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EQUATIONS OF THE MODULI OF POINTED CURVES IN THE INFINITE GRASSMANNIAN

J. M. MUÑOZ PORRAS & F. J. PLAZA MARTÍN

Abstract

The main result of this paper is the explicit computation of the equations defining the moduli space of triples (C, p, ϕ), where C is an integral and complete algebraic curve, p a smooth rational point and ϕ a certain isomorphism. This is achieved by introducing algebraically infinite Grassmannians, tau and Baker-Ahkiezer functions and by proving an Addition Formula for tau functions.

1. Introduction

The Krichever morphism gives an immersion of the moduli space, $\mathcal{M}_\infty$, of triples (C, p, ϕ) (where C is an algebraic curve, p a smooth point of C and ϕ a certain isomorphism) in the infinite Grassmannian (see [25], [24]). The aim of this paper is to give an explicit system of equations defining the subscheme $\mathcal{M}_\infty$ of $\mathrm{Gr}(V)$; this problem is solved in §6.

We have adopted an algebraic point of view ([4]) and most of the results of this paper are valid over arbitrary base fields. Consequently, we have included several paragraphs addressing certain aspects of the theory of soliton equations and infinite Grassmannians for arbitrary base fields (these facts are well known when the base field is $\mathbb{C}$, [24], [8], [16]). This allows us to give the foundations for a theory of soliton equations

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valid over arbitrary base fields, extending the previous results of G. Anderson ([2]) for the case of p-adic fields. We hope that the techniques developed in this paper will clarify the "arithmetic properties" of the theory of KP equations.

When the base field is $\mathbb{C}$, our equations for $\mathcal{M}_\infty$ (as a subscheme of the infinite Grassmannian, $\mathrm{Gr}(V)$) are equivalent to a system of infinite partial differential equations 6.9 which are different from the equations of the KP hierarchy. To clarify the relation between the KP hierarchy and our equations 6.9, let us consider the chain of closed immersions:

$$\mathcal{M}_\infty \overset{\text{Krichever}}{\hookrightarrow} \mathrm{Gr}(V) \overset{\text{Plücker}}{\hookrightarrow} \mathbb{P}^\infty$$

where $\mathbb{P}^\infty$ is a suitable infinite dimensional projective space. It is well known ([20], [25]) that the image of $\mathrm{Gr}(V)$ in $\mathbb{P}^\infty$ is defined by the Plücker equations, which are equivalent to the KP hierarchy ($\mathrm{char}(k) = 0$). Then, the image of $\mathcal{M}_\infty$ in $\mathbb{P}^\infty$ will be defined by the following system of differential equations:

$$\begin{cases} \text{the KP equations (given in Theorem 5.4),} \\ \text{the p.d.e.'s given in Theorem 6.9,} \\ \text{the p.d.e.'s given in Corollary 6.10.3.} \end{cases}$$

In particular, we deduce a characterization (6.10), in terms of partial differential equations, of the infinite formal series $\tau(t) \in \mathbb{C}\{\{t_1, t_2, \dots\}\}$, which are the τ-functions of a triple $(C, p, \phi) \in \mathcal{M}_\infty$.

Let us note that the results of Shiota ([22]) give a necessary and suficient condition for a theta function of a principally polarized abelian variety to be the theta function of a Jacobian, but do not solve the problem of characterizing, in terms of differential equations, the formal τ-functions defined by Jacobian theta functions; this problem is solved in Corollary 6.10.

The paper is organized as follows. A survey on infinite Grassmannians is given in §2. In §3 the algebraic analogue of the group of maps $S^1 \to \mathbb{C}^*$ of Segal-Wilson is constructed and interpreted as the Jacobian of the formal curve. The action of that group on the Grassmannian is used in §4 in order to introduce tau and Baker-Ahkiezer functions. At the end of this section the Addition Formulae for tau functions are stated and proved.

Equations defining the infinite Grassmannian of $k((z))$ in a suitable infinite dimensional projective space are computed in §5. Over an arbitrary field it is defined by the well known Bilinear Residue Identity (equation 5.2) whose proof is an easy consequence of Theorem 4.8. When the base field is $\mathbb{C}$, using Schur polynomials, this identity turns out to be equivalent to the KP Hierarchy.

The last section, §6, contains the main result of this paper. A relative generalization of the Krichever map allows us to obtain a closed immersion of the moduli functor of pointed curves, whose rational points are the triples (C, p, ϕ), into the infinite Grassmannian of $k((z))$. We then prove its representability and give a characterization that permits to compute its equations.

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2. Infinite Grassmannians and determinant bundles

2.A. Infinite Grassmannians

In order to define the "infinite" Grassmannian of a vector space V (over a field k) one should consider some extra structure on it. This structure consists of a family $\mathcal{B}$ of subspaces of V such that the following conditions hold:

1. $A, B \in \mathcal{B} \quad \Longrightarrow \quad A + B, A \cap B \in \mathcal{B}$,
2. $A, B \in \mathcal{B} \quad \Longrightarrow \quad \dim(A + B)/A \cap B < \infty$,
3. $\cap_{A \in \mathcal{B}} A = (0)$,
4. the canonical homomorphism $V \to \varprojlim_{A \in \mathcal{B}} V/A$ is an isomorphism,
5. the canonical homomorphism $\varinjlim_{A \in \mathcal{B}} A \to V/B$ is a surjection (for $B \in \mathcal{B}$).

Let us interpret these conditions. First, V is endowed with a linear topology where a family of neigbourhoods of (0) is precisely $\mathcal{B}$. Then, the last three claims mean that the topology is separated; V is complete and every finite dimensional subspace of V/B is a neigbourhood of (0).

Example 1. In the study of the moduli space of pointed curves the fundamental example is $V = k((z))$ and $\mathcal{B}$ consists of the set of subspaces

$A \subseteq V$ containing $z^n \cdot k[[z]]$ as a subspace of finite codimension (for an integer n).

Example 2. Other examples of pairs $(V, \mathcal{B})$ satisfying the above requeriments are (V an arbitrary k-vector space):

- $(V, \mathcal{B} := \{(0)\})$;
- V and $\mathcal{B}$ the set of all finite dimensional subspaces of V.

In the sequel, we shall fix a pair $(V, \mathcal{B})$ satisfying the above conditions and a subspace $V_+ \in \mathcal{B}$.

Following [4] (see also [16]), there exists a Grassmannian scheme $\mathrm{Gr}(V, \mathcal{B})$, whose rational points are the set:

$$\left\{ \begin{array}{c} \text{subspaces } L \subseteq V, \text{ such that } L \cap V_+ \\ \text{and } V/L + V_+ \text{ are of finite dimension} \end{array} \right\}$$

(which we shall call discrete subspaces of V) that coincides with the usual infinite Grassmannian defined by Pressley-Segal in [21], and Segal-Wilson in [24].

In order to construct this scheme, we shall give its functor of points $\underline{\mathrm{Gr}}(V, \mathcal{B})$ and prove that it is representable in the category of k-schemes. To this end, we need some notation: given a morphism $T \to S$ of k-schemes a sub-$\mathcal{O}_S$-module $B \subseteq V_S := V \otimes_k \mathcal{O}_S$, we denote:

- $\widehat{B}_T := \varprojlim\limits_{A \in \mathcal{B}} (B/(A_S \cap B) \underset{k}{\otimes} \mathcal{O}_T)$.
- $\widehat{(V/B)}_T := \varprojlim\limits_{A \in \mathcal{B}} ((V_S/(A_S + B)) \underset{k}{\otimes} \mathcal{O}_T)$.

Definition 2.1. Given a k-scheme S, a discrete submodule of $\widehat{V}_S$ is a sheaf of quasi-coherent $\mathcal{O}_S$-submodules $\mathcal{L} \subset \widehat{V}_S$ such that: $\mathcal{L}_T \subset \widehat{V}_T$ for every morphism $T \to S$; and, for each $s \in S$ there exist an open neighborhood U of s and a commensurable k-vector subspace $B \in \mathcal{B}$ such that $\mathcal{L}_U \cap \widehat{B}_U$ is free of finite type and $\widehat{V}_U/\mathcal{L}_U + \widehat{B}_U = 0$.

Definition 2.2. The Grassmannian functor of a pair $(V, \mathcal{B})$, $\underline{\mathrm{Gr}}(V, \mathcal{B})$, is the contravariant functor over the category of k-schemes defined by:

$$\underline{\mathrm{Gr}}(V, \mathcal{B})(S) = \left\{ \text{discrete sub-}\mathcal{O}_S\text{-modules of } \widehat{V}_S \right\}.$$

Remark 1. Note that if V is a finite dimensional k-vector space and $\mathcal{B} = \{(0)\}$, then $\underline{\mathrm{Gr}}(V, \{(0)\})$ is the usual Grassmannian functor defined by Grothendieck [11, I.9.7].

