$ into the real numbers. Define such a homomorphism by the formula

$$a \in \Omega^1_\rho(\Sigma, \mathfrak{su}(2)) \ \mapsto \mathfrak{f}_A(a) = \int_\Sigma \mathrm{tr}(a \wedge F_A),$$

where $F_A \in \Omega^2_\rho(\Sigma, \mathfrak{su}(2))$ is the curvature of the connection A. Due to the Bianchi identity, $\mathfrak{f}_A$ annihilates the tangent space to the $\mathcal{G}^\rho$–orbit through A,

$$T_A(\mathcal{G}^\rho \cdot A) = \{\, d_A\psi = d\psi + [A,\psi] \,|\, \psi \in \Omega^0_\rho(\Sigma, \mathfrak{su}(2)) \,\}. \tag{27}$$

Since $\mathfrak{f}$ is $\mathcal{G}^\rho$–equivariant, it can be thought of as being the pull-back of a 1–form $\mathfrak{f}$ on $\mathcal{B}^\rho$. The zeroes of $\mathfrak{f}$ on $\mathcal{B}^\rho$ are precisely the $\mathcal{G}^\rho$–orbits of the ρ–invariant flat connections.

Let $*$ denote the Hodge star operator associated with the chosen σ–invariant Riemannian metric on Σ, and

$$* : \Omega^p_\rho(\Sigma, \mathfrak{su}(2)) \to \Omega^{3-p}_\rho(\Sigma, \mathfrak{su}(2)) \tag{28}$$

its restriction to the ρ–invariant forms. The restriction (28) is well–defined due to the fact that σ is an orientation preserving isometric involution. The Riemannian metric on Σ defines an L^2–inner product on $\Omega^p_\rho(\Sigma, \mathfrak{su}(2))$ by the formula

$$\langle a, b\rangle = -\int_\Sigma \operatorname{tr}(a \wedge *b).$$

With respect to this metric, the formal adjoint d_A^* to the restriction of the operator d_A on the ρ–invariant forms is given by the usual formula, $d_A^* = - * d_A * : \Omega^p_\rho(\Sigma, \mathfrak{su}(2)) \to \Omega^{p-1}_\rho(\Sigma, \mathfrak{su}(2))$. Thus the tangent space to $\mathcal{B}^\rho$ at an orbit $[A]$, which is isomorphic to the orthogonal complement of $T_A(\mathcal{G}^\rho \cdot A)$ in $\Omega^1_\rho(\Sigma, \mathfrak{su}(2))$, is simply the vector space

$$\mathcal{T}_A = \{\, \omega \in \Omega^1_\rho(\Sigma, \mathfrak{su}(2)) \,|\, d_A^* a = - * d_A * = 0 \,\}.$$

If $A \in \mathcal{A}^\rho$ is a flat connection, the operator

$$*d_A : \mathcal{T}_A \to \mathcal{T}_A$$

is well–defined. In general, we compose the operator $*d_A$ with the orthogonal projection onto $\mathcal{T}_A$ to get the operator

$$\begin{aligned} \nabla \mathfrak{f}_A &: \mathcal{T}_A \to \mathcal{T}_A, \\ a &\mapsto *d_A a - d_A u(a), \end{aligned} \tag{29}$$

where $u(a)$ is the unique solution in $\Omega^1_\rho(\Sigma, \mathfrak{su}(2))$ of the equation

$$*d_A * d_A u(a) = *(F_A \wedge a - a \wedge F_A). \tag{30}$$

One easily checks that $u(a) \in \Omega^0_\rho(\Sigma, \mathfrak{su}(2))$; namely, if we plug the form $\operatorname{Ad}_\rho \sigma^* u(a)$ in the equation (30), we will get

$$\begin{aligned} * \, d_A * d_A(\operatorname{Ad}_\rho \sigma^* u(a)) &= \operatorname{Ad}_\rho \sigma^*(*d_A * d_A u(a)) \\ &= \operatorname{Ad}_\rho \sigma^*(*(F_A \wedge a - a \wedge F_A)) = *(F_A \wedge a - a \wedge F_A). \end{aligned}$$

Since the equation (30) has only one solution, we must have $\sigma^* u(a) = \operatorname{Ad}_\rho u(a)$.

The operator $\nabla \mathfrak{f}_A : \mathcal{T}_A \to \mathcal{T}_A$ is an elliptic operator on the closed manifold Σ. It follows from the standard elliptic theory that the L^2–completion of $\nabla \mathfrak{f}_A$ is a selfadjoint Fredholm operator whose domain is the L^2_1–Sobolev completion of $\mathcal{T}_A$. Its eigenvalues form a discrete subset of the real line which has no accumulation points, and which is unbounded in both directions. Each eigenvalue has finite multiplicity, compare with Lemma 1.1 of [36].

A non-degenerate zero of the 1–form $\mathfrak{f}$ on $\mathcal{B}^\rho$ is, by definition, the orbit of a flat connection in $\mathcal{A}^\rho$ for which $\ker \nabla \mathfrak{f}_A = 0$.

Lemma 17. *All zeroes of $\mathfrak{f}$ on $\mathcal{B}^\rho$ are non–degenerate.*

Proof. The statement follows from the observation that $\ker \nabla \mathfrak{f}_A = H^1_\rho(\Sigma; \operatorname{ad} A) \subset H^1(\Sigma; \operatorname{ad} A)$ for any flat $A \in \mathcal{A}^\rho$ and the fact that $H^1(\Sigma; \operatorname{ad} A) = 0$ for $\Sigma = \Sigma(p,q,r)$ and any irreducible flat connection A; see [13, Proposition 2.5]. q.e.d.

4.2 Definition of χ^ρ

The assignment to an orbit $[A] \in \mathcal{B}^\rho$ of the operator $\nabla \mathfrak{f}_A$ on the L^2–completion of $\mathcal{T}_A$ defines a smooth map from $\mathcal{B}^\rho$ into the Banach space of real, selfadjoint operators on a separable Hilbert space. Let $[A_0]$ and $[A_1]$ in $\mathcal{B}^\rho$ be (non–degenerate) zeroes of $\mathfrak{f}$. Consider a continuously differentiable path of operators $\nabla \mathfrak{f}_{A(t)}$ with $[A(0)] = [A_0]$ and $[A(1)] = [A_1]$. The one–parameter family of spectra of operators $\nabla \mathfrak{f}_{A(t)}$ can be thought of as a collection of spectral curves in the plane (see Figure 6) connecting the spectrum of $\nabla \mathfrak{f}_{A_0}$ to the spectrum of $\nabla \mathfrak{f}_{A_1}$. These curves are continuously differentiable functions of t, at least near zero. The number of eigenvalues which cross from "minus" to "plus" minus the number which cross from "plus" to "minus" is well–defined and finite along a generic path. This number is called the *spectral flow* of the family along the path; see [3].

The spectral flow only depends on the homotopy class rel $\{0, 1\}$ of the path $t \to [A(t)]$. Therefore, it defines a locally constant function on

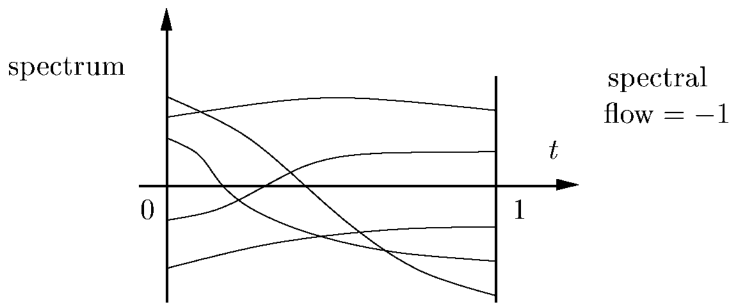

FIGURE 6

the space of continuous paths between $[A_0]$ and $[A_1]$. We denote this function by $sf^{\rho}([A_0],[A_1])$.

Lemma 18. *The integer $sf^{\rho}([A_0],[A_1])$ is independent of the homotopy class of the path between $[A_0]$ and $[A_1]$ modulo 4.*

Proof. Let us consider a path $A(t)$ in $\mathcal{A}^{\rho}$ connecting A_0 to a gauge equivalent of itself. It can be thought of as a connection in an SU(2)–bundle E (not necessarily trivial) on the 4–manifold $\Sigma \times S^1$, and then the spectral flow along the path $A(t)$ will be equal to the index of the restricted self–duality operator,

$$\mathcal{D}^{\rho}_A = d^*_A + d^-_A : \Omega^1_{\rho}(\Sigma \times S^1, \mathfrak{su}(2)) \to (\Omega^0 \oplus \Omega^2_-)_{\rho}(\Sigma \times S^1, \mathfrak{su}(2)).$$

The involution σ extends to $\Sigma \times S^1$ as the product involution $\sigma \times 1$ with the quotient space $S^3 \times S^1$ and the fixed point set $B' \subset S^3 \times S^1$ an embedded torus with the trivial normal bundle. According to [37, Theorem 18],

$$\operatorname{index} \mathcal{D}^{\rho}_A = 4c_2(E) - \frac{3}{2}(\chi - \tau)(S^3 \times S^1) + \chi(B') + \frac{1}{2}\, B'.\, B',$$

where χ stands for Euler characteristic, τ for signature (the sign in $(\chi - \tau)$ is different from that in [37] because we work with self-dual rather than anti-self-dual connections). Therefore, $\operatorname{index} \mathcal{D}^{\rho}_A = 0 \mod 4$.

q.e.d.

Now we are in a position to define the invariant $\chi^{\rho}(\Sigma)$ up to a $(\pm)$–sign. Pick $\alpha \in \mathcal{R}(\Sigma)$ and define

$$\chi^{\rho}(\Sigma) = \varepsilon_{\alpha} \cdot \sum_{\beta \in \mathcal{R}(\Sigma)} (-1)^{sf^{\rho}(\alpha,\beta)}, \quad \varepsilon_{\alpha} = \pm 1. \tag{31}$$

To define an overall sign of $\chi^\rho(\Sigma)$, it is sufficient to associate a sign to one particular flat connection orbit, $\alpha \in \mathcal{R}(\Sigma)$. In principle, we would like to take a path from the trivial connection θ to the connection α and set $\varepsilon_\alpha = sf^\rho(\theta, \alpha)$. The latter though is ill defined because θ is reducible, hence, is a degenerate zero of $\mathfrak{f}$. In what follows we describe a way around this problem.

To begin with, we replace the operator $\nabla\mathfrak{f}_A : \mathcal{T}_A \to \mathcal{T}_A$ by the operator

$$K_A^\rho = \begin{vmatrix} 0 & d_A^* \\ d_A & *d_A \end{vmatrix} : (\Omega^0 \oplus \Omega^1)_\rho (\Sigma, \mathfrak{su}(2)) \to (\Omega^0 \oplus \Omega^1)_\rho (\Sigma, \mathfrak{su}(2)). \tag{32}$$

We will use K_A^ρ to define the spectral flow $sf^\rho(\theta, \alpha)$. The advantage of K_A^ρ over $\nabla\mathfrak{f}_A$ is that the operator K_A^ρ and its spectrum depend continuously on A even if A is reducible. For any A, the operator K_A^ρ is an elliptic operator on Σ. Its L^2–completion is a self–adjoint Fredholm operator. It has pure point real spectrum without accumulation points, which is unbounded in both directions.

The operators $\nabla\mathfrak{f}_A$ and K_A^ρ are related as follows. At any irreducible connection $A \in \mathcal{A}^\rho$, the principal $\mathcal{G}^\rho$–bundle $\pi : \mathcal{A}^\rho \to \mathcal{B}^\rho$ induces the canonical exact sequence

$$0 \to T_1\mathcal{G}^\rho \xrightarrow{d} T_A\mathcal{A}^\rho \to \pi^* T_A\mathcal{B}^\rho \to 0.$$

The operator $d_A^* : T_A\mathcal{A}^\rho \to T_1\mathcal{G}^\rho$ provides a $\mathcal{G}^\rho$–equivariant splitting of this sequence, so that $\nabla\mathfrak{f}_A : T_A\mathcal{B}^\rho \to T_A\mathcal{B}^\rho$ extends to a $\mathcal{G}^\rho$–equivariant endomorphism of $T_1\mathcal{G}^\rho \oplus T_A\mathcal{A}^\rho = \Omega^0_\rho(\Sigma, \mathfrak{su}(2)) \oplus \Omega^1_\rho(\Sigma, \mathfrak{su}(2))$ given by the formula

$$K(\mathfrak{f})_A^\rho = \begin{pmatrix} 0 & d_A^* \\ d_A & \nabla\mathfrak{f}_A \end{pmatrix} : (\Omega^0 \oplus \Omega^1)_\rho(\Sigma, \mathfrak{su}(2)) \to (\Omega^0 \oplus \Omega^1)_\rho(\Sigma, \mathfrak{su}(2)).$$

By construction, $K(\mathfrak{f})_A^\rho$ has the same spectral flow as $\nabla\mathfrak{f}_A$. Note that $K(\mathfrak{f})_A^\rho = K_A^\rho$ if A is flat, and that the difference $K_A^\rho - K(\mathfrak{f})_A^\rho$ is a relatively compact perturbation of K_A^ρ. For a generic path $t \to A(t)$, the spectral flow of $K_{A(t)}^\rho$ equals $sf^\rho(\alpha, \beta)$ where $[A(0)] = \alpha$ and $[A(1)] = \beta$.

Lemma 19. *At the trivial connection θ, the operator K_θ^ρ has a 1–dimensional kernel given by the d_θ–constant elements in $\Omega^0_\rho(\Sigma, \mathfrak{su}(2))$.*

Proof. The kernel of K_θ^ρ consists of the harmonic, hence constant, functions $\phi : \Sigma \to \mathfrak{su}(2)$ which are ρ–invariant. If we think of ϕ as an element of $\mathfrak{su}(2)$, the latter condition means that $\mathrm{Ad}_\rho\,\phi = \phi$. Since $\rho^2 = -1$, $\phi = c \cdot \rho$ for some real constant c. Therefore, $\dim\ker K_\theta^\rho = 1$. q.e.d.

From now on, we identify the kernel of K_θ^ρ with

$$\ker K_\theta^\rho = H^0_\rho(\Sigma, \mathfrak{su}(2)) = \mathfrak{su}^\rho(2) = \{\, \xi \in \mathfrak{su}(2) \mid \mathrm{Ad}_\rho\,\xi = \xi \,\}.$$

Let $\Omega^1_\rho(\Sigma, \mathfrak{su}(2)) = \mathrm{im}\, d_\theta \oplus \ker d_\theta^*$ be the Hodge decomposition corresponding to the operator

$$d_\theta : \Omega^0_\rho(\Sigma, \mathfrak{su}(2)) \to \Omega^1_\rho(\Sigma, \mathfrak{su}(2)).$$

Note that the operator $*d_\theta : \ker d_\theta^* \to \ker d_\theta^*$ is invertible since Σ is an integral homology 3–sphere. Choose $a \in \ker d_\theta^*$ and define the following symmetric bilinear form on $\mathfrak{su}^\rho(2)$,

$$\tau_a(\xi, \eta) = \int_\Sigma \mathrm{tr}\,([\xi, a] \wedge *[\eta, (*d_\theta)^{-1} a]). \tag{33}$$

This form plays an important part in the perturbation theory of K_θ^ρ.

Lemma 20. *There exists $\varepsilon > 0$ such that, for any $a \in \ker d_\theta^*$ and any $0 \le s < \varepsilon$, the operator $K^\rho_{\theta+s\cdot a}$ has exactly one eigenvalue in the interval $(-\varepsilon, \varepsilon)$. To order s^3, the eigenvalue of $K^\rho_{\theta+s\cdot a}$ in $(-\varepsilon,\varepsilon)$ is $\lambda \cdot s^2$ where λ is the eigenvalue of the form τ_a.*

Proof. Let $a \in \Omega^1_\rho(\Sigma, \mathfrak{su}(2))$ be an $\mathfrak{su}(2)$–valued 1–form on Σ; then $K_{\theta+s\cdot a} = K_\theta + s \cdot B_a$ where

$$B_a = \begin{vmatrix} 0 & i_a^* \\ i_a & *i_a \end{vmatrix},$$

and $i_a : \Omega^0_\rho(\Sigma, \mathfrak{su}(2)) \to \Omega^1_\rho(\Sigma, \mathfrak{su}(2))$ is given by $i_a(\varphi) = [a, \varphi]$. Suppose that for a small $s \ge 0$,

$$K_{\theta+s\cdot a}\psi_s = \lambda_s\, \psi_s, \tag{34}$$

where $\lambda_0 = 0$ and $\psi_s = (\varphi_s, \omega_s) \in (\Omega^0 \oplus \Omega^1)_\rho(\Sigma, \mathfrak{su}(2))$. Up to the order s^3, $\lambda_s = \lambda_1\, s + \lambda_2\, s^2 + \dots$ and $\psi_s = \psi_0 + \psi_1\, s + \psi_2\, s^2 + \dots$ so that the equation (34) is of the form

$$K_\theta \psi_0 = 0, \tag{35}$$

$$K_\theta \psi_1 + B_a \psi_0 = \lambda_1 \psi_0, \tag{36}$$

$$K_\theta \psi_2 + B_a \psi_1 = \lambda_2 \psi_0 + \lambda_1 \psi_1. \tag{37}$$

Since K_θ^2 is the Laplace operator, equation (35) implies that $\psi_0 = (\varphi_0, 0)$ with $\varphi_0 \in \mathfrak{su}^\rho(2)$ a constant. Equation (36) now takes the form

$$\begin{cases} d_\theta^* \omega_1 = \lambda_1 \varphi_0, \\ d_\theta \varphi_1 + * d_\theta \omega_1 = -[a, \varphi_0]. \end{cases}$$

We apply d_θ and d_θ^* respectively to the first and second equations in the system. Since $d_\theta d_\theta^* \omega_1 = \lambda_1 d_\theta \varphi_0 = 0$, we get that $d_\theta^* \omega_1 = 0$ and $\lambda_1 = 0$. The second equation implies that $d_\theta^* d_\theta \varphi_1 = -d_\theta^* [a, \varphi_0] = -[d_\theta^* a, \varphi_0] = 0$. Therefore, $\psi_1 = (\varphi_1, \omega_1)$ where $\varphi_1 \in \mathfrak{su}^\rho(2)$ is a constant. Equation (37) is now of the form

$$\begin{cases} d_\theta^* \omega_2 + i_a^* \omega_1 = \lambda_2 \varphi_0, \\ d_\theta \varphi_2 + * d_\theta \omega_2 = -[a, \varphi_1] - i_a^* \omega_1, \end{cases}$$

therefore, for any $\xi \in \mathfrak{su}^\rho(2)$,

$$\langle i_a^* \omega_1, \xi \rangle = \langle \lambda_2 \varphi_0, \xi \rangle - \langle d_\theta^* \omega_2, \xi \rangle = \lambda_2 \langle \varphi_0, \xi \rangle,$$

which implies that $\langle \omega_1, i_a \xi \rangle = \lambda_2 \langle \varphi_0, \xi \rangle$. On the other hand, $* d_\theta \omega_1 = -i_a \varphi_0$, so that $\omega_1 = (* d_\theta)^{-1} (i_a \varphi_0)$, and

$$\lambda_2 \langle \varphi_0, \xi \rangle = -\langle (* d_\theta)^{-1} (i_a \varphi_0), i_a \xi \rangle.$$

In other words, the eigenvalues of $K_{\theta + s \cdot a}$ up to the order s^3 are of the form $\lambda \cdot s^2$ where λ is the eigenvalue of the bilinear form

$$(\varphi_0, \xi) \mapsto -\langle (* d_\theta)^{-1} (i_a \varphi_0), i_a \xi \rangle = \int_\Sigma \operatorname{tr} \left([(* d_\theta)^{-1} a, \varphi_0] \wedge * [a, \xi] \right),$$

which is isomorphic to (33). q.e.d.