Theorem 2.3. *The functor* $\underline{\mathrm{Gr}}(V, \mathcal{B})$ *is representable by a reduced and separated* k*-scheme* $\mathrm{Gr}(V, \mathcal{B})$*. The discrete submodule corresponding to the identity:*

$$Id \in \underline{\mathrm{Gr}}(V, \mathcal{B})\,(\mathrm{Gr}(V, \mathcal{B}))$$

will be called the **universal submodule** *and denoted by:*

$$\mathcal{L}_V \subset \widehat{V}_{\mathrm{Gr}(V,\mathcal{B})}.$$

Proof. The proof is modeled on the Grothendieck construction of finite Grassmannians [11]. Given a vector subspace $A \in \mathcal{B}$, define the functor $\underline{F_A}$ over the category of k-schemes by:

$$\underline{F_A}(S) = \{ \text{ sub-}\mathcal{O}_S\text{-modules } \mathcal{L} \subset \widehat{V}_S \text{ such that } \mathcal{L} \oplus \widehat{A}_S = \widehat{V}_S \ \}$$

and show that it is representable by an affine and integral k-scheme, F_A. The properties on $\mathcal{B}$ imply that $\{\underline{F_A}, A \in \mathcal{B}\}$ is a covering of $\underline{\mathrm{Gr}}(V, \mathcal{B})$ by open subfunctors (see [4]) and the result follows. q.e.d.

Remark 2. In this subsection infinite-dimensional Grassmannian schemes have been constructed in an abstract way. Choosing particular vector spaces $(V, \mathcal{B})$ we obtain different classes of Grassmannians. Two examples are relevant:

- $V = k((z))$, $V_+ = k[[z]]$. In this case, $\mathrm{Gr}(k((z)), k[[z]])$ is the algebraic version of the Grassmannian constructed by Pressley-Segal, and Segal-Wilson ([21], [24]). This Grassmannian is particularly well adapted for studying problems related to the moduli space of pointed curves (over arbitrary fields), to the moduli space of vector bundles and to the KP-hierarchy.

- Let $(X, \mathcal{O}_X)$ be a smooth, proper and irreducible curve over the field k and let V be the adeles ring over the curve and $V_+ = \prod_p \widehat{\mathcal{O}_p}$ (recall the first example). In this case $\mathrm{Gr}(V, \mathcal{B})$ is an adelic Grassmannian which will be useful for studying arithmetic problems over the curve X or problems related to the classification of vector bundles over a curve (nonabelian theta functions,...).

Instead of adelic Grassmannians, we could define Grassmannians associated with a fixed divisor on X in an analogous way. These adelic Grassmannians will also be of interest in the study of conformal field theories over Riemann surfaces in the sense of Witten ([26]).

2.B. Determinant bundles

In this subsection we recall from [4] the construction of the determinant bundle over the Grassmannian, in the sense of Knudsen and Mumford [14]. This allows us to define determinants algebraically and over arbitrary fields (for example for $k = \mathbb{Q}$ or $k = \mathbb{F}_q$).

We shall denote the Grassmannian $\mathrm{Gr}(V, \mathcal{B})$ simply by $\mathrm{Gr}(V)$.

Definition 2.4. For each $A \in \mathcal{B}$ and each $L \in \mathrm{Gr}(V)(S)$ we define a complex, $\mathcal{C}_A^\bullet(L)$, of $\mathcal{O}_S$-modules by:

$$\mathcal{C}_A^\bullet(L) \equiv \ldots \to 0 \to L \oplus \hat{A}_S \xrightarrow{\delta} \hat{V}_S \to 0 \to \ldots$$

δ being the addition homomorphism.

It is not difficult to prove that $\mathcal{C}_A^\bullet(L)$ is a perfect complex of $\mathcal{O}_S$-modules, and therefore its determinant and index are well defined. Recall that the index of a point $L \in \mathrm{Gr}(V)(S)$ is the locally constant function $i_L \colon S \to \mathbb{Z}$ defined by:

$$i_L(s) = \text{ Euler-Poincaré characteristic of } \mathcal{C}_{V_+}^\bullet(L) \otimes k(s)$$

$k(s)$ being the residual field of the point $s \in S$ (see [14] for details).

By easy calculation, we have that if V is a finite-dimensional k-vector space, $\mathcal{B} = \{(0)\}$ and $L \in \mathrm{Gr}(V)(S)$, then $i_L = \mathrm{codim}_k(L)$. And for the general case, the index of any rational point $L \in \mathrm{Gr}(V)(\mathrm{Spec}(k))$ is exactly:

$$\dim_k(L \cap V_+) - \dim_k V/(L + V_+).$$

Let $\mathrm{Gr}^n(V)$ be the subset over which the index takes values equal to $n \in \mathbb{Z}$. Then, the decomposition of $\mathrm{Gr}(V)$ in connected components is:

$$\mathrm{Gr}(V) = \coprod_{n \in \mathbb{Z}} \mathrm{Gr}^n(V).$$

Given a point $L \in \mathrm{Gr}(V)(S)$ and $A \in \mathcal{B}$, we denote by $\mathrm{Det}\, \mathcal{C}_A^\bullet(L)$ the determinant sheaf of the perfect complex $\mathcal{C}_A^\bullet(L)$ in the sense of [14].

Note that this determinant does not depend on A (up to isomorphisms), since for $A, B \in \mathcal{B}$ there exists a canonical isomorphism:

$$\operatorname{Det}^* \mathcal{C}_A^\bullet \xrightarrow{\sim} \operatorname{Det}^* \mathcal{C}_B^\bullet \otimes \wedge^{max}(A/A\cap B) \otimes \wedge^{max}(B/A\cap B)^*. \tag{2.5}$$

Hence we define the determinant bundle over $\mathrm{Gr}^0(V)$, Det_V, as the invertible sheaf:

$$\operatorname{Det} \mathcal{C}_{V_+}^\bullet(\mathcal{L}_V)$$

($\mathcal{L}_V$ being the universal submodule over $\mathrm{Gr}^0(V)$).

Observe that the choice of other subspace of $\mathcal{B}$, A, instead of V_+ might shift the labelling of the connected components by a constant and modify the determinant bundle by an isomorphism.

Now, if $L \in \mathrm{Gr}^0(V)$ is a rational point, and $L \cap V_+$ and $V/L + V_+$ are therefore k-vector spaces of the same dimension, then one has an isomorphism:

$$\mathrm{Det}_V(L) \simeq \wedge^{max}(L \cap V_+) \otimes \wedge^{max}(V/(L + V_+))^*.$$

That is, our determinant coincides, over the rational points, with the determinant bundles of Pressley-Segal and Segal-Wilson ([21], [24]). Furthermore, this construction gives the usual determinant bundle when V is finite-dimensional.

2.C. Sections of the determinant bundle

It is well known that the determinant bundle has no global sections. We shall therefore explicitly construct global sections of the dual of the determinant bundle over the connected component $\mathrm{Gr}^0(V)$ of index zero.

We use the following notation: $\wedge^\bullet E$ is the exterior algebra of a k-vector space E, $\wedge^r E$ its component of degree r, and $\wedge E$ is the component of higher degree when E is finite-dimensional. Given a perfect complex $\mathcal{C}^\bullet$ over k-scheme X, we shall write $\operatorname{Det}^* \mathcal{C}^\bullet$ to denote the dual of the invertible sheaf $\operatorname{Det} \mathcal{C}^\bullet$.

To explain how global sections of the invertible sheaf $\operatorname{Det}^* \mathcal{C}^\bullet$ can be constructed, recall that if $f : E \to F$ is a homomorphism between finite-dimensional k-vector spaces of equal dimension, then it induces a homomorphism:

$$\wedge(f) : k \to \wedge F \otimes (\wedge E)^*.$$

Thus, considering $E \xrightarrow{f} F$ as a perfect complex, $\mathcal{C}^\bullet$, over $\mathrm{Spec}(k)$, we have defined a **canonical section** $\wedge(f) \in H^0(\mathrm{Spec}(k), \operatorname{Det}^* \mathcal{C}^\bullet)$.

Let us now consider a perfect complex $\mathcal{C}^\bullet \equiv (E \xrightarrow{f} F)$ of sheaves of $\mathcal{O}_X$-modules over a k-scheme X, with Euler-Poincaré characteristic $\mathcal{X}(\mathcal{C}^\bullet) = 0$. Using the above argument, we construct a canonical section $det(f|_U) \in H^0(U, \mathrm{Det}^* \mathcal{C}^\bullet)$ for every open subscheme of X, U, over which $\mathcal{C}^\bullet$ is quasi-isomorphic to a complex of finitely-generated free modules.