This lemma justifies the following definition. Choose $a \in \Omega_\rho^1(\Sigma, \mathfrak{su}(2))$ for which τ_a is non–degenerate (the existence of such forms will follow from section 6.4). Fix $\alpha = [A] \in \mathcal{R}(\Sigma)$, and define ε_α in (31) equal $\operatorname{sign} \det(\tau_a)$ times the spectral flow of the family

$$t \mapsto (1 - t)\, K_{\theta + s \cdot a}^\rho + t\, K_A^\rho$$

for s small enough.

The invariant $\chi^\rho(\Sigma)$ is now well–defined by the formula (31) due to the fact that for any $a_1, a_2 \in \Omega_\rho^1(\Sigma, \mathfrak{su}(2))$ for which τ_{a_1} and τ_{a_2} are non–degenerate, the difference in $\operatorname{sign} \det(\tau_{a_1})$ and $\operatorname{sign} \det(\tau_{a_2})$ is given by the spectral flow of K_A^ρ between $\theta + s \cdot a_1$ and $\theta + s \cdot a_2$; see Lemma 2.9 of [36].

In a less formal way, one can think of $\chi^\rho(\Sigma)$ as the sum

$$\chi^\rho(\Sigma) = \sum_{\alpha \in \mathcal{R}(\Sigma)} (-1)^{sf^\rho(\theta,\alpha)},$$

where $sf^\rho(\theta, \alpha)$ is defined as follows. For a generic nearby connection θ' of θ, the operator $K_{\theta'}^\rho$ has one small non–zero eigenvalue. Define $p(\theta') = 1$ if this eigenvalue is negative, and $p(\theta') = 0$ otherwise. Then

$$sf^\rho(\theta, \alpha) = p(\theta') + sf^\rho(\theta', \alpha) \mod 4.$$

Geometrically, to define $sf^\rho(\theta, \alpha)$ we simply count the intersection points of the spectral curves with the straight line connecting the points $(0, -\varepsilon)$ and $(1, \varepsilon)$ where $\varepsilon > 0$ is chosen smaller than the absolute value of any non–zero eigenvalue of K_θ^ρ and K_α^ρ; see Figure 7.

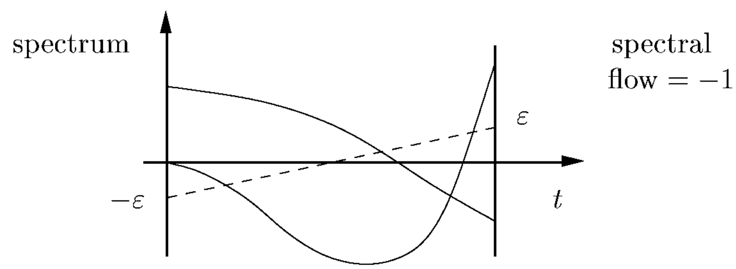

FIGURE 7

We further note that the invariant $\chi^\rho(\Sigma)$ is metric independent. The definition of a flat connection requires no metric, so the metric can only affect $\chi^\rho(\Sigma)$ through the spectral flow. In fact, it does not because the space of σ–invariant Riemannian metrics on Σ is path connected.

5. The invariant χ^ρ and Floer homology

In this section we express the invariant χ^ρ of Brieskorn homology spheres in terms of their Floer homology.

5.1 Casson invariant via gauge theory

C. Taubes gave in [36] a gauge–theoretical interpretation of the Casson's invariant. We shortly recall his construction in the special case of a Brieskorn homology sphere $\Sigma = \Sigma(p, q, r)$.

Let $E \to \Sigma$ be a trivialized SU(2)–bundle over Σ. For any connection A in E define the following elliptic differential operator

$$(38) \quad K_A = \begin{vmatrix} 0 & d_A^* \\ d_A & *d_A \end{vmatrix} : (\Omega^0 \oplus \Omega^1)(\Sigma, \mathfrak{su}(2)) \to (\Omega^0 \oplus \Omega^1)(\Sigma, \mathfrak{su}(2)),$$

where d_A stands for the covariant derivative with respect to A, and $*$ is the Hodge operator associated with a Riemannian metric on Σ. If the metric is σ–invariant and the connection A is ρ–invariant in the sense of Section 3.2, then the restriction of the operator K_A on the ρ–invariant forms $\Omega^*_\rho(\Sigma, \mathfrak{su}(2))$ coincides with the operator (32).

In proper Sobolev completions of $(\Omega^0 \oplus \Omega^1)(\Sigma, \mathfrak{su}(2))$ the operator (38) is Fredholm. Therefore, for any pair $\alpha, \beta \in \mathcal{B}$ of flat connections in E one can define the spectral flow $sf(\alpha, \beta)$ as follows. Let $A(t)$, $0 \le t \le 1$, be a path of connections in E such that $[A(0)] = \alpha$ and $[A(1)] = \beta$. Associated to $A(t)$ is a path of Fredholm operators $K_{A(t)}$. The spectral flow (α, β) is the net number of the eigenvalues of $K_{A(t)}$ which change from under $-\varepsilon$ to over ε as $A(t)$ varies from $A(0)$ to $A(1)$; here, $\varepsilon > 0$ is chosen smaller than the absolute value of any non–zero eigenvalue of $K_{A(0)}$ and $K_{A(1)}$, compare with the definition of $sf^\rho(\alpha, \beta)$ in Section 4.2.

Note that in our case of $\Sigma(p, q, r)$, all flat connections except for the trivial one are irreducible and non–degenerate. The latter means that $\ker K_\alpha = 0$ unless $\alpha \ne \theta$. The kernel of K_θ is 3–dimensional, while the kernel of K_θ^ρ is 1–dimensional.

The spectral flow $sf(\alpha, \beta)$ is well–defined modulo 8. C. Taubes proved in [36] that

$$\lambda(\Sigma) = \frac{1}{2} \sum_{\alpha \in \mathcal{R}(\Sigma)} (-1)^{sf(\theta, \alpha)},$$

where flat connections, as usual, are identified with SU(2)–representations via holonomy. Furthermore, R. Fintushel and R. Stern showed in [13] that for any $\alpha \in \mathcal{R}(\Sigma(p, q, r))$, the spectral flow $sf(\theta, \alpha)$ is even. Thus χ simply counts the irreducible representations.

For any oriented integral homology 3–sphere Σ, A. Floer defined in [14] a $(\mathbb{Z}/8)$–graded instanton homology theory $I_*(\Sigma)$ whose Euler characteristic equals $2 \cdot \lambda(\Sigma)$. In the special case of $\Sigma = \Sigma(p, q, r)$ this theory is of a particularly simple form. Namely, the group $I_n(\Sigma(p, q, r))$ is trivial for odd j, and is a free abelian group generated by all irreducible $\alpha \in \mathcal{R}(\Sigma(p, q, r))$ such that $sf(\theta, \alpha) = j \mod 8$ for even j.

5.2 Comparing the spectral flows

With any irreducible $\alpha \in \mathcal{R}(\Sigma(p,q,r))$ we have associated two numbers, $sf(\theta,\alpha) \mod 8$ and $sf^\rho(\theta,\alpha) \mod 4$. The aim of this section is to compare them.

Let α and β be flat (possibly trivial) connections in a trivial SU(2)–bundle E over $\Sigma = \Sigma(p,q,r)$. By pull-back we extend E to a trivial bundle (which is also denoted by E) over the infinite cylinder $\Sigma \times \mathbb{R}$. Let us choose a σ–invariant metric on Σ, and the corresponding product metric on $\Sigma \times \mathbb{R}$. Let further $\rho : E \to E$ be the lift of σ defined in (20), and $A(t)$ a path of ρ–invariant connections in E forming a ρ–invariant connection over $\Sigma \times \mathbb{R}$ vanishing in the $\mathbb{R}$–direction and equal to respectively α and β near the ends of $\Sigma \times \mathbb{R}$.

Let us consider the self–duality operator

$$\text{(39)}\quad \mathcal{D}_A = d_A^* \oplus d_A^- : \Omega^1(\Sigma \times \mathbb{R}, \mathfrak{su}(2)) \to (\Omega^0 \oplus \Omega^2_-)(\Sigma \times \mathbb{R}, \mathfrak{su}(2)),$$

and impose the global boundary condition (2.3) of Atiyah–Patodi–Singer [2]. This boundary value problem has well–defined index, $\operatorname{index} \mathcal{D}_A(\alpha,\beta)$.

The assignment to an $\mathfrak{su}(2)$–valued differential p–form ω on $\Sigma \times \mathbb{R}$ of the form $\rho^*\omega = \operatorname{Ad}_\rho \sigma^*\omega$ defines an involution on the space $\Omega^p(\Sigma \times \mathbb{R}, \mathfrak{su}(2))$, $p = 0,1,2,\ldots$ This involution splits all the spaces $\Omega^p(\Sigma \times \mathbb{R}, \mathfrak{su}(2))$, as well as the space $\Omega^2_-(\Sigma \times \mathbb{R}, \mathfrak{su}(2))$, in the direct sum of the (± 1)–eigenspaces of ρ^*. According to this splitting,

$$\text{(40)}\qquad \mathcal{D}_A = \mathcal{D}_A^+ \oplus \mathcal{D}_A^-.$$

We denote the operator $\mathcal{D}_A^+$ by $\mathcal{D}_A^\rho$, so that

$$\text{(41)}\qquad \mathcal{D}_A^\rho : \Omega^1_\rho(\Sigma \times \mathbb{R}, \mathfrak{su}(2)) \to (\Omega^0 \oplus \Omega^2_-)_\rho(\Sigma \times \mathbb{R}, \mathfrak{su}(2));$$

compare with (26). The Atiyah–Patodi–Singer index of the operator (41) will be denoted by $\operatorname{index} \mathcal{D}_A^\rho(\alpha,\beta)$.

Lemma 21. *Let $A(t)$ be a 1–parameter family of ρ–invariant connections connecting α to θ, where α is an irreducible flat connection on $\Sigma(p,q,r)$. Then*

$$\begin{aligned} sf(\theta,\alpha) &= -3 - \operatorname{index} \mathcal{D}_A(\alpha,\theta) \mod 8, \\ sf^\rho(\theta,\alpha) &= -1 - \operatorname{index} \mathcal{D}_A^\rho(\alpha,\theta) \mod 4. \end{aligned}$$

If $A(t)$ connects two irreducible flat connections, α and β, then

$$\begin{aligned} sf(\alpha,\beta) &= \operatorname{index} \mathcal{D}_A(\alpha,\beta) \mod 8 \quad \textit{and} \\ sf^\rho(\alpha,\beta) &= \operatorname{index} \mathcal{D}_A^\rho(\alpha,\beta) \mod 4. \end{aligned}$$

Proof. Both statements concerning sf are proved in [13, Lemma 3.2]; the other two follow by the same argument after one notes that $sf^{\rho}(\theta,\theta)=-1$. q.e.d.

Let $\operatorname{index}(\rho^*,\mathcal{D}_A)=\operatorname{tr}(\rho^*\mid\ker\mathcal{D}_A)-\operatorname{tr}(\rho^*\mid\operatorname{coker}\mathcal{D}_A)$. A standard argument proves the following result.

Lemma 22. *For any 1–parameter family $A(t)$ of ρ–invariant connections connecting an irreducible flat connection α to a connection β, which is either trivial or flat irreducible,*

$$\operatorname{index}\mathcal{D}^{\rho}_A(\alpha,\beta)=\frac{1}{2}\operatorname{index}\mathcal{D}_A(\alpha,\beta)+\frac{1}{2}\operatorname{index}(\rho^*,\mathcal{D}_A)(\alpha,\beta).$$

Proof. According to (40), the operator $\mathcal{D}_A$ splits in the direct sum $\mathcal{D}_A=\mathcal{D}^+_A\oplus\mathcal{D}^-_A$ with $\mathcal{D}^+_A=\mathcal{D}^{\rho}_A$. Therefore,

$$\begin{aligned}\operatorname{index}\mathcal{D}_A&=\operatorname{tr}(1\mid\ker\mathcal{D}_A)-\operatorname{tr}(1\mid\operatorname{coker}\mathcal{D}_A)\\&=\operatorname{tr}(1\mid\ker\mathcal{D}^+_A)+\operatorname{tr}(1\mid\ker\mathcal{D}^-_A)\\&\quad-\operatorname{tr}(1\mid\operatorname{coker}\mathcal{D}^+_A)-\operatorname{tr}(1\mid\operatorname{coker}\mathcal{D}^-_A),\\\operatorname{index}(\rho^*,\mathcal{D}_A)&=\operatorname{tr}(\rho^*\mid\ker\mathcal{D}_A)-\operatorname{tr}(\rho^*\mid\operatorname{coker}\mathcal{D}_A)\\&=\operatorname{tr}(1\mid\ker\mathcal{D}^+_A)-\operatorname{tr}(1\mid\ker\mathcal{D}^-_A)\\&\quad-\operatorname{tr}(1\mid\operatorname{coker}\mathcal{D}^+_A)+\operatorname{tr}(1\mid\operatorname{coker}\mathcal{D}^-_A).\end{aligned}$$

Adding these two formulas together completes the proof. q.e.d.

The G–index theorem for manifolds with boundary proved by H. Donnelly in [11], gives the following formula for $\operatorname{index}(\rho^*,\mathcal{D}_A)(\alpha,\beta)$,

$$\begin{aligned}\operatorname{index}(\rho^*,\mathcal{D}_A)(\alpha,\beta)&=\int_N\beta_0-\frac{1}{2}(h^{\rho}_{\beta}+\eta^{\rho}_{\beta}(0))(\Sigma)\\&\quad+\frac{1}{2}(-h^{\rho}_{\alpha}+\eta^{\rho}_{\alpha}(0))(\Sigma),\end{aligned}\tag{42}$$

where N is the fixed point set of the involution $\sigma\times 1:\Sigma\times\mathbb{R}\to\Sigma\times\mathbb{R}$, the integrand β_0 is a universal polynomial in characteristic forms, h^{ρ}_{α} is the trace of the map induced by ρ^* on $H^0(\Sigma,d_{\alpha})\oplus H^1(\Sigma,d_{\alpha})$, and $\eta^{\rho}_{\alpha}(0)$ is the η^{ρ}–invariant of the operator K_{α}; see (38).

More precisely, the η^{ρ}–invariant is defined as follows. We define the η^{ρ}–function of α by the formula

$$\eta^{\rho}_{\alpha}(s)=\sum_{\lambda\neq 0}\operatorname{sign}\lambda\cdot\operatorname{tr}(\rho^*\mid W^{\alpha}_{\lambda})\,|\lambda|^{-s},\tag{43}$$

where W_λ^α is the λ–eigenspace of the operator K_α, and $\operatorname{tr}(\rho^* \mid W_\lambda^\alpha)$ the trace of the operator ρ^* restricted to W_λ^α. Standard methods show that the series on the right converges for $\operatorname{Re}(s)$ large enough and has a meromorphic continuation to the entire complex s–plane such that $\eta_\alpha^\rho(0)$ is finite.