Since the construction is canonical, the functions:

$$g_{UV} := det(f|_U)|_{U\cap V} \cdot det(f|_V)|_{U\cap V}^{-1}$$

satisfy the cocycle condition. Then, these local sections glue $\{det(f|_U)\}$ and give a canonical global section:

$$\mathrm{Det}(f) \in H^0(X, \mathrm{Det}^* \mathcal{C}^\bullet)$$

(see [4] for more details). If the complex $\mathcal{C}^\bullet$ is acyclic, one has an isomorphism:

$$\begin{aligned} \mathcal{O}_X &\xrightarrow{\sim} \mathrm{Det}^* \mathcal{C}^\bullet \\ 1 &\mapsto det(f). \end{aligned}$$

Let us consider the perfect complex $\mathcal{C}_A^\bullet \equiv (\mathcal{L} \oplus A \xrightarrow{\delta_A} V)$ over $\mathrm{Gr}(V)$ defined in 2.B ($\mathcal{L}$ being the universal discrete submodule over $\mathrm{Gr}(V)$) for a given $A \in \mathcal{B}$. Since $\mathcal{C}_A^\bullet|_{F_A}$ is acyclic, one then has an isomorphism:

$$\begin{aligned} \mathcal{O}_X|_{F_A} &\xrightarrow{\sim} \mathrm{Det}^* \mathcal{C}_A^\bullet|_{F_A} \\ 1 &\mapsto s_A = det(\delta_A)|_{F_A}. \end{aligned}$$

By the above argument it is easy to prove that the section $s_A \in H^0(F_A, \mathrm{Det}^* \mathcal{C}_A^\bullet)$ can be extended in a canonical way to a global section of $\mathrm{Det}^* \mathcal{C}_A^\bullet$ over $\mathrm{Gr}^0(V)$, which will be called the **canonical section** ω_A **of** $\mathrm{Det}^* \mathcal{C}_A^\bullet$. (We restrict ourselves to cases where $F_A \subseteq \mathrm{Gr}^0(V)$, or, what amounts to the same, $\dim_k(A/A \cap V_+) - \dim_k(V_+/A \cap V_+) = 0$).

This result allows us to compute many global sections of $\mathrm{Det}_V^* = \mathrm{Det}\, \mathcal{C}_{V_+}^\bullet$ over $\mathrm{Gr}^0(V)$: given $A \in \mathcal{B}$ such that $F_A \subseteq \mathrm{Gr}^0(V)$, the isomorphism $\mathrm{Det}^* \mathcal{C}_A^\bullet \xrightarrow{\sim} \mathrm{Det}_V^*$ is not canonical (recall the formula 2.5). Therefore, the determination of an isomorphism $\mathrm{Det}^* \mathcal{C}_A^\bullet \xrightarrow{\sim} \mathrm{Det}_V^*$ depends on the choice of bases for the vector spaces $A/A \cap V_+$ and $V_+/A \cap V_+$. For a detailed discussion of the construction of sections see [4], [20]).

2.D. Computations for the infinite Grassmannian: $V = k((z))$

Let $(V, \mathcal{B})$ be as in Example 1 and take $V_+ = k[[z]]$.

Let $\mathcal{S}$ be the set of Young's diagrams (also called Maya or Ferrers diagrams) of virtual cardinal zero; or equivalently, the sequences $\{s_0, s_1, \dots\}$ of integer numbers satisfying the following conditions:

1. the sequence is strictly increasing,
2. there exists $s \in \mathbb{Z}$ such that $\{s, s+1, s+2, \dots\} \subseteq \{s_0, s_1, \dots\}$,
3. $\#(\{s_0, s_1, \dots\} - \{0, 1, \dots\}) = \#(\{0, 1, \dots\} - \{s_0, s_1, \dots\})$.

For notation's sake, we define $e_i := z^i$. For each $S \in \mathcal{S}$, let A_S be the vector subspace of V generated by $\{e_{s_i}, i \geq 0\}$. By the third condition one has:

$$\dim_k(A_S/A_S \cap V_+) = \dim_k(V_+/A_S \cap V_+),$$

and hence $A_S \in \mathcal{B}$ and $F_{A_S} \subseteq \mathrm{Gr}^0(V)$. Further, $\{F_{A_S}, S \in \mathcal{S}\}$ is a covering of $\mathrm{Gr}^0(V)$.

Define linear forms $\{e_i^*\}$ of V^* by the following condition:

$$e_i^*(e_j) := \delta_{ij}.$$

For each finite set of increasing integers, $J = \{j_1, \dots, j_r\}$, let us define $e_J := e_{j_1} \wedge \dots \wedge e_{j_r}$ and $e_J^* := e_{j_1}^* \wedge \dots \wedge e_{j_r}^*$.

Given $S \in \mathcal{S}$, choose $J, K \subseteq \mathbb{Z}$ such that $\{e_j\}_{j \in J}$ is a basis of $V_+/A_S \cap V_+$ and $\{e_k^*\}_{k \in K}$ of $(A_S/A_S \cap V_+)^*$. We have seen that tensor by $e_J \otimes e_K^*$ defines an isomorphism:

$$H^0(\mathrm{Gr}^0(V), \mathrm{Det}^* \mathcal{C}^\bullet_{A_S}) \xrightarrow{\otimes(e_J \otimes e_K^*)} H^0(\mathrm{Gr}^0(V), \mathrm{Det}_V^*).$$

Definition 2.6. For each $S \in \mathcal{S}$, Ω_S is the global section of Det_V^* defined by:

$$\Omega_S = \omega_{A_S} \otimes e_J \otimes e_K^*.$$

We shall denote by Ω_+ the canonical section of Det_V^*.

Let $\Omega(\mathcal{S})$ be the k-vector subspace of $H^0(\mathrm{Gr}^0(V), \mathrm{Det}_V^*)$ generated by the global sections $\{\Omega_S, S \in \mathcal{S}\}$.

We define the Plücker morphism:

$$\begin{aligned} \mathfrak{p}_V : \mathrm{Gr}^0(V) &\to \mathbb{P}\Omega(S)^* := \operatorname{Proj} \operatorname{Sym} \Omega(S), \\ L &\mapsto \{\Omega_S(L)\} \end{aligned}$$

as the morphism of schemes associated to the surjective sheaf homomorphism:

$$\Omega(S)_{\mathrm{Gr}^0(V)} \to \mathrm{Det}_V^*$$

by the universal property of $\mathbb{P}$.

Remark 3. Once the Plücker morphism is introduced, it can be proved that the Plücker equations are in fact the defining equations for $\mathrm{Gr}^0(V)$ when $\mathrm{char}(k) = 0$ ([20]).

This is the property that Sato used in [25] to define his Universal Grassmann Manifold (UGM); that is, a point of the UGM is a point of an infinite dimensional projective space (with countable many coordinates) satisfying all the Plücker relations.

Remark 4. For studing the index n connected component of the Grassmannian, $\mathrm{Gr}^n(V)$ the same constructions are applied since the homothety of $k((z))$ defined by z^{-n} induces isomorphisms:

$$\begin{gathered}\mathrm{Gr}^n(V) \xrightarrow{\sim} \mathrm{Gr}^0(V), \\ H^0(\mathrm{Gr}^0(V), \mathrm{Det}_V^*) \xrightarrow{\sim} H^0(\mathrm{Gr}^n(V), \mathrm{Det}_n^*),\end{gathered}$$

where Det_n^* is defined by $\mathrm{Det}^* \mathcal{C}^\bullet_{z^n \cdot V_+}$. Let us distinguish the global section given by the image of Ω_+ and denote it by Ω_+^n.

2.E. The Grassmannian of the dual space

Let $(V, \mathcal{B})$ be as usual. Consider V as a linear topological space. Now, a submodule $L \subseteq \widehat{V}_S$ (S a k-scheme) carries a linear topology: that given by $\{L \cap \widehat{A}_S\}_{A \in \mathcal{B}}$ as neigbourhoods of (0). We introduce the following notation:

$$\begin{aligned} L^* &:= \mathrm{Hom}_{\mathcal{O}_S}(L, \mathcal{O}_S), \\ L^c &:= \{f \in L^* \text{ continuous } \}, \\ L^\circ &:= \{f \in (\widehat{V}_S)^* \,|\, f|_L \equiv 0\}, \\ L^\diamond &:= \{f \in (\widehat{V}_S)^c \,|\, f|_L \equiv 0\}, \end{aligned}$$

where $\mathcal{O}_S$ has the discrete topology.

Observe that given two subspaces $A, B \in \mathcal{B}$ such that $B \subseteq A$ the following claims hold:

- there is a canonical isomorphism $A^\diamond/B^\diamond \xrightarrow{\sim} (A/B)^*$. (This implies that $(A^\diamond + B^\diamond)/A^\diamond \cap B^\diamond$ is finite dimensional.)
- $(A + B)^\diamond = A^\diamond \cap B^\diamond$.

- $(A \cap B)^\diamond = A^\diamond + B^\diamond$.

Consider the following family of subspaces of V^c:

$$\mathcal{B}^\diamond := \{A^\diamond \text{ where } A \in \mathcal{B}\}.$$

In order to make explicit the meaning of the expression "Grassmannian of the dual space", $(V^c, \mathcal{B}^\diamond)$, we need the following:

Lemma 2.7 ([20]).

1. $V^c = \varinjlim_{A\in\mathcal{B}} A^\diamond$;
2. $V = \varinjlim_{A\in\mathcal{B}} A$;
3. $\cap_{A\in\mathcal{B}} A^\diamond = (0)$;
4. $V^c = \varprojlim_{A\in\mathcal{B}} V^c/A^\diamond$.

The Lemma and these considerations imply the following:

Theorem 2.8. *The family $\mathcal{B}^\diamond$ satisfies the conditions of 2.A, and therefore the infinite Grassmannian of the pair $(V^c, \mathcal{B}^\diamond)$ exists.*

If a subspace $V_+ \in \mathcal{B}$ is chosen, then we shall consider the subspace $V_+^\diamond$ for the pair $(V^c, \mathcal{B}^\diamond)$.

Now, we shall construct a canonical isomorphism between the Grassmannian of V and that of V^c. The expression of this isomorphism for the rational points will be that given by incidence:

$$\begin{aligned} I : \operatorname{Gr}(V, \mathcal{B}) &\longrightarrow \operatorname{Gr}(V^c, \mathcal{B}^\diamond), \\ L &\longmapsto L^\diamond. \end{aligned}$$

Let $\mathcal{L}$ be the universal sheaf of $\operatorname{Gr}(V)$. Consider the following sub-$\mathcal{O}_{\operatorname{Gr}(V)}$-module of $\widehat{V}^c_{\operatorname{Gr}(V)}$ defined as:

$$\mathcal{L}^\diamond = \{\omega \in \widehat{V}^c_{\operatorname{Gr}(V)} \text{ such that } \omega|_{\mathcal{L}} \equiv 0\}.$$

Let us check that $\mathcal{L}^\diamond$ is in fact a $\operatorname{Gr}(V)$-valued point of $\operatorname{Gr}(V^c)$, and that it does induce the desired morphism. By the definition of the Grassmannian, this can be done locally. Recall that $\{F_A\}_{A\in\mathcal{B}}$ is a covering of $\operatorname{Gr}(V)$. Let $\mathcal{L}^\diamond|_{F_A}$ be the restriction of $\mathcal{L}^\diamond$ (as sub-$\mathcal{O}_{\operatorname{Gr}(V)}$-module of

$\widehat{V}^c_{\mathrm{Gr}(V)}$) to the open subscheme $F_A \hookrightarrow \mathrm{Gr}(V)$. Since $\widehat{V}_{F_A} \simeq \mathcal{L}|_{F_A} \oplus \widehat{A}_{F_A}$ canonically, one has:

$$\mathcal{L}^\diamond|_{F_A} \simeq (\widehat{A}_{F_A})^c$$

in a canonical way. Recalling the Definition 2.1, the conclusion follows.

To finish, we compute $I^* \operatorname{Det}_{V^c}$. Observe that there exists a canonical morphism of complexes of $\mathcal{O}_{\mathrm{Gr}(V)}$-modules (written vertically):

$$\begin{array}{ccc} \mathcal{L}_V^\diamond \oplus V_+^\diamond & \longrightarrow & V^* \\ \downarrow & & \downarrow \\ V^c & \longrightarrow & \mathcal{L}_V^* \oplus V_+^* \end{array}$$

where $\mathcal{L}_V$ is the universal submodule of $\mathrm{Gr}(V)$, and one easily checks that this is in fact a quasi-isomorphism. Since the inverse image of the universal submodule of $\mathrm{Gr}(V^c)$ is $\mathcal{L}_V^\diamond$, one has the following formulae:

$$I^* \operatorname{Det}_{V^*} \simeq \operatorname{Det}_V^*, \qquad I^*(i) = -i,$$

where i is the index function.

3. "Formal geometry" of local curves

3.A. Formal groups

We are first concerned with the algebraic analogue of the group Γ ([24, § 2.3]) of continuous maps $S^1 \to \mathbb{C}^*$ acting as multiplication operators over the Grassmannian. The main difference between our definition of the group Γ and the definitions offered in the literature ([24], [21]) is that in the algebro-geometric setting the elements $\sum_{-\infty}^{+\infty} g_k\, z^k$ with infinite positive and negative coefficients do not make sense as multiplication operators over $k((z))$. In this sense, our approach is close to that of [16].

The main idea for defining the algebraic analogue of the group Γ is to construct a (formal) scheme whose set of rational points is precisely the multiplicative group $k((z))^*$ (see [4]).

Definition 3.1. The contravariant functor, $\underline{k((z))}^*$, over the category of k-schemes with values in the category of commutative groups is defined by:

$$S \rightsquigarrow \underline{k((z))}^*(S) := H^0(S, \mathcal{O}_S)((z))^*.$$

Where for a k-algebra A, $A((z))^*$ is the group of invertible elements of the ring $A((z)) := A[[z]][z^{-1}]$ of Laurent series with coefficients in A.

Note that for each k-scheme S and $f \in \underline{k((z))}^*(S)$, the function:

$$\begin{aligned} &S \to \mathbb{Z} \\ &s \mapsto v_s(f) := \text{order of } f_s \in k(s)((z)) \end{aligned}$$

is locally constant. From this fact we deduce that given an irreducible affine k-scheme, $S = \operatorname{Spec}(A)$, we have that:

$$\begin{aligned} \underline{k((z))}^*(S) &= \coprod_{n\in\mathbb{Z}} \{f \in A((z))^* \,|\, v(f) = n\} \\ &= \coprod_{n\in\mathbb{Z}} \left\{ \begin{matrix} \text{series } a_{n-r}\, z^{n-r} + \dots + a_n\, z^n + \dots \text{ such that} \\ a_{n-r}, \dots, a_{n-1} \text{ are nilpotent and } a_n \in A^* \end{matrix} \right\}. \end{aligned}$$

Theorem 3.2. *The subfunctor $\underline{k((z))}^*_{red}$ of $\underline{k((z))}^*$ defined by:*

$$S \rightsquigarrow \underline{k((z))}^*_{red}(S) := \coprod_{n\in\mathbb{Z}} \left\{ z^n + \sum_{i>n} a_i\, z^i \quad a_i \in H^0(S, \mathcal{O}_S) \right\}$$

is representable by a group k-scheme whose connected component of the origin will be denoted by Γ_+.

Proof. It suffices to observe that the functor:

$$S \rightsquigarrow \left\{ z^n + \sum_{i>n} a_i\, z^i \quad a_i \in H^0(S, \mathcal{O}_S) \right\}$$

is representable by the scheme $\operatorname{Spec}(\varinjlim_l k[x_1, \dots, x_l]) = \varprojlim_l \mathbb{A}^l_k$, where the group law is given by the multiplication of series; that is:

$$\begin{aligned} &k[x_1, \dots] \to k[x_1, \dots] \otimes_k k[x_1, \dots], \\ &\qquad x_i \mapsto x_i \otimes 1 + \sum_{j+k=i} x_j \otimes x_k + 1 \otimes x_i. \end{aligned}$$

q.e.d.

Theorem 3.3. *Let $\underline{k((z))}^*_{nil}$ be the subfunctor of $\underline{k((z))}^*$ defined by:*

$$S \rightsquigarrow \underline{k((z))}^*_{nil}(S) := \coprod_{n>0} \left\{ \begin{matrix} \textit{finite series } a_n\, z^{-n} + \dots + a_1\, z^{-1} + 1 \\ \textit{such that } a_i \in H^0(S, \mathcal{O}_S) \textit{ are nilpotent} \end{matrix} \right\}.$$

There exists a formal group k-scheme Γ_- such that:

$$\operatorname{Hom}_{\textit{for-sch}}(S, \Gamma_-) = \underline{k((z))}^*_{nil}(S)$$

for every k-scheme S.

Proof. Note that Γ_- is the direct limit in the category of formal schemes ([11, **I**.10.6.3]) of the schemes representing the subfunctors:

$$S \rightsquigarrow \Gamma_-^n(S) = \left\{ \begin{array}{c} a_n\, z^{-n} + \cdots + a_1\, z^{-1} + 1 \text{ such that } a_i \in H^0(S, \mathcal{O}_S) \\ \text{and the n}^{\text{th}} \text{ power of the ideal } (a_1, \ldots, a_n) \text{ is zero} \end{array} \right\}.$$

And its associated ring (that is, as ringed space) is:

$$k\{\{x_1, \ldots\}\} = \varprojlim_n k[[x_1, \ldots, x_n]],$$

the morphisms of the projective system being:

$$\begin{aligned} k[[x_1, \ldots, x_{n+1}]] &\to k[[x_1, \ldots, x_n]], \\ x_i &\mapsto x_i \qquad \text{for } i = 1, \ldots, n-1, \\ x_{n+1} &\mapsto 0. \end{aligned}$$

It is now easy to show that $\Gamma_- = \operatorname{Spf}(k\{\{x_1, \ldots\}\})$ (with group law given by multiplication of series) satisfies the desired condition. We shall call the ring $k\{\{x_1, \ldots\}\}$ the ring of "infinite" formal series in infinite variables (which is different from the ring of formal series in infinite variables. For instance, $x_1 + x_2 + \ldots \in k\{\{x_1, \ldots\}\}$). q.e.d.

Remark 5. Our group scheme Γ is the algebraic analogue of the Γ group of Segal-Wilson [24]. Note that the indices "-" and "+" do not coincide with the Segal-Wilson notation. Replacing $k((z))$ by $k((z^{-1}))$, we obtain their notation.

Let us define the exponential maps for the groups Γ_- and Γ_+. Let $\mathbb{A}_n$ be the n dimensional affine space over $\operatorname{Spec}(k)$ with the additive group law, and $\hat{\mathbb{A}}_n$ the formal group obtained as the completion of $\mathbb{A}_n$ at the origin. We define $\hat{\mathbb{A}}_\infty$ as the formal group scheme $\varinjlim_n \hat{\mathbb{A}}_n$. Obviously it holds that:

$$\hat{\mathbb{A}}_\infty = \operatorname{Spf} k\{\{y_1, \ldots\}\}$$

with the additive group law.