The formula (42) will be further simplified by using the flat cobordism techniques of R. Fintushel and R. Stern [13]. Any Brieskorn homology 3–sphere $\Sigma(p,q,r)$ with $p,q,r \geq 2$, is the boundary of the (compactified) complex surface

$$V(p,q,r) = \{\, (x,y,z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = 0\,\},$$

which is non–singular except at the origin, and

$$\Sigma(p,q,r) = V(p,q,r) \cap S^5 \subset \mathbb{C}^3$$

is a smooth manifold. It is Seifert fibered over S^2 with the Seifert invariants $\{b; (p,b_1),(q,b_2),(r,b_3)\}$ satisfying the equation (6). For simplicity, we will sometimes denote the Seifert invariants $(p,b_1),(q,b_2),(r,b_3)$ by respectively $(a_1,b_1),(a_2,b_2),(a_3,b_3)$, and $\Sigma(p,q,r) = \Sigma(a_1,a_2,a_3)$ by Σ.

If we blow up the surface $V(p,q,r)$ once at the origin, we get another surface, W, which is non–singular except at three points. The singularities of W are just cones on the lens spaces $L(a_i,b_i)$, $i = 1,2,3$. Topologically, one can identify W with a 4–dimensional orbifold obtained as the mapping cylinder of the Seifert orbit map $\Sigma \to S^2$. The oriented boundary of W is Σ. Let W_0 denote W with open cones around the singularities removed. Then W_0 is a smooth manifold with the oriented boundary $\partial W_0 = \Sigma \cup -L(a_1,b_1) \cup -L(a_2,b_2) \cup -L(a_3,b_3)$ and the fundamental group

$$\pi_1 W_0 = \pi_1 \Sigma / \langle h \rangle = \langle x,y,z \mid x^p = y^q = z^r = xyz = 1 \rangle,$$

the (p,q,r)–triangle group. Since Σ is a homology sphere, there is a one–to–one correspondence between SU(2)–representations α of $\pi_1\Sigma$ and SO(3)–representations of $\pi_1 W_0$, which we again denote by α.

The involution (7) is the restriction on Σ of the complex conjugation involution, which we still call σ, on W. The involution on W, in its turn, restricts to involutions σ on W_0 and on each of the lens spaces $L(a_i,b_i)$. The latter can be described as follows. Let (a_i,b_i) be a pair of relatively prime positive numbers, and $S^3 \subset \mathbb{C}^2$ be defined by the equation $|z|^2 + |w|^2 = 1$. The group $\mathbb{Z}/a_i$ acts on S^3 by the formula

$$t^*(z,w) = (tz, t^{b_i} w), \quad t = e^{2\pi i j/a_i} \in \mathbb{C}, \quad 0 \leq j \leq a_i - 1. \tag{44}$$

This action is free, and its quotient is the lens space $L(a_i,b_i) = S^3/(\mathbb{Z}/a_i)$. The involution σ acting on $L(a_i,b_i)$ is induced by the complex conjugation on S^3, $(z,w) \mapsto (\bar{z},\bar{w})$. It turns $L(a_i,b_i)$ into a double branched cover of S^3 with branch set a two bridge link of type (a_i,b_i); see [6], Chapter 12.

Note that the manifold W_0 itself is a double branched cover of W_0/σ, which is a 4–ball with three disjoint 4–balls in its interior removed. The branch set is a surface N_0 in W_0/σ providing a cobordism between the Montesinos knot $k(p,q,r)$ on one side and three two bridge links of the types (a_i,b_i), $i=1,2,3$, on the other. The surface N_0 is homeomorphic to the real projective plane $\mathbb{R}P^2$ with four disjoint discs removed.

Let $\mathbb{V}_\alpha$ denote the flat SO(3)–bundle over W_0 determined by $\alpha \in \mathcal{R}(\Sigma)$ with the rotation numbers ℓ_1, ℓ_2, ℓ_3, see (11). When $\mathbb{V}_\alpha$ is restricted to a boundary component $L(a_i,b_i)$, it takes the form

$$S^3 \times_{\mathbb{Z}/a_i} \mathfrak{su}(2) \to L(a_i,b_i)$$

where $\mathbb{Z}/a_i$ acts on S^3 by the formula (44), and on $\mathfrak{su}(2)$ by sending $\xi \in \mathfrak{su}(2)$ to $\mathrm{Ad}_{\alpha(t)}\,\xi$, $t = \exp(2\pi i \ell_i j/a_i)$, $0 \le j \le a_i - 1$.

An SO(3)–bundle $\mathbb{V}_\alpha$ is classified by its second Stiefel–Whitney class $w_2 \in H^2(W_0,\mathbb{Z}/2) = \mathbb{Z}/2$ which is the obstruction to lifting it to an SU(2)–bundle. This means that $w_2(\mathbb{V}_\alpha) = 0$ iff $\alpha(h) = +1$. The involution σ acting on W_0 preserves the class w_2, $\sigma^* w_2 = w_2$.

The lift ρ of the involution σ acting on Σ; see (20), defines a lift Ad_ρ of σ acting on W_0, to the SO(3)–bundle $\mathbb{V}_\alpha$ so that all flat SO(3)–connections in $\mathbb{V}_\alpha$ are Ad_ρ–invariant. Equivariant differential forms, equivariant self–duality operators *etc.* are defined over W_0 and $L(a_i,b_i)$, $i=1,2,3$, in the same way as they were defined over $\Sigma(p,q,r)\times\mathbb{R}$ and $\Sigma(p,q,r)$.

Lemma 23. *Let $\alpha : \pi_1\Sigma(p,q,r) \to \mathrm{SU}(2)$ be an irreducible representation, and $\mathbb{V}_\alpha$ the induced* SO(3)*–bundle over W_0 with flat connection A_α induced by α. Then* $\mathrm{index}(\rho^*, \mathcal{D}_{A_\alpha})(W_0) = 0$.

Proof. Let α be an irreducible representation. Since

$$\mathrm{index}(\rho^*, \mathcal{D}_{A_\alpha})(W_0) = \mathrm{tr}(\rho^* \mid \ker \mathcal{D}_{A_\alpha}) - \mathrm{tr}(\rho^* \mid \mathrm{coker}\, \mathcal{D}_{A_\alpha}),$$

the result follows from the fact that both $\ker \mathcal{D}_{A_\alpha}$ and $\mathrm{coker}\,\mathcal{D}_{A_\alpha}$ on W_0 vanish; see Proposition 3.3 and the argument right after it in [13].

q.e.d.

Let α and β be irreducible flat connections. The Donnelly's theorem [11] can be applied again, this time to the manifolds W_0 and $-W_0$:

$$\begin{aligned} 0 = \int_{N_0} \beta_0 - \frac{1}{2}(h^{\rho}_{\alpha} + \eta^{\rho}_{\alpha}(0))(\Sigma) \\ + \frac{1}{2}\sum_{i=1}^{3} (-h^{\rho}_{\alpha} + \eta^{\rho}_{\alpha}(0))(L(a_i, b_i)), \end{aligned} \tag{45}$$

$$\begin{aligned} 0 = \int_{-N_0} \beta_0 + \frac{1}{2}(-h^{\rho}_{\beta} + \eta^{\rho}_{\beta}(0))(\Sigma) \\ - \frac{1}{2}\sum_{i=1}^{3} (h^{\rho}_{\beta} + \eta^{\rho}_{\beta}(0))(L(a_i, b_i)), \end{aligned} \tag{46}$$

where $-N_0$ stands for the fixed point set of the involution σ acting on $-W_0$. Adding (42), (45), and (46) together and noting that $h^{\rho}_{\alpha}(\Sigma) = h^{\rho}_{\beta}(\Sigma) = 0$ (because α and β are non–degenerate and irreducible), we get

$$\begin{aligned} \operatorname{index}(\rho^*, \mathcal{D}_A)(\alpha, \beta) = \int_{N_0 \cup N \cup -N_0} \beta_0 \\ - \frac{1}{2}\sum_{i=1}^{3} (h^{\rho}_{\beta} + \eta^{\rho}_{\beta}(0))(L(a_i, b_i)) \\ + \frac{1}{2}\sum_{i=1}^{3} (-h^{\rho}_{\alpha} + \eta^{\rho}_{\alpha}(0))(L(a_i, b_i)). \end{aligned} \tag{47}$$

By [11], the right-hand side of (47) is simply $\operatorname{index}(\rho^*, \mathcal{D}_B)(\overline{W}_0)$ where B is the connection over $\overline{W}_0 = W_0 \cup -W_0 \cong W_0 \cup \Sigma \times \mathbb{R} \cup -W_0$ built from flat connections A_α over W_0 and A_β over $-W_0$, respectively, and a 1–parameter family A of connections over Σ connecting α to β. Thus,

$$\operatorname{index}(\rho^*, \mathcal{D}_A)(\alpha, \beta) = \operatorname{index}(\rho^*, \mathcal{D}_B)(\overline{W}_0),$$

and the latter is given by the right-hand side of the formula (47) with the integral $\int_{N_0 \cup N \cup -N_0} \beta_0$ replaced by $\int_{N_0 \cup -N_0} \beta_0$.

Proposition 24. *Let α and β be irreducible flat connection on $\Sigma(p,q,r)$, and B the connection over $\overline{W}_0$ built as above from flat connections A_α and A_β and a 1–parameter family A of ρ–invariant connections over Σ connecting α to β. Then* $\operatorname{index}(\rho^*, \mathcal{D}_B)(\overline{W}_0) = 0 \mod 8$.

Proof. The formula (47) involves three kinds of terms, each of which *a priori* depends on the representation α and/or the manifold $\Sigma(p,q,r)$. An easy calculation with the group cohomology $H^0(L(a_i,b_i),\mathbb{V}_\alpha) = H^0(\mathbb{Z}/a_i,\mathbb{V}_\alpha)$ and $H^1(L(a_i,b_i),\mathbb{V}_\alpha) = H^1(\mathbb{Z}/a_i,\mathbb{V}_\alpha) = 0$ shows that $h^\rho_\alpha(L(a_i,b_i)) = h^\rho_\beta(L(a_i,b_i)) = -1$. Proposition 27 below assures that, for any lens space $L(a,b)$ and any (trivial or irreducible) representation $\alpha : \pi_1 L(a,b) \to \mathrm{SU}(2)$, the corresponding η^ρ-invariant vanishes.

It follows from Theorem 18 of [37] that, for any *closed* orientable manifold X with an orientation preserving involution σ, the integrand β_0 is a polynomial in the Euler class of the fixed point set F of σ and the Euler class of the normal bundle of F in X. Since β_0 is universal, we conclude in particular that $\int_{N_0\cup -N_0}\beta_0$ in (47) does not depend on the choice of α,β, or B.

Let us choose $\beta=\alpha$; then

$$\operatorname{index}\mathcal{D}^\rho_A(\alpha,\alpha) = sf^\rho(\alpha,\alpha) = 0 \mod 4$$

and

$$\operatorname{index}\mathcal{D}_A(\alpha,\alpha) = sf(\alpha,\alpha) = 0 \mod 8,$$

therefore,

$$\operatorname{index}(\rho^*,\mathcal{D}_A)(\alpha,\alpha) = 0 \mod 8,$$

see Lemma 22. On the other hand, the formula (47) implies that, modulo 8,

$$0 = \operatorname{index}(\rho^*,\mathcal{D}_A)(\alpha,\alpha) = \int_{N_0\cup -N_0}\beta_0 + 3,$$

so $\int_{N_0\cup -N_0}\beta_0 = -3 \mod 8$. Now, for *any* choice of flat irreducible α and β, we have $\operatorname{index}(\rho^*,\mathcal{D}_A)(\alpha,\beta) = -3+3 = 0 \mod 8$. q.e.d.

Corollary 25. *There is a universal constant* $c \mod 4$ *such that, for any irreducible flat connections* $\alpha,\beta\in\mathcal{R}(\Sigma(p,q,r))$,

$$\begin{aligned} sf^\rho(\theta,\alpha) &= \frac{1}{2}\cdot sf(\theta,\alpha) + c \mod 4 \quad \textit{and}\\ sf^\rho(\alpha,\beta) &= \frac{1}{2}\cdot sf(\alpha,\beta) \mod 4; \end{aligned}$$

in particular, $sf(\alpha,\beta)$ *is always even.*

Proof. Since $sf^\rho(\alpha,\beta) = \operatorname{index}\mathcal{D}^\rho_A(\alpha,\beta) \mod 4$ and $sf(\alpha,\beta) = \operatorname{index}\mathcal{D}_A(\alpha,\beta) \mod 8$, the result for irreducible α and β follows from Proposition 24 and Lemmas 21 and 22.

Since

$$-1 = sf^\rho(\theta,\alpha) + sf^\rho(\alpha,\theta) \mod 4$$

and

$$-1 = sf^\rho(\theta,\alpha) + sf^\rho(\alpha,\beta) + sf^\rho(\beta,\theta) \mod 4$$

for any flat irreducible connections α and β, we get that

$$sf^\rho(\alpha,\beta) = sf^\rho(\theta,\beta) - sf^\rho(\theta,\alpha) \mod 4, \quad \text{and similarly,}$$
$$sf(\alpha,\beta) = sf(\theta,\beta) - sf(\theta,\alpha) \mod 8.$$

Coupled with the fact that $sf^\rho(\alpha,\beta) = \frac{1}{2}\, sf(\alpha,\beta) \mod 4$, these imply

$$sf^\rho(\theta,\beta) - sf^\rho(\theta,\alpha) = \frac{1}{2}\, sf(\theta,\beta) - \frac{1}{2}\, sf(\theta,\alpha) \mod 4,$$

or, equivalently,

$$sf^\rho(\theta,\beta) - \frac{1}{2}\, sf(\theta,\beta) = sf^\rho(\theta,\alpha) - \frac{1}{2}\, sf(\theta,\alpha) \mod 4.$$

Therefore, the difference $sf^\rho(\theta,\alpha) - \frac{1}{2}\, sf(\theta,\alpha) \mod 4$ is the same for all flat irreducible representations α. We denote it by c. The exact value of c can be found by repeating the argument of Proposition 24 for the connection $\beta = \theta$. The answer will depend on $\operatorname{index}(\rho^*, \mathcal{D}_{A_\theta})(W_0)$, which is a rational homology invariant of W_0. This implies the universality of c in that it is independent of $\Sigma(p,q,r)$. q.e.d.

Corollary 26. *For any Brieskorn homology sphere, $\chi^\rho(\Sigma(p,q,r)) = \pm 2\,\nu(\Sigma(p,q,r))$ with a universal $(\pm)$–sign.*

5.3 The η^ρ–invariant for lens spaces

Let $L(a,b)$ be a lens space, and $\alpha : \pi_1 L(a,b) \to \mathrm{SO}(3)$ a representation of its fundamental group. In this section we compute the η^ρ–function of the operator

$$K_\alpha = \begin{vmatrix} 0 & d^*_\alpha \\ d_\alpha & *d_\alpha \end{vmatrix} : (\Omega^0 \oplus \Omega^1)(\Sigma, \mathfrak{su}(2)) \to (\Omega^0 \oplus \Omega^1)(\Sigma, \mathfrak{su}(2))$$

defined in (43) by the formula

$$\eta_\alpha^\rho(s) = \sum_{\lambda \neq 0} \operatorname{sign} \lambda \cdot \operatorname{tr}(\rho^* \mid W_\lambda^\alpha)\, |\lambda|^{-s}, \tag{48}$$

where W_λ^α is the λ–eigenspace of K_α.

Proposition 27. *For any lens space $L(a,b)$ and any representation $\alpha : \pi_1 L(a,b) \to \mathrm{SO}(3)$, the function (48) is identically zero, $\eta_\alpha^\rho(s) = 0$.*

First step in our proof is the following result, which is a version of Cancellation Lemma of [22].

Lemma 28. *The η^ρ–function (48) equals the η^ρ–function of the operator $*d_\alpha : \ker d_\alpha^* \to \ker d_\alpha^*$; in other words,*

$$\eta_\alpha^\rho(s) = \sum_{\lambda \neq 0} \operatorname{sign} \lambda \cdot \operatorname{tr}(\rho^* \mid V_\lambda^\alpha)\, |\lambda|^{-s}, \tag{49}$$

where $V_\lambda^\alpha \subset \ker d_\alpha^*$ is the λ–eigenspace of the operator $*d_\alpha$.

Proof. We can ignore the d_α–harmonic forms as we only consider non-zero eigenvalues in (48) and (49). By the Hodge decomposition, we can split the remaining forms into d_α–closed and d_α–coclosed, Ω_c^* and Ω_{cc}^*. Since $H^1(L(a,b), d_\alpha) = 0$, we have the following direct sum decomposition:

$$\Omega^1(L(a,b), \mathfrak{su}(2)) = \ker d_\alpha \,\oplus\, \ker d_\alpha^*.$$

Therefore, the spaces Ω_c^1 and Ω_{cc}^1 can be identified as $\Omega_c^1 = \ker d_\alpha$ and $\Omega_{cc}^1 = \ker d_\alpha^*$. The space Ω_{cc}^0 consists of those 0–forms

$$\phi \in \Omega^0(L(a,b), \mathfrak{su}(2))$$

for which $d_\alpha \phi \neq 0$.

A straightforward calculation shows that the decomposition

$$(\Omega_{cc}^0 \,\oplus\, \Omega_c^1) \,\oplus\, \Omega_{cc}^1$$

is stable with respect to the operator K_α, and that K_α restricted to Ω_{cc}^1 is isomorphic to the operator $*d_\alpha : \ker d_\alpha^* \to \ker d_\alpha^*$.

Let us restrict our attention to the space $\Omega_{cc}^0 \oplus \Omega_c^1$. There is a basis for Ω_{cc}^0 consisting of eigenforms of the Laplace operator $\Delta_\alpha = d_\alpha^* d_\alpha$. Let w be one of these. Then $\Delta_\alpha w = \lambda^2 w$ with some $\lambda > 0$. Consider the two elements of $\Omega_{cc}^0 \oplus \Omega_c^1$ defined by $(w, d_\alpha w/\lambda)$ and $(w, -d_\alpha w/\lambda)$. They are each eigenvectors of K_α restricted to $\Omega^0_{cc} \oplus \Omega^1_c$ with eigenvalues of opposite sign:

$$\begin{pmatrix} 0 & d^*_\alpha \\ d_\alpha & *d_\alpha \end{pmatrix} \begin{pmatrix} w \\ \pm d_\alpha w/\lambda \end{pmatrix} = \begin{pmatrix} \pm d^*_\alpha d_\alpha w/\lambda \\ d_\alpha w \end{pmatrix} = \begin{pmatrix} \pm\lambda w \\ d_\alpha w \end{pmatrix} = \pm\lambda \cdot \begin{pmatrix} w \\ \pm d_\alpha w/\lambda \end{pmatrix}.$$

Since the forms $(w, \pm d_\alpha w/\lambda)$ form a basis for $\Omega^0_{cc} \oplus \Omega^1_c$ we are finished. q.e.d.

Let V_λ denote the space of all $\mathfrak{su}(2)$–valued 1–forms ψ_λ on S^3 such that $d^*\psi_\lambda = 0,\ *d\psi_\lambda = \lambda\,\psi_\lambda$. Since $L(a,b)$ is a quotient of S^3 by a free action of the group $\mathbb{Z}/a$, the λ–eigenspace of the operator $*d_\alpha : \ker d^*_\alpha \to \ker d^*_\alpha$ is then a subspace in V_λ consisting of $\mathfrak{su}(2)$–valued 1–forms ψ on S^3 satisfying the equation

$$t^*\,\psi = \operatorname{Ad}_{\alpha(t)}\psi \quad \text{for all } t \in \mathbb{Z}/a.$$

The involution σ on $S^3 \subset \mathbb{C}^2$ induced by complex conjugation on $\mathbb{C}^2$ defines an involution on V_λ by the formula

$$\psi \mapsto \operatorname{Ad}_\rho \sigma^*\psi.$$

It descends to the involution

$$\rho^* : V^\alpha_\lambda \to V^\alpha_\lambda, \tag{50}$$

which is used in the formula (48) defining the η^ρ–function. The fact that the involution (50) is well–defined can be checked directly as follows. For any $\psi \in V^\alpha_\lambda$ we need to check that $t^*\,\rho^*\psi = \operatorname{Ad}_{\alpha(t)}\rho^*\psi$. Assume for simplicity that $\rho = j$; then

$$\begin{aligned} t^*\,\rho^*\psi &= \operatorname{Ad}_j t^*\sigma^*\psi = \operatorname{Ad}_j \sigma^*(t^{-1})^*\psi, \\ &= \operatorname{Ad}_j \sigma^* \operatorname{Ad}_{\alpha(t^{-1})}\psi, \qquad\qquad \text{since } \psi \in V^\alpha_\lambda, \\ &= \operatorname{Ad}_j \sigma^* \operatorname{Ad}_j \operatorname{Ad}_{\alpha(t^{-1})} \operatorname{Ad}_j \psi, \\ &= \sigma^* \operatorname{Ad}_{\alpha(t)} \operatorname{Ad}_j \psi = \operatorname{Ad}_{\alpha(t)}\rho^*\psi. \end{aligned}$$

For any angle ϕ between 0 and 2π and any positive integer c we define an operator $\xi_\phi(c)$ on $\mathbb{C}^2$ by the formula

$$\xi_\phi(c)\,:(z,w) \mapsto (e^{i\phi}z, e^{ic\phi}w). \tag{51}$$

An elementary calculation shows that $\xi_\phi(c)\sigma = \sigma\xi_\phi(c)\xi_{-2\phi}(c)$. The operator $\xi_\phi(c)$ induces by pull–back an operator $\xi^*_\phi(c)$ on $\mathfrak{su}(2)$–valued 1–forms on S^3 (with trivial action on the coefficients). Since $\xi^*_\phi(c)$ commutes with the $(\mathbb{Z}/a)$–action, it is well defined as an operator on V^α_λ for every λ. Moreover,

$$\rho^*\xi^*_\phi(c) = \xi^*_{-2\phi}(c)\xi^*_\phi(c)\rho^*. \tag{52}$$

Let $\phi = \pi$; then the operators ρ^* and $\xi^*_\pi(c)$ commute due to the fact that $\xi^*_{2\pi}(c) = 1^* = 1$. Moreover, $(\xi^*_\pi(c))^2 = \xi_{2\pi}(c) = 1$, hence $\xi^*_\pi(c)$ is an involution on V^α_λ. Let

$$V^\alpha_\lambda = V^\alpha_\lambda(+1, \xi^*_\pi(c)) \oplus V^\alpha_\lambda(-1, \xi^*_\pi(c))$$

be a decomposition of V^α_λ in the direct sum of the (± 1)–eigenspaces of $\xi^*_\pi(c)$. Then, of course,

$$\operatorname{tr}(\rho^* \,|\, V^\alpha_\lambda) = \operatorname{tr}(\rho^* \,|\, V^\alpha_\lambda(+1, \xi^*_\pi(c))) + \operatorname{tr}(\rho^* \,|\, V^\alpha_\lambda(-1, \xi^*_\pi(c))).$$

We claim that $\operatorname{tr}(\rho^* \,|\, V^\alpha_\lambda(-1, \xi^*_\pi(c))) = 0$ for any c. This follows from the fact that the following diagram

$$\begin{array}{ccc}
V^\alpha_\lambda(-1, \xi^*_\pi(c)) & \xrightarrow{\rho^*} & V^\alpha_\lambda(-1, \xi^*_\pi(c)) \\
{\scriptstyle \xi^*_{\pi/2}(c)}\downarrow & & {\scriptstyle \xi^*_{\pi/2}(c)}\downarrow \\
V^\alpha_\lambda(-1, \xi^*_\pi(c)) & \xrightarrow{\rho^*} & V^\alpha_\lambda(-1, \xi^*_\pi(c))
\end{array}$$

anticommutes, because, due to (52),

$$\rho^*\xi^*_{\pi/2}(c) = \xi^*_{-\pi}(c)\xi^*_{\pi/2}(c)\rho^* = -\xi^*_{\pi/2}(c)\rho^* \quad \text{on} \quad V^\alpha_\lambda(-1, \xi^*_\pi(c)).$$

We now restrict our attention to computation of $\operatorname{tr}(\rho^* \,|\, V^\alpha_\lambda(+1, \xi^*_\pi))$ where

$$V^\alpha_\lambda(+1, \xi^*_\pi) = \bigcap_{c=1}^{\infty} V^\alpha_\lambda(+1, \xi^*_\pi(c)).$$

Note that the number of different spaces on the right is finite. Since $\xi^*_\pi(c) = 1$ on the space $V^\alpha_\lambda(+1, \xi^*_\pi)$ for all c, the operator $\xi^*_{\pi/2}(c)$ defines an involution on it, which commutes with ρ^*,

$$\rho^*\xi^*_{\pi/2}(c) = \xi^*_{-\pi}(c)\xi^*_{\pi/2}(c)\rho^* = \xi^*_{\pi/2}(c)\rho^* \quad \text{on} \quad V^\alpha_\lambda(+1, \xi^*_\pi).$$

The space $V_\lambda^\alpha(+1,\xi_\pi^*))$ can be split in the direct sum of (± 1)–eigenspaces of $\xi_{\pi/2}^*(c)$, so that

$$\operatorname{tr}(\rho^* \mid V_\lambda^\alpha(+1,\xi_\pi^*)) = \operatorname{tr}(\rho^* \mid V_\lambda^\alpha(+1,\xi_{\pi/2}^*(c))) + \operatorname{tr}(\rho^* \mid V_\lambda^\alpha(-1,\xi_{\pi/2}^*(c))).$$

Again, $\operatorname{tr}(\rho^* \mid V_\lambda^\alpha(-1,\xi_{\pi/2}^*(c))) = 0$ due to the existence of the following anticommutative diagram:

$$\begin{array}{ccc}
V_\lambda^\alpha(-1,\xi_{\pi/2}^*(c)) & \xrightarrow{\rho^*} & V_\lambda^\alpha(-1,\xi_{\pi/2}^*(c)) \\
{\scriptstyle \xi_{\pi/4}^*(c)}\downarrow & & {\scriptstyle \xi_{\pi/4}^*(c)}\downarrow \\
V_\lambda^\alpha(-1,\xi_{\pi/2}^*(c)) & \xrightarrow{\rho^*} & V_\lambda^\alpha(-1,\xi_{\pi/2}^*(c))
\end{array}$$

Its anticommutativity can be checked by using the formula (52),

$$\begin{aligned}
\rho^*\xi_{\pi/4}^*(c) &= \xi_{-\pi/2}^*(c)\xi_{\pi/4}^*(c)\rho^*, \\
&= \xi_{\pi/2}^*(c)\xi_{\pi/4}^*(c)\rho^*, \qquad \text{since } \xi_{\pi/2}^*(c) \text{ is an involution,} \\
&= -\xi_{\pi/4}^*(c)\rho^*.
\end{aligned}$$

It is clear that next we can restrict the trace computation to the space

$$V_\lambda^\alpha(+1,\xi_{\pi/2}^*) = \bigcap_{c=1}^{\infty} V_\lambda^\alpha(+1,\xi_{\pi/2}^*(c)),$$

then use $\xi_{\pi/4}^*(c)$ as involutions, etc. Induction by the powers of 2 will finally show that

$$\operatorname{tr}(\rho^* \mid V_\lambda^\alpha) = \operatorname{tr}(\rho^* \mid V_\lambda^\alpha(+1))$$

where

$$V_\lambda^\alpha(+1) = \bigcap_{k=0}^{\infty} V_\lambda^\alpha(+1,\xi_{\pi/2^k}^*) = \bigcap_{k=0}^{\infty}\bigcap_{c=1}^{\infty} V_\lambda^\alpha(+1,\xi_{\pi/2^k}^*(c)).$$

Since the sums $\sum a_k/2^k,\ a_k \in \mathbb{Z}$, are dense in $\mathbb{R}$, all 1–forms $\psi \in V_\lambda^\alpha(+1)$ are S^1–invariant,

$$t^*\,\psi = \psi \quad \text{for all } t \in S^1 \text{ and } c \geq 1, \tag{53}$$

where the action of t^* is given by the formula (51). On the other hand,

$$t^* \psi = \mathrm{Ad}_{\alpha(t)} \psi \quad \text{for all } t \in \mathbb{Z}/a \tag{54}$$

with the action $(z,w) \mapsto (\exp(2\pi i j/a)z, \exp(2\pi i j b/a)w)$. The equations (53) and (54) together imply, in particular, that any $\psi \in V_\lambda^\alpha(+1)$ must be a $(+1)$–eigenvector of the operators $\mathrm{Ad}_{\alpha(t)}$. Suppose that α is not trivial, then ψ must be an S^1–invariant (in the sense of (53)) real valued 1–form on S^3 such that $d^*\psi = 0$ and $*d\psi = \lambda\psi$. The trace of ρ^* on such forms is obviously independent of α and $L(a,b)$.

If $\alpha = \theta$ is a trivial representation, then V_λ^θ consists of $\mathfrak{su}(2)$–valued 1–forms which are S^1–invariant (in the sense of (53)) and belong to V_λ. Again, $\mathrm{tr}(\rho^* \mid V_\lambda^\theta)$ is independent of $L(a,b)$.

The 3–sphere S^3 admits an orientation reversing involution induced by complex conjugation on the second coordinate in $\mathbb{C}^2$. It induces an isomorphism of the spaces $V_\lambda^\alpha(+1)$ and $V_{-\lambda}^\alpha(+1)$ commuting with ρ^*. Therefore, $\mathrm{tr}(\rho^* \mid V_\lambda^\alpha(+1)) - \mathrm{tr}(\rho^* \mid V_{-\lambda}^\alpha(+1)) = 0$ for any α and $\lambda \neq 0$, and the function (48) vanishes identically. This completes the proof of Proposition 27.

6. The invariant λ^ρ via gauge theory

In this section, we complete the proof of our main theorem by proving the following result (and fixing the signs).

Proposition 29. *Let $\Sigma = \Sigma(p,q,r)$ be a Brieskorn homology sphere, and λ^ρ and χ^ρ the invariants defined by (17) and (31). Then $\lambda^\rho(\Sigma) = 1/2 \cdot \chi^\rho(\Sigma)$.*

This result will be obtained by a modification of the Taubes argument of [36] in our ρ–invariant setting.

6.1 Gauge theory and Heegaard splitting

Let $\Sigma = M_1 \cup_{M_0} M_2$ be a σ–invariant Heegaard splitting of Σ of genus g defined in (9). It is convenient to enlarge both M_1 and M_2 to overlap (still being σ–invariant) and think of the 3–manifold $M_0 = M_1 \cap M_2$ as a thickened Riemann surface, $M_0 = R_0 \times [-1/2, 1/2]$.

Let E be a vector $\mathrm{SU}(2)$–bundle over Σ with a fixed trivialization, and E_k the trivial $\mathrm{SU}(2)$–bundle obtained by restricting E onto M_k, $k = 0,1,2$. For each k, we introduced in (21) the space $\mathcal{A}_k^\rho$ of the ρ–invariant smooth connections on E_k. These spaces are considered as pre–Hilbert manifolds with the L^2_1–Sobolev space structure. The inclusions

$$M_0 \xrightarrow{j_1,j_2} M_1, M_2 \xrightarrow{i_1,i_2} \Sigma$$

induce by pull-back the maps

$$\mathcal{A}^\rho \xrightarrow{I} \mathcal{A}^\rho_1 \times \mathcal{A}^\rho_2 \xrightarrow{J} \mathcal{A}^\rho_0 \times \mathcal{A}^\rho_0, \tag{55}$$

where $I = i_1^* \times i_2^*$ and $J = j_1^* \times j_2^*$. The sequence (55) is exact in that I is an embedding, J is a submersion, and $\mathrm{im}\,(I) = J^{-1}(\Delta)$ where $\Delta \subset \mathcal{A}^\rho_0 \times \mathcal{A}^\rho_0$ is the diagonal. Each of the arrows in (55) is equivariant with respect to the corresponding group action, $\mathcal{G}^\rho_k$.

Let $\mathcal{A}^{0,\rho} \subset \mathcal{A}^\rho$ and $\mathcal{A}^{0,\rho}_k \subset \mathcal{A}^\rho_k$ be the subsets of connections which restrict to irreducible connections on $R_0 \times [-1/4, 1/4] \subset M_0$. In each case, $\mathcal{A}^{0,\rho} \subset \mathcal{A}^\rho$, $\mathcal{A}^{0,\rho}_k \subset \mathcal{A}^\rho_k$ are open and dense and are the complements of sets of infinite codimension. Introduce the quotients $\mathcal{B}^\rho$, $\mathcal{B}^\rho_k$, $\mathcal{B}^{0,\rho}$, and $\mathcal{B}^{0,\rho}_k$ by the respective gauge groups. With the L^2_1–Sobolev structure, each of $\mathcal{B}^{0,\rho}$, $\mathcal{B}^{0,\rho}_k$ is naturally a pre–Hilbert manifold, and $\mathcal{A}^{0,\rho} \to \mathcal{B}^{0,\rho}$, $\mathcal{A}^{0,\rho}_k \to \mathcal{B}^{0,\rho}_k$ are principal bundles with the structure groups $\mathcal{G}^\rho$ and $\mathcal{G}^\rho_k$, compare with [36], Proposition 4.1. The tangent space to $\mathcal{B}^{0,\rho}_k$ at an orbit $[A]$ is the vector space

$$\mathcal{T}_{kA} = \{\, \omega \in \Omega^1_\rho(M_k, \mathfrak{su}(2)) \mid d_A^*\omega = 0 \;\text{ and }\; i^*(*\omega) = 0 \,\},$$

where $i : \partial M_k \to M_k$ is the inclusion.

Let us define another vector bundle, $\mathcal{L}_k \to \mathcal{B}^{0,\rho}_k$, $k = 0, 1, 2$, with the fiber at an orbit $[A] \in \mathcal{B}^{0,\rho}_k$ the vector space

$$\mathcal{L}_{kA} = \{\, \omega \in \Omega^1_\rho(M_k, \mathfrak{su}(2)) \mid d_A^*\omega = 0 \,\}.$$

Give $\mathcal{L}_{kA}$ the structure of pre–Hilbert space by using the L^2–inner product on $\Omega^1_\rho(M_k, \mathfrak{su}(2))$. Note that the bundle $\mathcal{L}_k$ is different from the tangent bundle $\mathcal{T}_k$, and that the orbits of irreducible flat connections on E_k are in fact zeroes of a smooth section $\mathfrak{f}_k$ of $\mathcal{L}_k$. The section $\mathfrak{f}_k$ in question assigns $[\, A, *F_A \,] \in \mathcal{L}_{kA}$ to each $[A] \in \mathcal{B}^{0,\rho}_k$. The covariant derivative $\nabla \mathfrak{f}_k$ of $\mathfrak{f}_k$, $k = 0, 1, 2$, is the linear map

$$\begin{gathered} \nabla \mathfrak{f}_{kA} : \mathcal{T}_{kA} \to \mathcal{L}_{kA}, \\ a \mapsto (\nabla \mathfrak{f}_{kA})(a) = *d_A a - d_A u_k(a), \end{gathered} \tag{56}$$

where $u_k(a) \in \Omega^0_\rho(M_k, \mathfrak{su}(2))$ is the unique solution of the equation

$$d_A^* d_A u_k(a) = *(F_A \wedge a - a \wedge F_A), \quad i^* u_k(a) = 0;$$

compare with (30).