Definition 3.4. If the characteristic of k is zero, the exponential map for Γ_- is the following isomorphism of formal group schemes:

$$\begin{aligned} \hat{\mathbb{A}}_\infty &\overset{\exp}{\to} \Gamma_-, \\ \{a_i\}_{i>0} &\mapsto \exp(\sum_{i>0} a_i\, z^{-i}). \end{aligned}$$

This is the morphism induced by the ring homomorphism:

$$\begin{aligned} k\{\{x_1,\dots\}\} &\xrightarrow{\exp^*} k\{\{y_1,\dots\}\}, \\ x_i &\mapsto \text{ coefficient of } z^{-i} \text{ in the series } \exp(\sum_{j>0} y_j\, z^{-j}). \end{aligned}$$

Definition 3.5. If the characteristic of k is positive, the exponential map for Γ_- is the following isomorphism of formal schemes:

$$\begin{aligned} \hat{\mathbb{A}}_\infty &\to \Gamma_-, \\ \{a_i\}_{i>0} &\mapsto \prod_{i>0}(1 - a_i\, z^{-i}), \end{aligned}$$

which is the morphism induced by the ring homomorphism:

$$\begin{aligned} k\{\{x_1,\dots\}\} &\xrightarrow{\exp^*} k\{\{y_1,\dots\}\}, \\ x_i &\mapsto \text{ coefficient of } z^{-i} \text{ in the series } \prod_{i>0}(1 - a_i\, z^{-i}). \end{aligned}$$

Note that this latter exponential map is not a isomorphism of groups. Considering over $\hat{\mathbb{A}}_\infty$ the law group induced by the isomorphism, exp, of formal schemes, we obtain the Witt formal group law.

The exponential map for Γ_+ is defined in a analogous way; one only has to replace z^{-i} by z^i in the above expressions. (See [5] for the connection of these definitions and the Cartier-Dieudonné theory.) The following property gives the structure of $\underline{k((z))}^*$.

Theorem 3.6. *The natural morphism of functors of groups over the category of k-schemes:*

$$\underline{\Gamma_-} \times \underline{\mathbb{G}_m} \times \underline{\Gamma_+} \to \underline{k((z))}^*$$

is injective and for $char(k) = 0$ gives an isomorphism with $\underline{k((z))}^*_0$ (the connected component of the origin in the functor of groups $\underline{k((z))}^*$). The functor on groups $\underline{k((z))}^*_0$ is therefore "representable" by the (formal) k-scheme:

$$\Gamma = \Gamma_- \times \mathbb{G}_m \times \Gamma_+.$$

3.B. "Formal geometry"

It should be noted that the formal group scheme Γ_- has properties formally analogous to the Jacobians of algebraic curves: one can define formal Abel maps and prove formal analogues of the Albanese property of the Jacobians of smooth curves.

Let $\hat{C} = \mathrm{Spf}(k[[t]])$ be a formal curve. The Abel morphism of degree 1 is defined as the morphism of formal schemes:

$$\phi : \hat{C} \to \Gamma_-$$

given by the $\hat{C}$-valued point of Γ_-:

$$\phi(t) = (1 - \frac{t}{z})^{-1} = 1 + \sum_{i>0} \frac{t^i}{z^i},$$

that is, the morphism induced by the ring homomorphism:

$$\begin{aligned} k\{\{x_1, \dots\}\} &\to k[[t]], \\ x_i &\mapsto t^i. \end{aligned}$$

Note that the Abel morphism is the algebro-geometric version of the function $q_\xi(z)$ used by Segal-Wilson ([24] page 32) to study the Baker-Akhiezer function.

Let us explain further why we call ϕ the "Abel morphism" of degree 1. If $char(k) = 0$, composing ϕ with the inverse of the exponential map, affords:

$$\bar{\phi} : \hat{C} \xrightarrow{\phi} \Gamma_- \xrightarrow{\exp^{-1}} \hat{\mathbb{A}}_\infty,$$

and since $(1-\frac{t}{z})^{-1} = \exp(\sum_{i>0} \frac{t^i}{i\, z^i})$ (see [24, p. 33]), $\bar{\phi}$ is the morphism defined by the ring homomorphism:

$$\begin{aligned} k\{\{y_1, \dots\}\} &\to k[[t]], \\ y_i &\mapsto \frac{t^i}{i}, \end{aligned}$$

or in terms of the functor of points:

$$\begin{aligned} &\hat{C} \xrightarrow{\bar{\phi}} \hat{\mathbb{A}}_\infty, \\ &t \mapsto \{t, \frac{t^2}{2}, \frac{t^3}{3}, \dots\}. \end{aligned}$$

Observe that given the basis $\omega_i = t^i\, dt$ of the differentials $\Omega_{\hat{C}} = k[[t]]dt$, $\bar{\phi}$ can be interpreted as the morphism defined by the "abelian integrals" over the formal curve:

$$\bar{\phi}(t) = \left(\int_0^t \omega_0, \int_0^t \omega_1, \dots, \int_0^t \omega_i, \dots \right),$$

which coincides precisely with the local equations of the Abel morphism for smooth algebraic curves over the field of complex numbers.

Remark 6. The above introduced notation is also motivated by the following two facts:

- (Γ_-, ϕ) satisfies the Albanese property for $\hat{C}$; that is, every morphism $\psi : \hat{C} \to X$ in a commutative group scheme (which sends the unique rational point of $\hat{C}$ to the $0 \in X$) factors through the Abel morphism and a homomorphism of groups $\Gamma_- \to X$. This property follows from the following fact: the direct limit $\varinjlim_n S^n\widehat{C}$ exists (as a formal scheme) and it is naturally isomorphic to Γ_-).

- Observe that for each element:

$$u \in \Gamma_-(S) \subseteq \underline{k((z))^*}(S) = H^0(S, \mathcal{O}_S)((z))^*$$

we can define a fractionary ideal of the formal curve $\hat{C}_S$ by: $I_u = u \cdot \mathcal{O}_S((z))$. We can therefore interpret the formal group Γ_- as a kind of Picard scheme over the formal curve $\hat{C}$ (see [7], [6]). The universal element of Γ_- is the invertible element of $\underline{k((z))}^*(\Gamma_-)$ given by:

$$v(x, z) = 1 + \sum_{i \geq 1} x_i \, z^{-i} \in k((z)) \hat{\otimes} k\{\{x_1, x_2, \dots\}\}.$$

This universal element will be the formal analogue of the universal invertible sheaf for the formal curve $\hat{C}$.

4. τ-functions and Baker-Akhiezer functions

The first part of this section is devoted to defining the τ-function and the Baker-Akhiezer function algebraically over an arbitrary base field k. We then prove an analogue for τ of the Addition Formulae for theta functions that will allow a characterization of the Baker-Akhiezer function, which is quite close to proposition 5.1 of [24].

Following on with the analogy between the groups Γ and Γ_- and the Jacobian of the smooth algebraic curves, we shall perform the well-known constructions for the Jacobians of the algebraic curves for the formal curve $\hat{C}$ and the group Γ: Poincaré bundle over the dual Jacobian and the universal line bundle over the Jacobian. In the formal case, these constructions are essentially equivalent to defining the τ-functions and the Baker-Akhiezer functions.

Let us denote by $\mathrm{Gr}^0(V)$ index-zero connected component of the Grassmannian of $V = k((z))$ and by Γ the group $\Gamma_- \times \mathbb{G}_m \times \Gamma_+$. Let

$$\Gamma \times \mathrm{Gr}^0(V) \xrightarrow{\mu} \mathrm{Gr}^0(V)$$

be the action of Γ over the Grassmannian by homotheties. We define the Poincaré bundle over $\Gamma \times \mathrm{Gr}^0(V)$ as the invertible sheaf:

$$\mathcal{P} = \mu^* \operatorname{Det}_V^*,$$

$p_2 : \Gamma \times \mathrm{Gr}^0(V) \longrightarrow \mathrm{Gr}^0(V)$ being the natural projection.

For each point $U \in \mathrm{Gr}^0(V)$, let us define the Poincaré bundle over $\Gamma \times \Gamma$ associated with U by:

$$\mathcal{P}_U = (1 \times \mu_U)^* \mathcal{P} = m^*({\mu_U}^* \operatorname{Det}_V^*),$$

where $\mu_U : \Gamma \to \Gamma(U) \subset \mathrm{Gr}^0(V)$ is the action of Γ on the orbit of U, and $m : \Gamma \times \Gamma \to \Gamma$ is the group law.

The sheaf of τ-functions of a point $U \in \mathrm{Gr}^0(V)$ (not necessarily a geometric point), $\widetilde{\mathcal{L}_\tau}(U)$, is the invertible sheaf over $\Gamma \times \{U\}$ defined by:

$$\widetilde{\mathcal{L}_\tau}(U) = \mathcal{P}|_{\Gamma \times \{U\}}.$$

The restriction homomorphism induces the following homomorphism between global sections:

$$H^0\left(\Gamma \times \mathrm{Gr}^0(V), \mu^* \operatorname{Det}_V^*\right) \to H^0\left(\Gamma \times \{U\}, \widetilde{\mathcal{L}_\tau}(U)\right). \tag{4.1}$$

Definition 4.2. The τ-function of the point $U \in \mathrm{Gr}^0(V)$ over Γ is defined as the image $\widetilde{\tau}_U$ of the section $\mu^*\Omega_+$ by the homomorphism 4.1 where Ω_+ is the global section defined in 2.6.