Proposition 30. *The operator $\nabla \mathfrak{f}_{kA}$ is a bounded operator from $\mathcal{T}_{kA}$ to $\mathcal{L}_{kA}$ when these spaces are completed in the Sobolev L^2_1 and L^2 norms, correspondingly. This operator is Fredholm of index $2g-1$ for $k = 1, 2$, and of index $4g - 2$ for $k = 0$.*

Proof. The assertion that $\nabla \mathfrak{f}_{kA}$ is Fredholm is a standard elliptic theory result. The index of $\nabla \mathfrak{f}_{kA}$ is independent of the connection A. We choose A to be the trivial connection θ, and reduce the calculations to equivariant de Rham theory. The index of $\nabla \mathfrak{f}_{kA}$ turns out to be equal to minus one half of the Euler characteristic of the cohomology $H^*_\rho(\partial M_k, \mathfrak{su}(2)) = H^*_\rho(M_k, d_\theta)$; compare with [36], Lemma A.2. The result now is a corollary of the following lemma. q.e.d.

Lemma 31. *The vector spaces $H^0_\rho(M_k, d_\theta)$, $k = 0, 1, 2$, are 1–dimensional. The vector spaces $H^1_\rho(M_1, d_\theta)$, $H^1_\rho(M_2, d_\theta)$ and $H^1_\rho(M_0, d_\theta)$ have dimensions $2g, 2g$ and $4g$, respectively.*

Proof. The first assertion can be proved by the same argument as in Lemma 19. We prove the second statement of the lemma for $H^1_\rho(M_0, d_\theta)$; for the other two spaces the proof is similar. Let us identify $H^1_\rho(M_0, d_\theta)$ with the space of ρ–invariant 1–cocycles on $\pi_1 M_0$, i.e., the maps $\xi : \pi_1 M_0 \to \mathfrak{su}(2)$ such that $\xi(x \cdot y) = \xi(x) + \xi(y)$ and $\xi(\sigma_* x) = \operatorname{Ad}_\rho \xi(x)$. Since the action of $\pi_1 M_0$ on $\mathfrak{su}(2)$ is trivial, all the coboundaries vanish, and the cocycles factor through $H_1(M_0)$.

Let us choose a basis $t_1, \dots, t_{2g}$ in $\pi_1 M_0$ such that

$$\sigma_* : \pi_1 M_0 \to \pi_1 M_0$$

acts by reversing the generators t_i, $\sigma_*(t_i) = t_i^{-1}$. The involution σ_* acts on the first homology group $H_1(M_0)$ as minus identity, and the 1–cocycles ξ are then simply the linear maps $\xi : H_1(M_0) \to \mathfrak{su}(2)$ such that $-\xi(t) = \operatorname{Ad}_\rho \xi(t)$. Since the (-1)–eigenspace of Ad_ρ acting on $\mathfrak{su}(2)$ is 2–dimensional, and $\dim H^1(M_0) = 2g$, we get that $\dim H^1_\rho(M_0, d_\theta) = 4g$. q.e.d.

We observe that $\mathcal{R}(\Sigma) \subset \mathcal{B}^{0,\rho}$ and $\mathcal{R}^\rho(M_k) \subset \mathcal{B}^{0,\rho}_k$, which means that irreducible flat connections A on E and E_k restrict to irreducible connections on $R_0 \times [-1/4, 1/4]$. The latter follows from the fact that the inclusions $R_0 \to M_k \to \Sigma$ induce epimorphisms on the fundamental groups, $\pi_1 R_0 \to \pi_1 M_k$ and $\pi_1 R_0 \to \pi_1 \Sigma$.

Proposition 32. *The representation space $\mathcal{R}^\rho(M_k)$ is an embedded submanifold in $\mathcal{B}_k^{0,\rho}$, $k = 0, 1, 2$.*

Proof. It is sufficient to show that $\dim \ker \nabla \mathfrak{f}_{kA} = \operatorname{index} \nabla \mathfrak{f}_{kA}$ for $[A] \in \mathcal{R}^\rho(M_k)$. Any $\omega \in \operatorname{coker} \nabla \mathfrak{f}_{kA}$ would satisfy the conditions $d_A \omega = 0, d_A^* \omega = 0, i^* \omega = 0$, and $\sigma^* \omega = \operatorname{Ad}_\rho \omega$. Thus, $[\omega] \in H^1_\rho(M_k, \partial M_k; d_A)$. Since $H^1_\rho(M_k, \partial M_k; d_A) \subset H^1(M_k, \partial M_k; d_A)$, which vanishes by Lemma 4.10 of [36], we get $[\omega] = 0$. Therefore, $\omega = d_A \mu$ for $\mu \in \Omega^0_\rho(M_k, \mathfrak{su}(2))$ with $d_A^* d_A \mu = 0$ and, as A is irreducible, $\mu = 0$ on ∂M_k. This means that both μ and ω vanish on M_k and hence $\operatorname{coker} \nabla \mathfrak{f}_{kA} = 0$. q.e.d.

The maps I and J in (55) commute with the actions of respective gauge groups. They induce the sequence

$$\mathcal{B}^{0,\rho} \xrightarrow{I} \mathcal{B}_1^{0,\rho} \times \mathcal{B}_2^{0,\rho} \xrightarrow{J} \mathcal{B}_0^{0,\rho} \times \mathcal{B}_0^{0,\rho}, \tag{57}$$

which is exact in the following sense (see [36], Lemma 4.2).

Lemma 33. *In the sequence (57), I is an embedding, J is a submersion, and* $\operatorname{im} I = J^{-1}(\Delta)$ *where $\Delta \subset \mathcal{B}_0^{0,\rho} \times \mathcal{B}_0^{0,\rho}$ is the diagonal.*

On the level of tangent spaces, (57) gives the following sequence of linear maps

$$0 \to \mathcal{T}_A \xrightarrow{\Phi} \mathcal{T}_{1,A_1} \times \mathcal{T}_{2,A_2} \xrightarrow{\Psi} \mathcal{T}_{0,A_0} \to 0, \tag{58}$$

where $(A_1, A_2) = I(A)$, $(A_0, A_0) = J(A_1, A_2)$, and $\Phi = I_*$, $\Psi = (j_1^*)_* - (j_2^*)_*$. The map I_* has the following explicit description: at $[A] \in \mathcal{B}^{0,\rho}$,

$$I_*(a) = (i_1^* a - d_{A_k} \phi_1(a),\ i_2^* a - d_{A_k} \phi_2(a)),$$

where $\phi_k(a) \in \Omega^0_\rho(M_k, \mathfrak{su}(2))$, $\alpha = 1, 2$, is the unique solution to the Neumann boundary problem

$$d_{A_k}^* d_{A_k} \phi_k(a) = 0 \quad \text{and} \quad * \, (i_k^* a - d_{A_k} \phi_k(a))|_{\partial M_k} = 0.$$

For $\alpha = 1, 2$, the map $(j_k^*)_*$ at $[A_k] \in \mathcal{B}_k^{0,\rho}$ is given by the formula

$$(j_k^*)_*(a_k) = j_k^* a_k - d_{A_0} \psi_k(a_k) \quad \text{where}$$
$$d_{A_0}^* d_{A_0} \psi_k(a_k) = 0 \quad \text{and} \quad * \, (j_k^* a_k - d_{A_0} \psi_k(a_k))|_{\partial M_0} = 0.$$

The inclusion maps in (55) also induce the following sequence

$$0 \to \mathcal{L}_A \xrightarrow{\Phi} \mathcal{L}_{1,A_1} \times \mathcal{L}_{2,A_2} \xrightarrow{\Psi} \mathcal{L}_{0,A_0} \to 0, \tag{59}$$

where $\underline{\Phi}(f) = (i_1^* f, i_2^* f)$ and $\underline{\Psi}(f_1, f_2) = j_1^* f_1 - j_2^* f_2$. The following statement is an easy modification of Lemmas 4.3 and 4.4 of [36].

Lemma 34. *Each of the sequences (58) and (59) is an exact Fredholm complex.*

6.2 A spectral interpretation of the intersection number

Let $[A]$, $[A'] \in \mathcal{B}^{0,\rho}$ be two points in $\mathfrak{f}^{-1}(0)$, and $j_1^* i_1^*[A] = [A_0]$ and $j_1^* i_1^*[A'] = [A'_0]$ be the corresponding points in $\mathcal{R}^\rho(M_1) \cap \mathcal{R}^\rho(M_2) \subset \mathcal{R}^\rho(M_0)$. A calculation of the relative sign difference between the intersection numbers of $\mathcal{R}^\rho(M_1) \cap \mathcal{R}^\rho(M_2)$ at the points $[A_0]$ and $[A'_0]$ can be made as follows.

Choose a path $\lambda_0 : [0,1] \to \mathcal{R}^\rho(M_0)$ of (the gauge equivalence classes of) ρ–invariant flat connections on M_0 between $[A_0]$ and $[A'_0]$. Since $\mathcal{R}^\rho(M_2)$ is path connected, we may assume that λ_0 extends to a path of ρ–invariant flat connections on M_2, that is, $\lambda_0 = j_2^* \lambda_2$ with $\lambda_2 : [0,1] \to \mathcal{R}^\rho(M_2)$. We can extend it to a path of ρ–invariant connections on the entire manifold Σ. These connections cannot be made flat on the entire Σ since $\mathcal{R}^\rho(\Sigma)$ is discrete. In other words, we construct a path $\lambda : [0,1] \to \mathcal{B}^{0,\rho}$ of ρ–invariant connections such that $\lambda_2 = i_2^* \lambda$, and $\lambda(0) = [A], \lambda(1) = [A']$. The path λ restricts on M_1 to a path which we call λ_1, so $\lambda_1 = i_1^* \lambda$.

The operators $\nabla \mathfrak{f}_0|_{\lambda_0}$ define a path of bounded Fredholm operators of index $4g-2$. The vector bundle $\ker(\nabla \mathfrak{f}_0|_{\lambda_0})$ can be identified with the restriction $\lambda_0^* T\mathcal{R}^\rho(M_0)$ of the tangent bundle of $\mathcal{R}^\rho(M_0)$ to the path λ_0. Similarly, the vector bundle $\ker(\nabla \mathfrak{f}_2|_{\lambda_2})$ is identified with $\lambda_2^* T\mathcal{R}^\rho(M_2)$.

Over the path $\lambda_1 : [0,1] \to \mathcal{B}_1^{0,\rho}$, the operators $\nabla \mathfrak{f}_1|_{\lambda_1}$ define a path of bounded Fredholm operators of index $2g-1$. At the endpoints of λ_1, the spaces $\ker(\nabla \mathfrak{f}_1|_{\lambda_1(t)})$, $t = 0, 1$, can be identified with the corresponding tangent spaces of $\mathcal{R}^\rho(M_1)$. This ensures that $\operatorname{coker}(\nabla \mathfrak{f}_1|_{\lambda_1(0)}) = 0$ and $\operatorname{coker}(\nabla \mathfrak{f}_1|_{\lambda_1(1)}) = 0$. If the cokernel of some $\nabla \mathfrak{f}_1|_{\lambda_1(t)}$ is not trivial, one can perturb the path of operators rel$\{0,1\}$ to get a path of bounded Fredholm operators of index $2g-1$ with trivial cokernels at each t; see Lemma 5.7 of [36]. Note that the orientation of the vector bundle $\ker(\nabla \mathfrak{f}_1|_{\lambda_1})$ at the endpoints $t = 0$ and $t = 1$ is consistent with the orientation on the corresponding tangent spaces to $\mathcal{R}^\rho(M_1)$. It can be checked by choosing a homotopy rel$\{\,0{,}1\,\}$ of the path λ_1 to a path which lies in $\mathcal{R}^\rho(M_1)$. Such a homotopy exists due to the fact that $\pi_1\,\mathcal{B}_1^{0,\rho} = \pi_0\,\mathcal{G}_1^\rho = [\,M_1, \mathrm{SU}(2)\,]_{\mathbb{Z}/2} = 1$.

Let $V \to [0,1]$ be the vector bundle

$$V = \ker(\nabla\mathfrak{f}_1|_{\lambda_1}) \oplus \ker(\nabla\mathfrak{f}_2|_{\lambda_2}),$$

and $h : V \to \lambda_0^* T\mathcal{R}^\rho(M_0)$ the L^2–orthogonal projection in $\lambda_0^*\mathcal{T}_0$ onto $\lambda_0^* T\mathcal{R}^\rho(M_0)$. Then h is a bundle homomorphism, and is an orientation preserving isomorphism $V \cong \lambda_0^*(T\mathcal{R}^\rho(M_1) \oplus T\mathcal{R}^\rho(M_2))$ at the endpoints. Let $\det h$ denote the corresponding section of the real line bundle

$$\Lambda^{4g-2}(\lambda_0^* T\mathcal{R}^\rho(M_0)) \otimes \Lambda^{4g-2}(V).$$

Note that $(\det h)^{-1}(0)$ is supported away from $\{\,0{,}1\,\}$. If $\det h$ is not transverse to the zero section, perturb h rel $\{\,0{,}1\,\}$ to h' with $\det h'$ being transverse. The intersection numbers at $[A_0]$ and $[A_0']$ coincide if and only if the cardinality of $(\det h')^{-1}(0)$ is even. Our next step is to represent the cardinality of $(\det h')^{-1}(0)$ modulo 2 as the spectral flow of a family of Fredholm operators.

Let $[A] \in \mathcal{B}^{0,\rho}$, $[A_k] = i_k^*[A]$, and $[A_0] = j_k^*[A_k]$ for $k = 1, 2$. Introduce the pre–Hilbert spaces

$$\mathcal{E}_A^0 = \mathcal{T}_{1,A_1} \oplus \mathcal{T}_{2,A_2} \oplus \mathcal{L}_{0,A_0}, \quad \mathcal{E}_A^1 = \mathcal{L}_{1,A_1} \oplus \mathcal{L}_{2,A_2} \oplus \mathcal{T}_{0,A_0},$$

and the operator

$$\begin{aligned} H_A : \mathcal{E}_A^0 &\to \mathcal{E}_A^1, \\ (a_1, a_2, u) &\mapsto (\nabla\mathfrak{f}_{1,A_1}(a_1),\ \nabla\mathfrak{f}_{2,A_2}(a_2), \\ &\qquad\qquad Y(a_1, a_2) - \nabla\mathfrak{f}_{0,A_0}^*(u)), \end{aligned} \tag{60}$$

where the adjoint operator, $\nabla\mathfrak{f}_{0,A_0}^* : \mathcal{L}_{0,A_0} \to \mathcal{T}_{0,A_0}$, is defined via integration by parts,

$$\langle \nabla\mathfrak{f}_{0,A_0}^*(u), a\rangle_{L^2} = \int_{\partial M_0} \mathrm{tr}(u \wedge a) - \int_{M_0} \mathrm{tr}(d_A u \wedge a),$$

and the identification $\mathcal{T}_{0,A_0}^* = \mathcal{T}_{0,A_0}$ is provided by the metric

$$\langle \psi, \psi' \rangle_{A_0} = \langle d_{A_0}\psi, d_{A_0}\psi' \rangle_{L^2} + \langle \psi, \psi' \rangle_{L^2}; \tag{61}$$

see [36, Appendix], for details. The map Y is the map

$$\Psi : \mathcal{T}_{1,A_1} \oplus \mathcal{T}_{2,A_2} \to \mathcal{T}_{0,A_0}$$

of (58) followed by the map $\mathcal{T}_{0,A_0} \to \mathcal{T}^*_{0,A_0}$ given by the L^2–pairing, and the map $\mathcal{T}^*_{0,A_0} \to \mathcal{T}_{0,A_0}$ given by the pairing in the metric (61). Thus the assignment of $Y(a_1, a_2) \in \mathcal{T}_{0,A_0}$ to a pair $(a_1, a_2) \in \mathcal{T}_{1,A_1} \oplus \mathcal{T}_{2,A_2}$ defines a compact operator. The operator H_A is a bounded Fredholm operator of index 0 since it is a compact perturbation of the Fredholm, index 0, operator

$$(a_1, a_2, u) \mapsto (\nabla\mathfrak{f}_{1,A_1}(a_1), \nabla\mathfrak{f}_{2,A_2}(a_2), -\nabla\mathfrak{f}^*_{0,A_0}(u));$$

see Proposition 30.

We now return to the section $\det h$. A pair

$$(a_1, a_2) \in \ker(\nabla\mathfrak{f}_{1,A_1}) \oplus \ker(\nabla\mathfrak{f}_{2,A_2})$$

belongs to $(\det h)^{-1}(0)$ if and only if the L^2–orthogonal projection of $\Psi(a_1, a_2)$ onto $\ker(\nabla\mathfrak{f}_{0,A_0})$ is zero, which is the same as to say that $Y(a_1, a_2) - \nabla\mathfrak{f}^*_{0,A_0}(u) = 0$ for some u. The latter means precisely that the triple (a_1, a_2, u) belongs to the kernel of H_A. The following proposition is now apparent.

Proposition 35. *Let $[A], [A'] \in \mathcal{B}^{0,\rho}$ be two points in $\mathfrak{f}^{-1}(0)$. The intersection numbers at $[A_0] = j_1^* i_1^*[A]$ and $[A_0'] = j_1^* i_1^*[A']$ in*

$$\mathcal{R}^\rho(M_1) \cap \mathcal{R}^\rho(M_2) \subset \mathcal{R}^\rho(M_0)$$

coincide if and only if the spectral flow of a path of operators $H_{A(t)}$ connecting H_A to $H_{A'}$ is even, $sf_H([A], [A']) = 0 \mod 2$.