Obviously $\widetilde{\tau}_U$ is not a function over $\Gamma \times \{U\}$ since the invertible sheaf $\widetilde{\mathcal{L}_\tau}(U)$ is not trivial, but this definition is essentially the τ-function defined by M. and Y. Sato and Segal-Wilson ([25], [24]). Their τ-function is obtained by restricting the invertible sheaf $\widetilde{\mathcal{L}_\tau}(U)$ to the subgroup $\Gamma_- \subset \Gamma$.

¿From [4] we know that the invertible sheaf over Γ_-:

$$\mathcal{L}_\tau(U) = \widetilde{\mathcal{L}_\tau}(U)|_{\Gamma_- \times \{U\}}$$

is trivial, and that:

$$\sigma_0(g) = g \cdot \delta_U$$

where $g \in \Gamma_-$ and δ_U is a non-zero element in the fibre of $\mathcal{L}_\tau(U)$ over the point $(1, U)$ of $\Gamma \times \{U\}$, is a global section of $\mathcal{L}_\tau(U)$ without zeroes; and, therefore, gives a trivialization.

With respect to this, the global section of $\mathcal{L}_\tau(U)$ defined by $\widetilde{\tau}_U$ is identified with the function $\tau_U \in \mathcal{O}(\Gamma_-) = k\{\{x_1, \dots\}\}$ given by Segal-Wilson [24]:

$$\tau_U(g) = \frac{\widetilde{\tau}_U}{\sigma_0} = \frac{\mu^* \Omega_+}{\sigma_0} = \frac{\Omega_+(gU)}{g \cdot \delta_U}.$$

Observe that the τ-function τ_U is not a series of infinite variables but rather an element of the ring $k\{\{x_1, \dots\}\}$.

Remark 7. In the literature ([25], [3], [15]) one also finds another definition of the τ-function: for a geometric point $\widetilde{U} \in \mathrm{Det}_V$ in the fibre of $U \in \mathrm{Gr}^0(V)$, the $\bar{\tau}$-function of $\widetilde{U}$ is defined as the element $\bar{\tau}(U) \in H^0\,(\mathrm{Det}_V^*)^* \otimes k(U)$, ($k(U)$ being the residual field of U).

The deep relationship between both definitions emerges through the so called bosonization isomorphism. To introduce this isomorphism certain preliminaries are necessary.

Since the subgroup Γ_+ of Γ acts freely on $\mathrm{Gr}^0(V)$, the orbits of the rational points of $\mathrm{Gr}^0(V)$ under the action of Γ_+ are isomorphic to Γ_+ (as schemes). Let X be the orbit of $V_- = z^{-1} \cdot k[z^{-1}] \subset V$ under Γ_+. The restrictions of Det_V and Det_V^* to X are trivial invertible sheaves. Bearing in mind that the points of X are k-vector subspaces of V whose intersection with V_+ is zero, one has that the section Ω_+ of Det_V^* defines a canonical trivialization of Det_V^* over X.

Now, the **bosonization isomorphism** is the canonical isomorphism:

$$\Omega(S) \longrightarrow \mathcal{O}(\Gamma_+)$$

where $\Omega(S)$ is defined in 2.6, induced by the restriction homomorphism:

$$B : H^0(\mathrm{Gr}^0(V), \mathrm{Det}_V^*) \to H^0(X, \mathrm{Det}_V^* \mid_X)$$

and the isomorphism:

$$H^0(X, \mathrm{Det}_V^* \mid_X) \xrightarrow{\sim} \mathcal{O}(\Gamma_+)$$

associated to the trivialization defined by Ω_+.

Finally, note that there exists an isomorphism of k-vector spaces:

$$\mathcal{O}(\Gamma_+)^* = k[x_1, \dots]^* \to \mathcal{O}(\Gamma_-) = k\{\{x_1, x_2, \dots\}\}$$

identifying the Schur polynomial F_S with the linear form:

$$F_{S'} \mapsto F_S(F_{S'}) = \delta_{S,S'};$$

for details see the first chapter of [17]. Now, the composition of the homomorphism B^* (the dual of B) and the above isomorphism gives:

$$\widetilde{B}^* : \mathcal{O}(\Gamma_+)^* = k\{\{x_1, x_2, \dots\}\} \longrightarrow H^0\left(\mathrm{Det}_V^*\right)^*.$$

The relationship between τ_U and $\bar{\tau}(\widetilde{U})$ is: $\widetilde{B}^*(\tau_U) = \bar{\tau}(\widetilde{U})$ up to a non-zero constant.

The transformations of vertex operators under this isomorphism can now be explicitely computed when the characteristic is zero ([8]). In our approach, vertex operators are to be understood as the formal Abel morphisms which will be studied below.

Once we have defined the τ-function algebraically, we can define the Baker-Akhiezer functions using formula 5.14 of [24]; this procedure is used by several authors. However, we prefer to continue with the analogy with the classical theory of curves and Jacobians, and to define the Baker-Akhiezer functions as a formal analogue of the universal invertible sheaf of the Jacobian.

Let us consider the composition of morphism:

$$\widetilde{\beta} : \hat{C} \times \Gamma \times \mathrm{Gr}^0(V) \overset{\phi \times Id}{\longrightarrow} \Gamma \times \Gamma \times \mathrm{Gr}^0(V) \overset{m \times Id}{\longrightarrow} \Gamma \times \mathrm{Gr}^0(V),$$

$\phi : \hat{C} = \mathrm{Spf}\, k[[z]] \to \Gamma$ being the Abel morphism (taking values in $\Gamma_- \subset \Gamma$) and $m : \Gamma \times \Gamma \to \Gamma$ the group law.

Definition 4.3. The sheaf of Baker-Akhiezer functions is the invertible sheaf over $\hat{C} \times \Gamma \times \mathrm{Gr}^0(V)$ defined by:

$$\widetilde{\mathcal{L}}_{BA} = (\phi \times Id)^*(m \times Id)^*\mathcal{P}.$$

Let us define the sheaf of Baker-Akhiezer functions at a point $U \in \mathrm{Gr}^0(V)$ as the invertible sheaf:

$$\widetilde{\mathcal{L}}_{BA}(U) = \widetilde{\mathcal{L}}_{BA}|_{\hat{C}\times\Gamma\times\{U\}} = \widetilde{\beta_U}^* \widetilde{\mathcal{L}_\tau}(U),$$

where $\widetilde{\beta_U}^*$ is the following homomorphism between global sections:

$$H^0(\Gamma \times \{U\}, \widetilde{\mathcal{L}_\tau}(U)) \overset{\widetilde{\beta_U}^*}{\longrightarrow} H^0(\hat{C} \times \Gamma \times \{U\}, \widetilde{\mathcal{L}}_{BA}(U)).$$

By the definitions, $\widetilde{\mathcal{L}}_{BA}(U)|_{\hat{C}\times\Gamma_-\times\{U\}}$ is a trivial invertible sheaf over $\hat{C}\times\Gamma_-$.

Definition 4.4. The Baker-Akhiezer function of a point $U\in\mathrm{Gr}^0(V)$ is $\psi_U = v^{-1}\cdot\beta_U^*(\tau_U)$ where β^* is the restriction homomorphism:

$$H^0\left(\Gamma\times\{U\},\widetilde{\mathcal{L}_\tau}(U)\right)\xrightarrow{\beta^*}H^0\left(\hat{C}\times\Gamma\times\{U\},\widetilde{\mathcal{L}}_{BA}(U)\right)$$

induced by $\widetilde{\beta_U}^*$.

Note that from the definition one has the following expression for the Baker-Akhiezer function:

$$\psi_U(z,g)=v(g,z)^{-1}\cdot\frac{\tau_U\left(g\cdot\phi_1(z)\right)}{\tau_U(g)} \tag{4.5}$$

and that the Baker-Ahkiezer function of $V_-=z^{-1}\,k[z^{-1}]$ is the universal invertible element v^{-1} (see remark 6).

When the characteristic of the base field k is zero, we can identify Γ_- with the additive group scheme $\hat{\mathbb{A}}_\infty$ through the exponential. Therefore, the latter expression is the classical expression for the Baker-Akhiezer functions ([24, 5.16]):

$$\psi_U(z,t)=\exp(-\sum t_i\,z^{-i})\cdot\left(\frac{\tau_U(t+[z])}{\tau_U(t)}\right),$$

where $[z]=(z,\frac{1}{2}z^2,\frac{1}{3}z^3,\dots)$, $t=(t_1,t_2,\dots)$ and, through the exponential map, $v(t,z)=\exp(\sum t_i\,z^{-i})$.

For the general case, we obtain explicit expressions for ψ_U as a function over $\hat{C}\times\hat{\mathbb{A}}_\infty$ but considering in $\hat{\mathbb{A}}_\infty$ the group law, $*$, induced by the exponential 3.5:

$$\psi_U(z,t)=v(t,z)^{-1}\cdot\frac{\tau_U\left(t*\phi(z)\right)}{\tau_U(t)}.$$

Remark 8. Note that our definitions of τ-functions and Baker-Akhiezer functions are valid over arbitrary base schemes. One then has the notion of τ-functions and Baker-Akhiezer functions for families of elements of $\mathrm{Gr}^0(V)$ and, if we consider the Grassmannian of $\mathbb{Z}((z))$, we then have τ-functions and Baker-Akhiezer functions of the rational points of $\mathrm{Gr}\left(\mathbb{Z}((z))\right)$, and the geometric properties studied in this paper have a translation into arithmetic properties of the elements of $\mathrm{Gr}\left(\mathbb{Z}((z))\right)$. The results stated by Anderson in [2] are a particular case of a much more general setting valid not only for p-adic fields but also for arbitrary global numbers field.