6.3 Deformation of the H–family

The goal of this subsection is to prove that, for any pair of flat $[A]$, $[A'] \in \mathcal{B}^{0,\rho}$,

$$sf_H([A], [A']) = sf^\rho([A], [A']) \mod 2. \tag{62}$$

The left- and right-hand sides of (62) are, respectively, the spectral flows of the paths of operators $H_{A(t)}$ and $\nabla\mathfrak{f}_{A(t)}$; see (60) and (29). We will construct a homotopy of one path into the other keeping the spectral flow modulo 2 unchanged. To begin with, we compare the operators H_A and $\nabla\mathfrak{f}_A$ for A flat.

Lemma 36. *Let $[A] \in \mathcal{B}^{0,\rho}$ be a point in $\mathfrak{f}^{-1}(0)$; then*

$$\ker H_A = \ker(\nabla \mathfrak{f}_A) = 0.$$

Proof. The kernel of H_A consists of the triples

$$(a_1, a_2, u) \in \ker(\nabla \mathfrak{f}_{1,A_1}) \oplus \ker(\nabla \mathfrak{f}_{2,A_2}) \oplus \operatorname{coker}(\nabla \mathfrak{f}_{0,A_0})$$

such that the L^2–projection of $\Psi(a_1, a_2)$ onto $\ker(\nabla \mathfrak{f}_{0,A_0})$ vanishes. Suppose that $\mathfrak{f}(A) = 0$. Then

$$(\nabla \mathfrak{f}_{k,A_k})(a_k) = *d_{A_k} a_k \quad \text{for any} \quad k = 0, 1, 2;$$

see (56). If $(a_1, a_2) \in \ker(\nabla \mathfrak{f}_{1,A_1}) \oplus \ker(\nabla \mathfrak{f}_{2,A_2})$, then $\Psi(a_1, a_2)$ lies in $\ker(\nabla \mathfrak{f}_{0,A_0})$, because

$$\begin{aligned}
(\nabla \mathfrak{f}_{0,A_0})(\Psi(a_1, a_2)) &= *d_{A_0}(j_1^* a_1 - d_{A_0}\psi_1) - *d_{A_0}(j_2^* a_2 - d_{A_0}\psi_2), \\
&\quad \text{see (58)}, \\
&= j_1^*(*d_{A_1} a_1) - j_2^*(*d_{A_2} a_2) - *d_{A_0}^2 \psi_1 + *d_{A_0}^2 \psi_2, \\
&= 0.
\end{aligned}$$

Therefore,

$$\ker H_A = \{\, (a_1, a_2) \in \ker(\nabla \mathfrak{f}_{1,A_1}) \oplus \ker(\nabla \mathfrak{f}_{2,A_2}) \mid \Psi(a_1, a_2) = 0 \,\}.$$

Lemma 34 provides a unique $a \in \mathcal{T}_A$ such that $(a_1, a_2) = \Phi(a)$ and $a \in \ker(\nabla \mathfrak{f}_A)$. The latter can be seen as follows. Since $\Phi(a) = (a_1, a_2)$, we have that $a_k = i_k^* a - d_{A_k}\phi_k$ for $k = 1, 2$, and

$$\begin{aligned}
0 = *d_{A_k} a_k &= *d_{A_k}(i_k^* a - d_{A_k}\phi_k) \\
&= *d_{A_k}(i_k^* a) = i_k^*(*d_A a).
\end{aligned}$$

The form $(\nabla \mathfrak{f}_A)(a) = *d_A a$ restricts as zero to both M_1 and M_2, hence, $*d_A a = 0$. q.e.d.

For each $[A] \in \mathcal{B}^{0,\rho}$ we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathcal{T}_A \arrow[r, "\Phi"] \arrow[d, "\nabla \mathfrak{f}_A"] & \mathcal{T}_{A_1} \oplus \mathcal{T}_{A_2} \arrow[r, "\Psi"] \arrow[d, "\nabla \mathfrak{f}_{1,A_1} \oplus \nabla \mathfrak{f}_{2,A_2}"] & \mathcal{T}_{A_0} \arrow[r] \arrow[d, "\nabla \mathfrak{f}_{A_0}"] & 0 \arrow[d] \\
0 \arrow[r] & \mathcal{L}_A \arrow[r, "\underline{\Phi}"] & \mathcal{L}_{A_1} \oplus \mathcal{L}_{A_2} \arrow[r, "\underline{\Psi}"] & \mathcal{L}_{A_0} \arrow[r] & 0
\end{tikzcd}$$

The maps Φ, Ψ and $\underline{\Phi}, \underline{\Psi}$ have been defined, respectively, in (58) and (59). According to Lemma 34, the rows of this diagram are exact. The operators $\nabla \mathfrak{f}_A$ and $\nabla \mathfrak{f}_{k,A_k}$, $k = 0, 1, 2$, are Fredholm operators of indices $\operatorname{index} \nabla \mathfrak{f}_A = 0$, $\operatorname{index} \nabla \mathfrak{f}_{k,A_k} = 2g - 1$, $k = 1, 2$, and $\operatorname{index} \nabla \mathfrak{f}_{0,A_0} = 4g-2$; see Proposition 30. One can easily check right from the definitions that the diagram commutes if A is a flat connection. Unfortunately, this is not the case for a general A. However, the diagram can be made into commutative by compactly perturbing the operators $\nabla \mathfrak{f}_{1,A_1} \oplus \nabla \mathfrak{f}_{2,A_2}$ and $\nabla \mathfrak{f}_{0,A_0}$. More precisely, one can construct operators

$$\begin{aligned} &Q_{k,A_k} : \mathcal{T}_{1,A_1} \oplus \mathcal{T}_{2,A_2} \to \mathcal{L}_{1,A_1}, \ k = 1, 2, \quad \text{and} \\ &Q_{0,A_0} : \mathcal{T}_{0,A_0} \to \mathcal{L}_{0,A_0}, \end{aligned} \tag{63}$$

and the operator $Q_A : \mathcal{E}^0_A \to \mathcal{E}^1_A$,

$$Q_A(a_1, a_2, u) = (Q_{1,A_1}(a_1),\ Q_{2,A_2}(a_2),\ Y(a_1, a_2) - Q^*_{0,A_0}(u)) \tag{64}$$

with the following significance.

Lemma 37. *For every $[A] \in \mathcal{B}^{0,\rho}$, there exist bounded operators (63) such that the operator Q_A given by (64) is a Fredholm operator of index 0. Moreover, the following hold:*

1. *The operators $Q_A - H_A$, $(Q_{1,A_1} \oplus Q_{2,A_2}) - (\nabla \mathfrak{f}_{1,A_1} \oplus \nabla \mathfrak{f}_{2,A_2})$, and $Q_{0,A_0} - \nabla \mathfrak{f}_{0,A_0}$ are relatively compact operators; they vanish whenever A is flat.*

2. *The assignment of Q_A to A defines a smooth section, Q, over $\mathcal{B}^{0,\rho}$ of the Banach space bundle* $\operatorname{Fred}_0(\mathcal{E}^0, \mathcal{E}^1)$*, whose fiber over A consists of Fredholm operators $\mathcal{E}^0_A \to \mathcal{E}^1_A$ of index 0. The sections Q and H are homotopic rel $\mathfrak{f}^{-1}(0)$.*

3. *The following diagram is a commutative diagram of Fredholm bundle maps over $\mathcal{B}^{0,\rho}$,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T} & \xrightarrow{\Phi} & \mathcal{T}_1 \oplus \mathcal{T}_2 & \xrightarrow{\Psi} & \mathcal{T}_0 & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \nabla \mathfrak{f}} & & \downarrow{\scriptstyle Q_1 \oplus Q_2} & & \downarrow{\scriptstyle Q_0} & & \downarrow \\ 0 & \longrightarrow & \mathcal{L} & \xrightarrow{\underline{\Phi}} & \mathcal{L}_1 \oplus \mathcal{L}_2 & \xrightarrow{\underline{\Psi}} & \mathcal{L}_0 & \longrightarrow & 0 \end{array}$$

where the rows are exact, and the indices of both $Q_1 \oplus Q_2$ and Q_0 are $4g - 2$.

Proof. The entire construction of the operators Q_{k,A_k} in [36, pp. 574–577], works in the equivariant setting, so we omit the proof. q.e.d.

Proposition 38. *For any pair of flat connections* $[A],[A'] \in \mathcal{B}^{0,\rho}$, $sf_H([A],[A']) = sf^\rho([A],[A']) \mod 2$.

Proof. Let $A, A' \in \mathcal{A}^\rho$ be any pair of flat connections representing $[A],[A'] \in \mathcal{B}^{0,\rho}$. Choose a smooth path $[A(t)]$, $t \in [0,1]$, from $[A]$ to $[A']$. Since the paths $Q_{A(t)}$ and $H_{A(t)}$ are homotopic rel $\{0,1\}$, their spectral flows agree,

$$sf_H([A],[A']) = sf_Q([A],[A']) \mod 2.$$

We wish to show now that $sf_Q([A],[A']) = sf^\rho([A],[A']) \mod 2$. To this end, choose a splitting $\Theta : \mathcal{T}_0 \to \mathcal{T}_1 \oplus \mathcal{T}_2$ with the property that $\Psi \circ \Theta = 1$. Likewise, choose a splitting $\underline{\Theta} : \mathcal{L}_0 \to \mathcal{L}_1 \oplus \mathcal{L}_2$ with the property that $\underline{\Psi} \circ \underline{\Theta} = 1$. Since the rows in the commutative diagram above are exact, we get the commutative diagram

$$\begin{array}{ccc}
\mathcal{T} \oplus \mathcal{T}_0 & \xrightarrow{\Phi \oplus \Theta} & \mathcal{T}_1 \oplus \mathcal{T}_2 \\
\Big\downarrow {\scriptstyle \nabla\mathfrak{f} \oplus Q_0} & & \Big\downarrow {\scriptstyle Q_1 \oplus Q_2} \\
\mathcal{L} \oplus \mathcal{L}_0 & \xrightarrow{\underline{\Phi} \oplus \underline{\Theta}} & \mathcal{L}_1 \oplus \mathcal{L}_2
\end{array}$$

where $\Phi \oplus \Theta$ and $\underline{\Phi} \oplus \underline{\Theta}$ are isomorphisms.

Since $Q_A = H_A$ for any flat connection A, $\operatorname{coker} Q_0 = 0$ at the endpoints of the path $A(t)$. We may assume that in fact $\operatorname{coker} Q_{0,A_0(t)} = 0$ for all t; see Lemma 5.7 of [36]. Due to Proposition 30, $\ker Q_{0,A_0(t)} = \mathbb{R}^{4g-2}$, and we can consider the family of operators,

$$\nabla\mathfrak{f} \oplus Q_{0,A_0(t)} \oplus \Pi : \mathcal{T}_{A(t)} \oplus \mathcal{T}_{0,A_0(t)} \to \mathcal{L}_{A(t)} \oplus \mathcal{L}_{0,A_0(t)} \oplus \mathbb{R}^{4g-2},$$

where $\Pi : \mathcal{T}_{0,A_0(t)} \to \mathbb{R}^{4g-2}$ is the orthogonal projection to $\ker Q_{0,A_0(t)}$. Since $\ker(Q_{0,A_0(t)} \oplus \Pi) = 0$, the spectral flows of $\nabla\mathfrak{f}_{A(t)}$ and $\nabla\mathfrak{f} \oplus Q_{0,A_0(t)} \oplus \Pi$ coincide.

In its turn, the spectral flow of $\nabla\mathfrak{f} \oplus Q_{0,A_0(t)} \oplus \Pi$ equals the spectral flow of the path of operators $C(t)$ making the following diagram commutative,

$$\begin{array}{ccc}
\mathcal{T} \oplus \mathcal{T}_0 & \xrightarrow{\Phi \oplus \Theta} & \mathcal{T}_1 \oplus \mathcal{T}_2 \\
\Big\downarrow{\scriptstyle \nabla\mathfrak{f} \oplus Q_0} & & \Big\downarrow{\scriptstyle Q_1 \oplus Q_2} \\
\mathcal{L} \oplus \mathcal{L}_0 \oplus \mathbb{R}^{4g-2} & \xrightarrow{\Phi \oplus \Theta \oplus 1} & \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathbb{R}^{4g-2}
\end{array}$$

Thus, for any pair $(a_1, a_2) \in \mathcal{T}_{1,A_1(t)} \oplus \mathcal{T}_{2,A_2(t)}$, we have that

$$C(t)(a_1, a_2) = (Q_{1,A_1(t)}(a_1, a_2),\, Q_{2,A_2(t)}(a_1, a_2),\, \Pi(\Psi(a_1, a_2))).$$

This path of operators has the same spectral flow as the path $Q_{A(t)}$ defined by (64); compare with the proof of Proposition 35. q.e.d.

6.4 Comparing the signs

In this section we prove that the signs of $\chi^\rho(\Sigma)$ and $\lambda^\rho(\Sigma)$ agree. Recall that our definition of $\chi^\rho(\Sigma)$ made use of the symmetric bilinear form τ_a on $\mathfrak{su}^\rho(2)$ defined in (33) by the formula

$$\tau_a(\xi, \eta) = \int_\Sigma \operatorname{tr}([\xi, a] \wedge *[\eta, (*d_\theta)^{-1} a]).$$

Here, $a \in \ker d_\theta^*$ is an $\mathfrak{su}(2)$–valued 1–form on Σ. Since $*d_\theta : \ker d_\theta^* \to \ker d_\theta^*$ is invertible (due to the fact that Σ is an integral homology sphere), we may assume that $a = *d_\theta b,\ b \in \ker d_\theta^*$. Moreover, both ξ and η in $\mathfrak{su}^\rho(2)$ are proportional to ρ; see Lemma 19, so it is sufficient to compute $\tau_a(\xi, \eta)$ for $\xi = \eta = \rho$. In this case,

$$\tau_a(\rho, \rho) = \int_\Sigma \operatorname{tr}([b, \rho] \wedge *[a, \rho]), \quad a = *d_\theta b. \tag{65}$$

Our immediate goal is to construct such a that τ_a is non-degenerate. To do that, we use an equivariant Heegaard decomposition of Σ,

$$\Sigma = M_1 \cup M_2, \quad M_1 \cap M_2 = M_0 = R_0 \times [-1/2, 1/2],$$

where M_0 is a "thickened" Riemann surface. The Mayer–Vietoris isomorphism, $H^1(M_1) \oplus H^1(M_2) = H^1(M_0)$ induces an isomorphism in ρ–equivariant cohomology,

$$\Psi : H^1_\rho(M_1, d_\theta) \oplus H^1_\rho(M_2, d_\theta) \to H^1_\rho(M_0, d_\theta). \tag{66}$$

If we think of $H^1_\rho(M_k, d_\theta)$, $k = 0, 1, 2$, as the space of d_θ–harmonic ρ–invariant differential 1–forms ω on M_k such that $i^*_{\partial M_k}(*\omega) = 0$, the isomorphism Ψ is given by the formula

$$\Psi(v_1, v_2) = j_1^* v_1 - d_\theta \eta_1 - j_2^* v_2 + d_\theta \eta_2,$$

where η_1, η_2 are ρ–invariant harmonic $\mathfrak{su}(2)$–valued functions on M_0 such that

$$i^*_{\partial M_0} * (j_k^* v_k - d_\theta \eta_k) = 0, \;\; k = 1, 2.$$

The vector spaces $H^1_\rho(M_0, d_\theta)$ and $H^1_\rho(M_k, d_\theta)$, $k = 1, 2$, have dimensions $4g$ and $2g$, respectively; see Lemma 31.

A 1–form a making the form τ_a non–degenerate will be constructed after choosing elements

$$\omega_1 \in \Psi(H^1_\rho(M_1, d_\theta) \oplus 0)$$

and

$$\omega_2 \in \Psi(0 \oplus H^1_\rho(M_2, d_\theta)).$$

Write $\Psi^{-1}(\omega_1 - \omega_2) = (\alpha_1, \alpha_2)$; then α_1 restricts to M_0 as $\omega_1 + d_\theta \eta_1$, and α_2 restricts to M_0 as $\omega_2 + d_\theta \eta_2$, with η_1 and η_2 harmonic. Set

$$a = *d_\theta b \quad \text{where} \quad b = (1 - \beta)\,\alpha_1 + \beta\,\alpha_2 \tag{67}$$

and $\beta : [-1/2, 1/2] \to [0, 1]$ is a cut-off function such that $\beta = 0$ on $[-1/2, -1/4]$ and $\beta = 1$ on $[1/4, 1/2]$. Then

$$a = *d_\theta b = - * \beta' \,(dt \wedge (\alpha_1 - \alpha_2)),$$

and

$$\begin{aligned}
\tau_a(\rho, \rho) &= -\int_{M_0} \operatorname{tr}([\rho, (1-\beta)\alpha_1 + \beta\alpha_2] \wedge \beta'(dt \wedge [\rho, \alpha_1 - \alpha_2])) \\
&= \int_{M_0} \beta' \cdot dt \wedge \operatorname{tr}([\rho, \alpha_1] \wedge [\rho, \alpha_2]) \\
&= \int_{M_0} dt \wedge \operatorname{tr}([\rho, \omega_1] \wedge [\rho, \omega_2]).
\end{aligned}$$

The last expression is positive if $\omega_2 = -\omega_1^*$, where ω_1^* is the dual of ω_1 with respect to the non–degenerate symplectic pairing on $H^1_\rho(M_0, d_\theta)$,

$$\langle v, v' \rangle = -\int_{M_0} dt \wedge \operatorname{tr}(v \wedge v'),$$

and $\omega_1 \in H^1_\rho(M_1, d_\theta)$ is chosen in general position.