The fundamental property of the τ-function is the analogue of the Addition Formulae.

Let ϕ_N be the Abel morphism of degree N ($N > 0$); that is, the morphism $\hat{C}^N \to \Gamma_-$ given by $\prod_{i=1}^N (1 - \frac{x_i}{z})^{-1}$, where $\hat{C}^N = \operatorname{Spf}(A)$ ($A = k[[x_1, \dots, x_N]]$). Let $\Gamma_- = \operatorname{Spf} k\{\{t_1, \dots\}\}$ and observe that, in this setting, t_i should be interpreted as the coefficient of z^{-i} in $\prod(1 - \frac{x_j}{z})^{-1}$.

For simplicity's sake, U_A will denote the point $U \hat{\otimes} A \in \operatorname{Gr}^0(V)(A)$ for a rational point $U \in \operatorname{Gr}(V)$, and V (resp. V_+) will denote $\widehat{V}_A$ (resp. $(\widehat{V_+})_A$).

Now, for our τ-function we shall give formal analogues of the addition formula of [25], of Corollary 2.19 of [9] for theta functions, and of Lemma 4.2 of [12] for τ-funtions. Let us begin with an explicit computation for τ.

Lemma 4.6. *Let U be a rational point of $\operatorname{Gr}^0(V)$. Assume that $V/V_+ + z^N \cdot U = 0$ and $V_+ \cap z^N \cdot U$ is N-dimensional, and let $\{f_1, \dots, f_N\}$ be a basis of the latter. Then:*

$$\phi_N^* \tau_U = \frac{1}{\prod_{i<j}(x_i - x_j)} \cdot det \begin{pmatrix} f_1(x_1) & \dots & f_1(x_N) \\ \vdots & & \vdots \\ f_N(x_1) & \dots & f_N(x_N) \end{pmatrix}$$

as functions on $\hat{C}^N$ (up to elements of k^*).

Proof. By the very definition of the τ-function and the properties of Det, it follows that $\phi_N^* \tau_U$ equals the determinant of the inverse image of the complex:

$$\mathcal{L} \to (V/V_+)_{\operatorname{Gr}^0(V)}$$

by the morphism $\hat{C}^N \to \Gamma_- \to \operatorname{Gr}^0(V)$, which is precisely:

$$\mathcal{C}^\bullet \equiv g \cdot U_A \to A((z))/A[[z]] = V/V_+.$$

Let us define the following homomorphism of A-modules:

$$\begin{aligned} \alpha_N : A[[z]] &\to A^N, \\ f(z) &\mapsto (f(x_1), \dots, f(x_N)), \end{aligned}$$

whose kernel is the ideal generated by $\prod_{i=1}^N (z - x_i)$. One thus has the following exact sequence of complexes of A-modules:

$$\begin{tikzcd} 0 \arrow[r] & \prod(z - x_i) \cdot V_+ \arrow[r] \arrow[d, "\beta"] & V_+ \arrow[r] \arrow[d, "(\pi, \alpha_N)"] & V_+/\prod(z - x_i) \cdot V_+ \arrow[r] \arrow[d, "\bar{\alpha}_N"] & 0 \\ 0 \arrow[r] & V/z^N \cdot U_A \arrow[r] & (V/z^N \cdot U_A) \oplus A^N \arrow[r] & A^N \arrow[r] & 0 \end{tikzcd}$$

The complex on the middle (right-hand side resp.) will be denoted by $\mathcal{C}_1^\bullet$ ($\mathcal{C}_2^\bullet$ resp.). Further, note that the complex on the left-hand side is quasi-isomorphic to $\mathcal{C}^\bullet$.

Observe that:

- $\operatorname{Det}(\mathcal{C}_2^\bullet)$ is isomorphic to the ideal of the diagonals (as an A-module), and the section $det(\bar{\alpha}_N)$ is exactly $\prod_{i<j}(x_i - x_j)$,
- $det(\beta) = det((\pi, \alpha_N)) \cdot det(\bar{\alpha}_N)^{-1}$.

Moreover, we also have another exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_+ \cap z^N \cdot U_A & \longrightarrow & V_+ & \longrightarrow & V_+/V_+ \cap z^N \cdot U_A & \longrightarrow & 0 \\ & & {\scriptstyle \alpha_N^U}\big\downarrow & & {\scriptstyle (\alpha_N,\pi)}\big\downarrow & & {\scriptstyle \simeq}\big\downarrow & & \\ 0 & \longrightarrow & A^N & \longrightarrow & A^N \oplus V/z^N \cdot U_A & \longrightarrow & V/z^N \cdot U_A & \longrightarrow & 0 \end{array}$$

Let $\mathcal{C}_3^\bullet$ denote the complex on the right-hand side. The hypothesis implies that the non-trivial homomorphism of the complex $\mathcal{C}_3^\bullet$ is an isomorphism. They also imply that $\operatorname{Det}(\mathcal{C}_3^\bullet) \in k^*$.

¿From both these exact sequences one has the following relation:

$$det(\beta) = \frac{det(\alpha_N^U)}{\prod_{i<j}(x_i - x_j)} \qquad \text{(up to an element of } k^*\text{)},$$

In order to compute $det(\alpha_N^U)$, let us choose $\{f_1, \dots, f_N\}$ a basis of $V_+ \cap z^N \cdot U_A$ (as an A-module). The matrix associated to α_N^U is now:

$$\begin{pmatrix} f_1(x_1) & \dots & f_1(x_N) \\ \vdots & & \vdots \\ f_N(x_1) & \dots & f_N(x_N) \end{pmatrix}$$

and the formula follows. q.e.d.

This proof implies that (under the same conditions) $\phi_N^* \tau_{hU} = \prod_i h(x_i) \cdot \phi_N^* \tau_U$, where $h \in \Gamma_+$, and also the following theorem.

Theorem 4.7 (Addition Formulae [25]). *Let $U \in \operatorname{Gr}^0(V)$ be a rational point; then, for all $n \leq N$ (natural numbers) the functions:*

$$\left\{ \prod_{j<k} (x_{i_k} - x_{i_j}) \cdot \phi_{i_1 \dots i_n}^* \tau_U \right\}_{0<i_1<\dots<i_n\leq N}$$

satisfy the Plücker equations. Here, $\phi_{i_1 \dots i_n}$ denotes the morphism $\hat{C}^N \to \Gamma_-$ given by $\prod_j (1 - \frac{x_{i_j}}{z})^{-1}$.

We finish this section with a characterization of the Baker-Akhiezer function. The importance of this result will be clear in the next section. Further, this characterization will show the close relationship between the Baker-Akhiezer of a point $U \in \mathrm{Gr}^0(V)$ and a basis of U as k-vector space. (Compare with Proposition 5.1 of [24] and Proposition 4.8 of [15].)

Theorem 4.8. *Let $U \in \mathrm{Gr}^0(V)$ be a rational point. Then:*

$$\psi_U(z,t) = z \cdot \sum_{i \geq 1} \psi_U^{(i)}(z) p_i(t),$$

where $\psi_U^{(i)}(z) \in U$ and $p_i(t) \in k\{\{t_1, \dots\}\}$ is independent of U.

Furthermore, let U be in F_{A_S} for a sequence S and g be

$$\prod_{j=1}^{n} (1 - \frac{x_j}{z})^{-1} \in \Gamma_-.$$

Define $\mathbb{Z} - S = \{s_1, s_2, \dots\}$ (as in 2.D). Then $\psi_U^{(i)}(z)$ has a pole in $z = 0$ of order s_i, and $p_i(x)$ is a homogeneous polynomial in the x's of degree $i-1$.

Proof. Since $\Gamma_- = \operatorname{Spf} k\{\{t_1, \dots\}\}$ is the direct limit of the symmetric products of $\hat{C}$ (recall the relation between the x's and t's), let us compute $\psi_U|_{\hat{C}^N}$ using 4.6 for $N >> 0$. Choose N such that $z^{-N-i}k[[z]] \cap U$ is $N+i$-dimensional for $i = 0, 1$, and let $\{f_1(z), \dots, f_N(z)\}$ be a basis of $z^{-N}k[[z]] \cap U$, and $\{f_1(z), \dots, f_{N+1}(z)\}$ of $z^{-N-1}k[[z]] \cap U$. Without loss of generality, we can assume that f_i has a pole of order s_i at $z = 0$.