Thus, to prove that the signs of $\lambda^\rho(\Sigma)$ and $\chi^\rho(\Sigma)$ agree we need to check that the orientations of the vector spaces $\ker(\nabla\mathfrak{f}_{1,\theta+s\cdot a}) \oplus \ker(\nabla\mathfrak{f}_{2,\theta+s\cdot a})$ and $\ker(\nabla\mathfrak{f}_{0,\theta+s\cdot a})$ agree for a chosen as in (67) and all $s > 0$ sufficiently small. The vector spaces in question are oriented as follows. When a connection $A \in \mathcal{A}^\rho$ restricts to one or more of M_k, $k = 0, 1, 2$, so that $[A] \in \mathcal{R}^\rho(M_k)$, then $\ker(\nabla\mathfrak{f}_{k,\theta+s\cdot a})$ is naturally identified with $T_{[A]}\mathcal{R}^\rho(M_k)$. Following a generic path from A to $\theta + s\cdot a$, we orient $\ker(\nabla\mathfrak{f}_{k,\theta+s\cdot a})$.

This method for computing the orientations actually works because θ is a limit point of each of $\mathcal{R}^\rho(M_k)$, $k = 0, 1, 2$, hence one can find connections A of the form $\theta + s\cdot a'$, for small s, which restrict to one of M_k so that $[A] \in \mathcal{R}^\rho(M_k)$. The kernels of $\nabla\mathfrak{f}_{k,\theta+s\cdot a'}$ for small s can be controlled by identifying them with the kernels of certain homomorphisms independent of s. We first consider the general case with a' unrestricted.

Lemma 39. *Let $a' \in \Omega^1_\rho(\Sigma, \mathfrak{su}(2))$ be fixed. Introduce the homomorphisms $G_k : H^1_\rho(M_k, d_\theta) \to \mathfrak{su}^\rho(2)^*$, by the formula*

$$G_k(v)(\xi) = -\int_{M_k} \operatorname{tr}([\xi, a'] \wedge *v), \quad k = 1, 2.$$

If G_k are surjective, then for all $s > 0$ sufficiently small,

$$\ker(\nabla\mathfrak{f}_{k,\theta+s\cdot a'}) \cong \ker G_k \subset H^1_\rho(M_k, d_\theta), \quad k = 1, 2.$$

Proof. Let us fix $k = 1$; the proof for $k = 2$ is completely analogous. Let us consider the linear Fredholm operator

$$L_0 = \begin{pmatrix} 0 & d_\theta^* \\ d_\theta & *d_\theta \end{pmatrix} : E_1 \to E_2,$$

where E_1 and E_2 are the following Hilbert spaces:

$$E_1 = \{ (\phi_0, \phi_1) \in L^2_1(\Omega^0_\rho(M_1, \mathfrak{su}(2))) \oplus L^2_1(\Omega^1_\rho(M_1, \mathfrak{su}(2))) \mid \\ i^*_{\partial M_1}(\phi_0) = 0,\ i^*_{\partial M_1}(*\phi_1) = 0 \},$$
$$E_2 = L^2(\Omega^0_\rho(M_1, \mathfrak{su}(2))) \oplus L^2(\Omega^1_\rho(M_1, \mathfrak{su}(2))).$$

The kernel of $\nabla\mathfrak{f}_{1,\theta+s\cdot a'}$ can be identified with the kernel of the perturbed operator, $L_0 + s\cdot u$, where u is a compact operator defined from the equation

$$L_0 + s \cdot u = \begin{pmatrix} 0 & d^*_{\theta+s\cdot a'} \\ d_{\theta+s\cdot a'} & *d_{\theta+s\cdot a'} \end{pmatrix}. \tag{68}$$

The kernel of $L_0 + s \cdot u$ can be computed by using techniques from linear perturbation theory. Namely, let $h[u]\,:\,\ker L_0 \to \operatorname{coker} L_0$ be a linear map sending $v \in \ker L_0$ to the orthogonal projection of $u(v)$ to $\operatorname{coker} L_0$. Suppose that this projection is surjective. Then, for all s sufficiently small, the kernel of $L_0 + s \cdot u$ is isomorphic to $h[u]^{-1}(0) \subset \ker L_0$. The latter isomorphism is obtained as follows: for each $v \in \ker(L_0 + s \cdot u)$, there exists a unique $w \in h[u]^{-1}(0) \subset \ker L_0$ such that $v - w$ is L^2–orthogonal in E_1 to $\ker L_0$.

It follows from de Rham theory that

$$\ker L_0 = H^1_\rho(M_1, d_\theta)$$

and

$$\operatorname{coker} L_0 = H^0_\rho(M_1, d_\theta) = \mathfrak{su}^\rho(2).$$

According to (68), the map

$$h[u]\,:\,H^1_\rho(M_1, d_\theta) \to H^0_\rho(M_1, d_\theta)$$

sends $\phi_1 \in H^1_\rho(M_1, d_\theta)$ to the orthogonal projection of $- * [\,a, *\phi_1\,]$ to $H^0_\rho(M_1, d_\theta)$. Therefore, $\phi_1 \in h[u]^{-1}(0)$ if and only if $- * [\,a, *\phi_1\,]$ is orthogonal to $\ker d_\theta = \mathfrak{su}^\rho(2)$, i.e., for any $\xi \in \mathfrak{su}^\rho(2)$,

$$\begin{aligned} 0 &= -\int_{M_1} \operatorname{tr}(\xi \cdot [a, *\phi_1]) \\ &= -\int_{M_1} \operatorname{tr}([\xi, a] \wedge *\phi_1) \\ &= \quad G_1(\phi_1)(\xi). \end{aligned}$$

Thus, if G_1 is surjective, the perturbation theory identifies $\ker \nabla \mathfrak{f}_{1,\theta+s\cdot a'}$ with $\ker G_1$. q.e.d.

A similar argument proves the following result.

Lemma 40. *Let $a' \in \Omega^1_\rho(\Sigma, \mathfrak{su}(2))$ be fixed. Introduce the homomorphisms $G_0, G_0'\,:\,H^1_\rho(M_0, \mathfrak{su}(2)) \to \mathfrak{su}^\rho(2)^*$ defined by the formulas*

$$G_0(v)(\xi) = -\int_{M_0} \operatorname{tr}([\xi, a'] \wedge *v) \quad \textit{and}$$

$$G_0'(v)(\xi) = -\int_{M_0} dt \wedge \operatorname{tr}([\xi, a'] \wedge *v).$$

Suppose that both G_0 and G_0' are surjective. Then for all s sufficiently small,

$$\ker(\nabla \mathfrak{f}_{0,\theta+s\cdot a'}) = \ker G_0' \oplus \ker G_0 \subset H^1_\rho(M_0, d_\theta). \tag{69}$$

Orient S^2_ρ and S^1_ρ so that when $\theta + s \cdot a'$ restricts to M_1 to lie in $\mathcal{R}^\rho(M_1)$, then the orientation of $\ker(\nabla \mathfrak{f}_{1,\theta+s\cdot a'}) = H^1_\rho(M_1, d_\theta)/\operatorname{im} G_1^*$ agrees with the orientation which is obtained by identifying $\ker(\nabla \mathfrak{f}_{1,\theta+s\cdot a'})$ with $T_{[\theta+s\cdot a']}\mathcal{R}^\rho(M_1)$. Thus, whenever $\theta + s \cdot a'$ restricts to M_2 to lie in $\mathcal{R}^\rho(M_2)$, the orientation of $\ker(\nabla \mathfrak{f}_{2,\theta+s\cdot a'}) = H^1_\rho(M_2, d_\theta)/\operatorname{im} G_2^*$ agrees with the orientation of $\ker(\nabla \mathfrak{f}_{2,\theta+s\cdot a'}) = T_{[\theta+s\cdot a']}\mathcal{R}^\rho(M_2)$. Likewise, when $\theta + s \cdot a'$ restricts to a flat connection in $\mathcal{R}^\rho(M_0)$, the orientations of $H^1_\rho(M_0, d_\theta)/\operatorname{im}(G_0' \oplus G_0)^*$ and $T_{[\theta+s\cdot a']}\mathcal{R}^\rho(M_0)$ agree.

Hence, as a' varies in $\Omega^1_\rho(\Sigma, \mathfrak{su}(2))$ keeping $G_0' \oplus G_0$ and G_1, G_2 surjective, the orientations on $\ker(\nabla \mathfrak{f}_{k,\theta+s\cdot a'})$, $k = 0, 1, 2$, are determined by the maps $G_0' \oplus G_0$ and G_1, G_2. We need to prove now that these orientations agree if $a' = a$ as in (67) with $\omega_2 = -\omega_1^*$.

With the orientations of S^2_ρ and S^1_ρ fixed as above, the isomorphism

$$\Psi : H^1(M_1, d_\theta) \oplus H^1(M_2, d_\theta) \to H^1(M_0, d_\theta)$$

is orientation preserving because $\Psi = \Phi \oplus 1$ where

$$\Phi : H^1(M_1) \oplus H^1(M_2) \to H^1(M_0)$$

is the usual Mayer–Vietoris isomorphism, and $1 : T_1 S^2_\rho \to T_1 S^2_\rho$ is the identity homomorphism. We use the isomorphism Ψ to make the following identifications,

$$H^1(M_1, d_\theta) \oplus H^1(M_2, d_\theta) = h_1 \oplus h_2 \quad \text{and} \quad H^1(M_0, d_\theta) = h_1 \oplus h_2$$

where $h_1 = \Psi(H^1(M_1, d_\theta) \oplus 0)$ and $h_2 = \Psi(0 \oplus H^1(M_2, d_\theta))$. The homomorphisms G_1, G_2, and $G_0' \oplus G_0$ are then given by the following formulas (possibly after some generic deformations preserving the orientations; compare with [36, Lemma 7.3 and Lemma 7.5]),

$$\begin{gathered} G_1 \oplus G_2,\ G_0' \oplus G_0 : h_1 \oplus h_2 \to \mathfrak{su}^\rho(2)^* \oplus \mathfrak{su}^\rho(2)^* \\ (G_1 \oplus G_2)(v_1, v_2) = (G_1(v_1), G_2(v_2)), \quad \text{and} \\ (G_0' \oplus G_0)(v_1, v_2) = (G_1(v_1) + G_2(v_2), G_1(v_1) - G_2(v_2)). \end{gathered}$$

Therefore, $\ker(G_0' \oplus G_0)$ and $\ker(G_1 \oplus G_2)$ have opposite orientations if oriented by the requirement that the following isomorphisms be orientation preserving,

$$\mathfrak{su}^\rho(2)^* \oplus \mathfrak{su}^\rho(2)^* \oplus \ker G_1 \oplus \ker G_2 = h_1 \oplus h_2,$$
$$\mathfrak{su}^\rho(2)^* \oplus \mathfrak{su}^\rho(2)^* \oplus \ker(G_0' \oplus G_0) = h_1 \oplus h_2.$$

The orientation on $\ker G_1 \oplus \ker G_2$ was chosen, however to make the isomorphism

$$\mathfrak{su}^\rho(2)^* \oplus \ker G_1 \oplus \mathfrak{su}^\rho(2)^* \oplus \ker G_2 = h_1 \oplus h_2$$

orientation preserving. Since $\dim \ker G_1 = 2g - 1$ is odd, it implies that $\Psi : \ker G_1 \oplus \ker G_2 \to \ker(G_0' \oplus G_0)$ is orientation preserving.

6.5 Proof of the main theorem

It follows from Corollary 16, Corollary 26 and Proposition 29 that

$$\nu(\Sigma(p,q,r)) = \pm\frac{1}{8}\,\operatorname{sign} k(p,q,r) \tag{70}$$

with a universal $(\pm)$–sign. To fix the sign, we compute the left- and the right-hand sides for $(p,q,r) = (2,3,7)$. According to [13], the group $I_n(\Sigma(2,3,7))$ is $\mathbb{Z}$ if $n = 2, 6$, and is trivial otherwise. Therefore, $\nu(\Sigma(2,3,7)) = 1$. On the other hand, standard methods show that $\operatorname{sign} k(2,3,7) = 8$; see Figure 10. Thus the sign in (70) is plus. This completes the proof.

7. Application to the $\bar{\mu}$–invariant

In this section we define the invariants of Neumann and Siebenmann for integral homology 3–spheres of plumbing type and prove that they coincide. Their common value is usually referred to as the $\bar{\mu}$–invariant. It follows right from Siebenmann's definition of the $\bar{\mu}$–invariant and Theorem 1 that $\bar{\mu}(\Sigma) = \nu(\Sigma)$ for all Brieskorn homology spheres Σ. The question remains whether the same is true for arbitrary plumbed integral homology spheres. Note that our definitions of both Neumann's and Siebenmann's invariants differ from the original ones by a factor of $1/8$.

In should be also mentioned that one can define both Neumann's and Siebenmann's invariants for arbitrary $\mathbb{Z}/2$–homology 3–spheres of plumbing type, and prove that they coincide by exactly the same methods as the ones used in this section.

7.1 Definition of Neumann's $\bar{\mu}$–invariant

The principal S^1–bundles over S^2 are classified by the Euler class $e \in H^2(S^2;\mathbb{Z}) = \mathbb{Z}$. Let the Hopf fibration of S^3 over S^2 be represented by $e = -1$. This fixes the orientations. Denote the associated D^2–bundles indexed by $e \in \mathbb{Z}$ as $Y(e)$.

Let Γ be a connected plumbing graph, that is a connected graph with no cycles, each of whose vertices v_i carries an integer weight e_i, $i = 1, \dots, s$. Associated to each vertex v_i is the D^2–bundle $Y(e_i)$. If the vertex v_i has d_i edges connected to it in the graph Γ, we choose d_i disjoint discs in the base of $Y(e_i)$ and call the disc bundle over the j–th disc B_{ij}, so that $B_{ij} = D^2 \times D^2$. When two vertices, v_i and v_k, are connected by an edge, we identify B_{ij} with $B_{k\ell}$ by exchanging the base and fiber coordinates. This pasting operation is called plumbing, and the resulting smooth 4–manifold $P(\Gamma)$ is said to be obtained by plumbing according to Γ.

The manifold $P(\Gamma)$ is simply connected but, generally speaking, not parallelizable. Its intersection form in the natural basis associated to plumbing (namely, the basis represented by the zero-sections of the plumbed bundles) is given by the matrix $A(\Gamma) = (a_{ij})_{i,j=1,\dots,s}$ with the entries

$$a_{ij} = \begin{cases} e_i, & \text{if } i = j; \\ 1, & \text{if } i \text{ is connected to } j \text{ by an edge}; \\ 0, & \text{otherwise.} \end{cases} \tag{71}$$

Disconnected graphs are also allowed. Namely, if $\Gamma = \Gamma_0 + \Gamma_1$ is a disjoint union of Γ_0 and Γ_1, then $P(\Gamma)$ is the boundary connected sum $P(\Gamma_0) \natural P(\Gamma_1)$.

If Γ is a plumbing graph as above, then $\Sigma(\Gamma) = \partial P(\Gamma)$ is an integral homology sphere if and only if $\det A(\Gamma) = \pm 1$, in which case it is called a *homology sphere of plumbing type* or a *plumbed homology sphere*. Any Seifert fibered homology sphere is of plumbing type, with a star-shaped plumbing graph; see [27]. For any plumbed homology sphere $\Sigma(\Gamma)$, there exists a unique homology class $w \in H_2(P(\Gamma);\mathbb{Z})$ satisfying the following two conditions. First, w is *characteristic*, that is (dot represents intersection number)

$$w.x = x.x \mod 2 \quad \text{for all } x \in H_2(P(\Gamma);\mathbb{Z}), \tag{72}$$

and second, all the coordinates of w are either 0 or 1 in the natural basis of $H_2(P(\Gamma);\mathbb{Z})$. We call w *the integral Wu class* for $P(\Gamma)$. It is proved in [25] that the integer

$$\bar{\mu}(\Sigma) = \frac{1}{8}\,(\operatorname{sign} P(\Gamma) - w.w) \tag{73}$$

only depends on $\Sigma(\Gamma)$ and not on Γ. We call it *the Neumann $\bar{\mu}$–invariant.* One can easily see that condition (72) implies that for no two adjacent vertices in Γ the corresponding coordinates of w are both equal to 1. Therefore, the class w can be chosen to be spherical, hence $\bar{\mu}(\Sigma) = \mu(\Sigma) \mod 2$ where $\mu(\Sigma)$ is the Rohlin invariant.

Another description of the $\bar{\mu}$–invariant is as follows. In the natural basis of $H_2(P(\Gamma);\mathbb{Z})$, the defining property (72) translates to the linear system

$$\sum_{i=1}^{s} a_{ij}\varepsilon_j = a_{ii} \mod 2, \quad i = 1,\dots,s. \tag{74}$$

Due to the fact that $\det A(\Gamma) = 1 \mod 2$, this system has a unique solution $(\varepsilon_1,\dots,\varepsilon_s)$ over $\mathbb{Z}/2$. The $\bar{\mu}$–invariant is then given by the formula

$$\bar{\mu}(\Sigma) = \frac{1}{8}\left(\operatorname{sign} P(\Gamma) - \sum_{\varepsilon_i=1} a_{ii}\right).$$

The $\bar{\mu}$–invariant is easy to compute. The splicing additivity; see [12] and [31], reduces the problem to the computation of $\bar{\mu}(\Sigma(a_1,\dots,a_n))$. There is an explicit formula for the latter; see [25] and [27], Theorem 6.2. Namely, in the following two cases,

(1) all a_i are odd;

(2) exactly one of the a_i is even, say a_1,

the $\bar{\mu}$-invariant is given respectively by

$$\bar{\mu}(\Sigma(a_1,\dots,a_n) = \sum_{i=1}^{n}(c(a_i,b_i) + \operatorname{sign} b_i) - 1, \tag{75}$$

$$\text{and}\ \ \bar{\mu}(\Sigma(a_1,\dots,a_n) = \sum_{i=1}^{n} c(a_i - b_i, a_i) - 1, \tag{76}$$

where $c(a,b)$ is the function described below and, in the second case, we are to choose the Seifert invariants so that $(a_i - b_i)$ is odd for all i.

The integers $c(a,b)$ are defined for coprime integer pairs (a,b) with a odd. They are uniquely determined by the recursion:

$$\begin{aligned} &c(a,\pm 1) = 0 \text{ for any odd } a,\\ &c(a \pm 2b, b) = c(a,b),\\ &c(a, b+a) = c(a,b) + \operatorname{sign} b(b+a),\\ &c(a,b) = -c(-a,b) = -c(a,-b). \end{aligned}$$

Several equivalent definitions of $c(a,b)$ are available; see [27]. For example,

$$\begin{aligned} c(a,b) &= \sum (-1)^i \,\#\, \{ \, k \mid 0 < k < b \, , \; i < \frac{ka}{b} < i+1 \, \} \\ &= -\frac{1}{b} \sum_{\eta^b = -1} \frac{(\eta+1)(\eta^a+1)}{(\eta-1)(\eta^a-1)} \, , \\ &= b - 1 - 4 \cdot \#\{ \, 1 \le i \le \frac{b-1}{2} \mid \frac{b-1}{2} < ai < b \mod b \, \}, \; b \text{ odd.} \end{aligned}$$

7.2 Preferred involutions on plumbed manifolds

J. Montesinos observed in [23] that on any homology 3–sphere Σ of plumbing type, there exists a preferred class I_Σ of orientation preserving involutions such that, for $\sigma \in I_\Sigma$, the orbit space Σ/σ is S^3 and the fixed point set is a knot k_σ in S^3, which is called a knot of plumbing type. The goal of this section is to describe the involutions $\sigma \in I_\Sigma$.

Let D^2 be the unit disc in $\mathbb{C}$,

$$D^2 = \{ \, z \in \mathbb{C} \mid |z| \le 1 \, \},$$

and

$$\sigma : D^2 \times D^2 \to D^2 \times D^2$$

the complex conjugation involution, $(z,w) \mapsto (\bar z, \bar w)$. The boundary S^3 of $D^2 \times D^2$ can be represented by stereographic projection onto $\mathbb{R}^3 \cup \infty$ in such a way that σ induces on it the 180°-rotation about the y–axis. In this representation $\partial D^2 \times D^2$ is a regular neighborhood X of the unit circle in the xy–plane, the belt–sphere is the z–axis, and the belt–tube is $Y = S^3 \setminus \operatorname{int} X$.

The fixed point set of the involution σ on $D^2 \times D^2$ is the unit disc $B = D^1 \times D^1 \subset D^2 \times D^2$. The orbit space of σ is a 4–dimensional ball

D^4, so that the projection $p : D^2 \times D^2 \to D^4$ is a 2–fold branched cover of D^4 with the branch set the 2–disc $B' = p(B)$. The restriction

$$p\,|_{\partial(D^2\times D^2)}\colon\ \partial(D^2 \times D^2) \to S^3$$

is the standard double covering over the trivial knot $\partial B' \subset S^3$.

The branch set $B' \subset D^4$ can be described as follows. Let $S^3 = X \cup Y$ as above. The restriction of p onto X turns X into a double branched cover of the 3–ball $X' = X/\sigma$ with the branch set consisting of the arcs I and I' shown on the left in Figure 8.

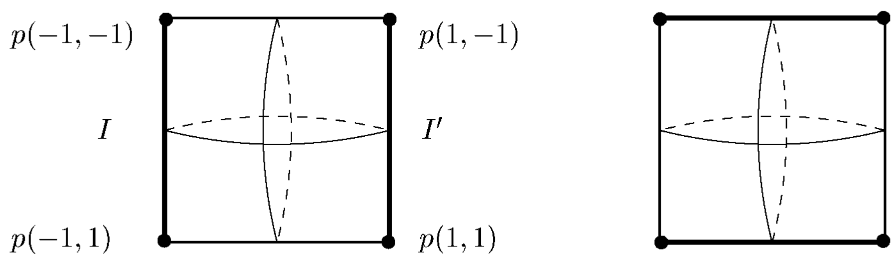

FIGURE 8

These two arcs represent the intersection $B' \cap X' \subset S^3$. Similarly, $Y' = Y/\sigma$ is a 3–ball with the branch set the arcs shown in Figure 8 on the right. The 3–balls X' and Y' are glued together to form S^3, the arcs in X' and Y' being glued along their endpoints into the trivial knot $\partial B' \subset S^3$. Now, let B'' be a 2–disc in S^3 with $\partial B'' = \partial B'$. Then the branch set $B' \subset D^4$ can be viewed as a 2–disc obtained by pushing the interior of B'' radially into D^4 so that $B' \cap S^3 = \partial B' = \partial B''$.

The complex conjugation involution on $D^2 \times D^2$ can be extended over the 2–disc bundles $Y(e)$ over S^2, which are the building blocks for the plumbed manifolds. The 4–manifold $Y(e)$ can be obtained by pasting two copies of $D^2 \times D^2$,

$$Y(e) = D^2 \times D^2 \cup_{h_e} D^2 \times D^2,$$

along $\partial D^2 \times D^2$ where $h_e : \partial D^2 \times D^2 \to \partial D^2 \times D^2$ is given by the formula

$$(e^{i\phi}, \lambda e^{i\psi}) \mapsto (e^{-i\phi}, \lambda e^{i(\psi - e\phi)}), \quad 0 \le \phi, \psi < 2\pi,\ 0 \le \lambda \le 1.$$

The complex conjugation involution σ on $D^2 \times D^2$ obviously commutes with the map h_e, so σ can be extended to an involution (also denoted by σ) on $Y(e)$. The quotient $Y(e)/\sigma$ is glued from two 4–balls along a 3–ball, so that $Y(e)/\sigma = D^4$. The branch set in D^4 can be described as follows.

Let us represent $Y(e)/\sigma = D^4$ as

$$D^4 = D^4 \cup_{D^3} \mathrm{Cyl}\,(\bar{h}_e) \cup_{D^3} D^4,$$

where $\mathrm{Cyl}\,(\bar{h}_e)$ is the mapping cylinder of the map $\bar{h}_e : D^3 \to D^3$ induced by h_e on the quotient of $\partial D^2 \times D^2$. The branch set in $Y(e)/\sigma$ then consists of four bands glued together; see Figure 9 : two untwisted bands in the two copies of D^4, and two more bands in $\mathrm{Cyl}\,(\bar{h}_e)$ generated by the arcs I' and I; see Figure 8. One can easily see that the former is untwisted while the latter one is twisted e half–twists in the right–hand direction (in case e is positive).

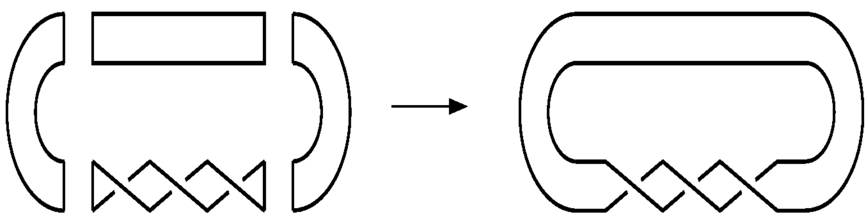

FIGURE 9

Let $P(\Gamma)$ be a plumbed manifold obtained by plumbing D^2–bundles $Y(e_i)$ according to a connected plumbing tree Γ. The involutions σ on the manifolds $Y(e_i)$ fit together to form an involution on $P(\Gamma)$ whose orbit space is D^4, and the branch set $B(\Gamma)$ is obtained by plumbing twisted bands (like the one in Figure 9) according to the tree Γ; see Figure 10 and [34].

In particular, this construction represents the boundary Σ of a plumbed manifold $P(\Gamma)$ as a double branched cover of S^3 branched over the boundary of the surface $B(\Gamma)$. It follows from the Smith theorem that the fixed point set of an involution acting on a $\mathbb{Z}/2$–homology sphere is itself a $\mathbb{Z}/2$–homology sphere. Therefore, if Σ is an integral homology sphere, the boundary of the surface $B(\Gamma)$ is a knot, which we denote by k_σ.

In the special case of Seifert fibered integral homology spheres $\Sigma(a_1, \dots, a_n)$, one can do plumbing holomorphically to obtain $P(\Gamma)$ as a complex manifold with holomorphic $\mathbb{C}^*$–action, see [28], representing a resolution of the singularity of $f^{-1}(0)$ where $f : \mathbb{C}^n \to \mathbb{C}^{n-2}$ is a

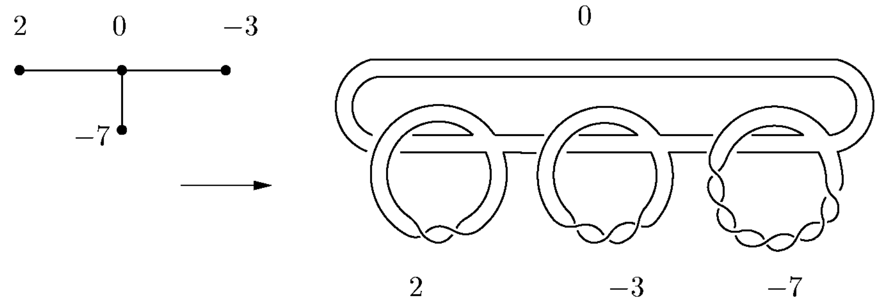

FIGURE 10

map of the form

$$f(z_1, \dots, z_n) = \left(\sum_{k=1}^{n} b_{1,k} z_k^{a_k}, \dots, \sum_{k=1}^{n} b_{n-2,k} z_k^{a_k} \right) \tag{77}$$

with sufficiently general coefficient matrix $(b_{i,j})$, $b_{i,j} \in \mathbb{R}$; see [27]. The involution σ on $P(\Gamma)$ will then be the complex conjugation involution. Therefore, any Seifert fibered homology sphere $\Sigma(a_1, \dots, a_n)$ is a double branched cover of S^3 with the branch set the knot $k_\sigma = k(a_1, \dots, a_n)$, which is usually called a Montesinos knot; see [23]. A Brieskorn homology sphere $\Sigma(p, q, r)$ is a Seifert fibered homology sphere with $n = 3$ fibers, and it is a double branched cover of S^3 with branch set $k(p, q, r)$.

7.3 Definition of Siebenmann's $\tilde{\rho}$–invariant

Let Σ be a $\mathbb{Z}/2$–homology sphere of plumbing type, and σ an involution on Σ from the preferred class I_Σ of involutions described in the previous section. Its orbit space Σ/σ is S^3, and the branch set is a knot $k_\sigma \subset S^3$. L. Siebenmann observed in [34] that one can use Waldhausen's classification to show that if σ' is also in I_Σ, the knot $k_{\sigma'}$ is related to k_σ by a sequence of mutations. The knot signature is a mutation invariant, thus the integers $\operatorname{sign} k_\sigma$, $\sigma \in I_\Sigma$, all coincide. L. Siebenmann calls this common value the $\tilde{\rho}$–invariant, so that

$$\tilde{\rho}(\Sigma) = \frac{1}{8} \operatorname{sign} k_\sigma.$$

The fact that $\tilde{\rho}(\Sigma) = \mu(\Sigma) \mod 2$ can be proved as follows. Fix $\sigma \in I_\Sigma$ and let F be an (orientable) Seifert surface of the knot k_σ pushed radially into D^4. Let W be the double branched cover of D^4

with the branch set F. This manifold is parallelizable, its boundary is Σ, therefore, $\operatorname{sign} W = \mu(\Sigma) \mod 2$. On the other hand, $\operatorname{sign} W = \operatorname{sign} k_\sigma$; see e.g. [17]. Therefore, $\tilde{\rho}$ is a lift of the Rohlin invariant to the integers defined for all integral homology 3–spheres Σ of plumbing type.

7.4 The invariants of Neumann and Siebenmann coincide

Now we have two lifts of the Rohlin invariant μ into the integers defined for all plumbed integral homology spheres, one by W. Neumann and the other by L. Siebenmann. We prove in this section that these lifts in fact coincide.

Proposition 41. *For any integral homology sphere Σ of plumbing type, $\bar{\mu}(\Sigma) = \tilde{\rho}(\Sigma)$.*

As we have mentioned before, the invariants $\bar{\mu}(\Sigma)$ and $\tilde{\rho}(\Sigma)$ are in fact defined for all $\mathbb{Z}/2$–homology spheres of plumbing type, and one can show that they are also equal to each other for such spheres.

Proof. Let Σ be a plumbed homology sphere, $\Sigma = \partial P(\Gamma)$. The 4–manifold $P(\Gamma)$ is a double branched cover of D^4 whose branch set is the surface $F = B(\Gamma) \subset D^4$ such that $\partial F \cap S^3 = k_\sigma$. The surface F is orientable if and only if all the weights in the plumbing tree Γ are even.

Suppose that Γ has only even weights. Then, since the intersection form of $P(\Gamma)$ is even, the Wu class w vanishes, and

$$\begin{aligned}\tilde{\rho}(\Sigma) &= \frac{1}{8}\operatorname{sign} k_\sigma \\ &= \frac{1}{8}\operatorname{sign} P(\Gamma) \\ &= \bar{\mu}(\Sigma).\end{aligned}$$

Now suppose that Γ has at least one odd weight. According to C. Gordon and R. Litherland [17, Theorem 3 and Corollary 5],

$$\operatorname{sign} k_\sigma = \operatorname{sign} P(\Gamma) + \frac{1}{2}\nu(F).$$

The number $\nu(F)$ is defined as minus the knot linking number, $\nu(F) = -\operatorname{lk}(k_\sigma, k_\sigma^F)$ where k_σ^F is a parallel copy of the knot k_σ missing the surface F. Thus to complete the proof we only need to show that

$$\frac{1}{2}\operatorname{lk}(k_\sigma, k_\sigma^F) = w.w,$$

where w is the Wu class of the manifold $P(\Gamma)$.

Let us first color in black the surface F, and orient its boundary $\partial F = k_\sigma$. In a knot projection there are two types of double points shown in Figure 11.

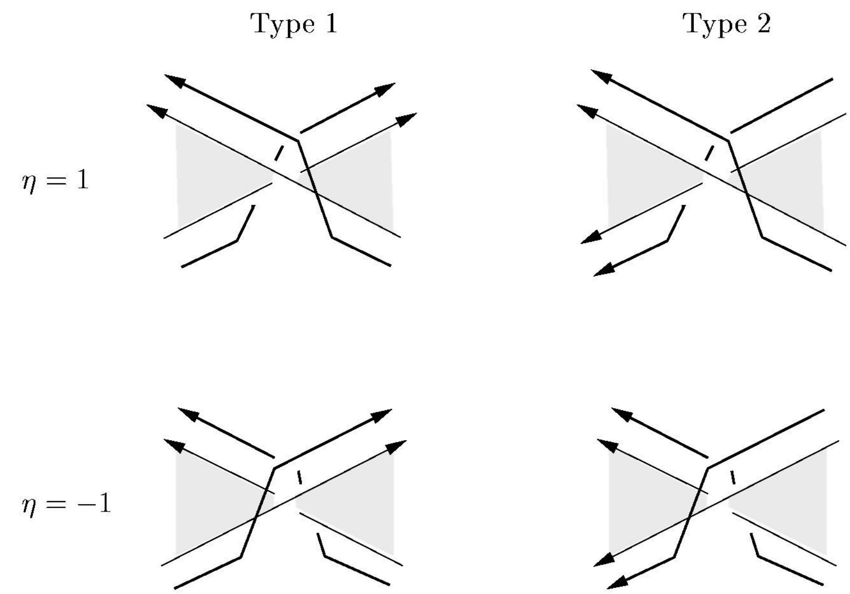

FIGURE 11

One can easily see that $1/2 \cdot \mathrm{lk}(k_\sigma, k_\sigma^F)$ is the sum of the incidence numbers η of double points of type 2. In a regular projection of k_σ, like the one seen in Figure 10, the double points only occur because of twisting the bands, and all the double points on the same band have the same type and the same incidence number.

Let v_i be a vertex in Γ weighted by a_{ii} (remember that the a_{ij}'s are the entries of the matrix $A(\Gamma)$; see (71)). Let $\varepsilon_i = 1$ if the double points on the band corresponding to v_i are of type 2, and $\varepsilon_i = 0$ otherwise. Then

$$\frac{1}{2}\,\mathrm{lk}(k_\sigma, k_\sigma^F) = \sum_{i=1}^{s} \varepsilon_i a_{ii}.$$

On the other hand, by comparing the orientations on the boundary of the i-th band with the orientations on the boundaries of bands adjacent to it one can easily check that

$$\sum_{j=1}^{s} \varepsilon_j a_{ij} = a_{ii} \mod 2, \quad i = 1, \dots, s.$$

Therefore, the vector $w = (\varepsilon_1, \dots, \varepsilon_s)$ is the Wu vector of the matrix $A(\Gamma)$, and $w.w = \sum \varepsilon_i a_{ii}$. q.e.d.

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TULANE UNIVERSITY, NEW ORLEANS