Recall that for $g = \prod_{j=1}^{n}(1 - \frac{x_j}{z})^{-1}$ the Baker-Akhiezer function is:

$$\psi_U(z,g) = g(z)^{-1} \cdot \frac{\tau_U(g \cdot \phi(z))}{\tau_U(g)}. \tag{4.9}$$

Let $\phi_{1,N}$ be the morphism $\hat{C} \times \hat{C}^N \to \Gamma_-$ given by $g \cdot (1 - \frac{\bar{z}}{z})^{-1}$, and ϕ_N the morphism $\hat{C}^N \to \Gamma_-$ given by g. Here, the ring of $\hat{C}^N$ is $k[[x_1, \dots, x_N]]$, and the ring of $\hat{C} \times \hat{C}^N$ is $k[[\bar{z}]] \hat{\otimes} k[[x_1, \dots, x_N]]$.

By Lemma 4.6 one sees that:

$$\frac{\phi_{1,N}^* \tau_U}{\phi_N^* \tau_U} = (-1)^N \cdot \prod_{i=1}^{N} (\bar{z} - x_i)^{-1} \cdot \left(\bar{f}_{N+1}(\bar{z}) \prod_i x_i + \sum_{i=1}^{N} \bar{f}_i(\bar{z}) p_i(x_1, \dots, x_N) \right)$$

where $\bar{f}_i(z) = z^{N+1} f_i(z)$, and p_i is a symmetric polynomial. Now, from the above expression for the Baker-Akheizer function, one has:

$$\psi_U(z,t)|_{\hat{C}^N} = z \cdot \sum_{i=1}^{N+1} f_i(z) p_i(x_1, \dots, x_N),$$

and therefore $\psi_U(z,t) = z \cdot \sum_{i>0} f_i(z) p_i(t)$, where $f_i(z) \in U$ and $p_i(t) \in k\{\{t_1, \dots\}\}$. Further, since $\{\psi_U(z,t)|_{\hat{C}^N}\}_{N \in \mathbb{N}}$ is an element of an inverse limit and $p_{N+1}(x)|_{\hat{C}^N}$ is known, one can compute p_i explicitly. From the choice of f_i and the properties of p_i one gets the conclusion. q.e.d.

Now, all the above definitions and results on tau, and BA functions can be generalized to the case of $U \in \mathrm{Gr}^n(V)$, a point in an arbitrary connected component.

Definition 4.10 (τ and BA functions for arbitrary points). Let U be a point of $\mathrm{Gr}^n(V)$ ($n \in \mathbb{Z}$). Define its τ-function by Definition 4.2 but replacing Ω_+ by Ω_+^n (see Remark 4). Define its Baker-Akhiezer function by formula 4.5.

Observe that from the definition of τ-function, we have that:

$$\tau_U(g) = \frac{\Omega_+^n(gU)}{g \cdot \delta_U} = \frac{\Omega_+(g \cdot z^{-n} U)}{g \cdot \delta_U}.$$

Furthermore, Lemma 4.6 can be generalized in the following way: let U be a rational point of $\mathrm{Gr}^n(V)$. Assume that $V/V_+ + z^{N-n} \cdot U = 0$ and $V_+ \cap z^{N-n} \cdot U$ is N-dimensional, and let $\{f_1, \dots, f_N\}$ be a basis of the latter. One then has that:

$$\phi_N^* \tau_U = \frac{1}{\prod_{i<j}(x_i - x_j)} \cdot det \begin{pmatrix} f_1(x_1) & \dots & f_1(x_N) \\ \vdots & & \vdots \\ f_N(x_1) & \dots & f_N(x_N) \end{pmatrix}$$

as functions on $\hat{C}^N$ (up to elements of k^*).

Similarly, the very definition of Baker-Ahkiezer function implies that:

$$\psi_U(z,t) = \psi_{z^{-n}U}(z,t)$$

at a point $U \in \mathrm{Gr}^n(V)$. Moreover, the proof of Theorem 4.8 shows that in this case:

$$\psi_U(z,t) = z^{1-n} \cdot \sum_{i \geq 1} \psi_U^{(i)}(z) p_i(t), \tag{4.11}$$

where $\psi_U^{(i)}(z) \in U$, and $p_i(t) \in k\{\{t_1, \dots\}\}$ is independent of U.

5. Bilinear identity and KP hierarchy

This section has two well defined parts. In the first part the famous Residue Bilinear Identity is deduced, while in the second one the equivalence with the KP hierarchy is shown, which was already known (see [8]). Nevertheless, the importance lies not in the result but in the proofs, since the methods used will also allow us to prove the fundamental theorems of our paper.

The essential ingredient comes from the relationship between $\mathrm{Gr}(k((z)))$ and $\mathrm{Gr}(k((z))^c)$ outlined in 2.E. However, in our case there exists a metric on V; namely, that induced by the residue pairing:

$$(f, g) = \mathrm{Res}_{z=0}(f \cdot g) dz.$$

Let us denote by $\bar{z}^i$ the element of V^c such that $\bar{z}^i(z^j) = \delta_{ij}$. One can formally write $V^c = k((\bar{z}^{-1}))$ (as k-vector spaces), and it may be seen that the homomorphism induced by the residue:

$$\begin{aligned} V &\to V^c, \\ z^i &\mapsto \bar{z}^{-i-1} \end{aligned}$$

is in fact an isomorphism, and sends V_+ to $V_+^\diamond$. It therefore induces an isomorphism $\mathrm{Gr}(V^c, \mathcal{B}^\diamond) \xrightarrow{\sim} \mathrm{Gr}(V, \mathcal{B})$.

The composition of the latter isomorphism and the isomorphism I constructed in 2.E gives a non-trivial automorphism of the Grassmannian:

$$\begin{aligned} R : \mathrm{Gr}(V) &\longrightarrow \mathrm{Gr}(V), \\ L &\longmapsto L^\perp. \end{aligned}$$

Trivial calculation shows that $R^* \operatorname{Det}_V \simeq \operatorname{Det}_V$, and that the index of a point $L \in \mathrm{Gr}(V)$ is exactly the opposite of the index of $R(L) = L^\perp$. Therefore, this induces an involution:

$$R^* : H^0(\mathrm{Gr}^n(V), \operatorname{Det}_n^*) \to H^0(\mathrm{Gr}^{-n}(V), \operatorname{Det}_{-n}^*).$$

It is now straightforward to see that $R^* \Omega_+^{-n} = \Omega_+^n$.

The last remarkable fact about R is that for a given point $U \in \mathrm{Gr}(V)$ the morphism:

$$\mu_U : \Gamma \to \mathrm{Gr}(V)$$

(μ_U being that induced by the action of Γ on $\mathrm{Gr}(V)$) is equivariant with respect to "passing to the inverse" in Γ and R in $\mathrm{Gr}(V)$. In other words:

$$(g \cdot U)^{\perp} = g^{-1} \cdot U^{\perp}.$$

¿From the latter two observations, one has trivially

$$\Omega^n_+(g \cdot U^{\perp}) = \Omega^n_+((g^{-1} \cdot U)^{\perp}) = \Omega^{-n}_+(g^{-1} \cdot U),$$

and hence

$$\tau_{U^{\perp}}(g) = \tau_U(g^{-1}).$$

This motivates us to give the following:

Definition 5.1. The adjoint Baker-Akheizer function of a point $U \in \mathrm{Gr}(V)$ is:

$$\psi^*_U(z, g) = \psi_{U^{\perp}}(z, g^{-1}).$$

(Recall that: $\psi_{U^{\perp}}(z, g) = g^{-1}\frac{\tau_U(g^{-1}*\phi(z)^{-1})}{\tau_U(g^{-1})})$.

Note that in characteristic zero, the definition gives:

$$\psi^*_U(z, t) = \exp(\sum_{i>0} t_i\, z^{-i})\frac{\tau_U(t - [z])}{\tau_U(t)}.$$

¿From the very definition of the Baker-Akhiezer function and formula 4.11, it follows that:

$$\big(\psi_{U^{\circ}}(\bar{z}^{-1}, t')\big)\,(\psi_U(z, t)) \;=\; 0,$$

or, in other words:

$$\mathrm{Res}_{z=0}\, \psi_U(z, t)\psi_{U^{\perp}}(z, t')\frac{dz}{z^2} \;=\; 0.$$

Using the adjoint Baker-Akhiezer function of U as $\psi_{U^{\perp}}$, the latter equation can be rewritten as:

$$\mathrm{Res}_{z=0}\, \psi_U(z, t)\psi^*_U(z, t')\frac{dz}{z^2} \;=\; 0. \tag{5.2}$$

Theorem 5.3. *Let U and U' be two rational points of the Grassmannian and let us assume they have the same index. Then, it holds that:*

$$\mathrm{Res}_{z=0}\, \psi_U(z, t)\psi^*_{U'}(z, t')\frac{dz}{z^2} \;=\; 0 \qquad \Longleftrightarrow \qquad U = U'.$$

Proof. The converse is already shown. For the direct one, observe that the identity and formula 4.11 imply that: $U'^{\perp} \subseteq U^{\perp}$. Recalling that both have the same index and that the metric is non-degenerate, the result follows. q.e.d.

When the base field is the field of complex numbers, $\mathbb{C}$, it is well known that the above results (valid over arbitrary fields) turn out to be equivalent to the KP hierarchy (see [8], [10]).

The goal now is to show how the KP hierarchy is obtained from the Residue Bilinear Identity (when $k = \mathbb{C}$). In this procedure we shall use Schur polynomials and their properties in a very fundamental way. However, the advantage of this is that we can deal in a similar way with the residue condition that characterizes the moduli space of pointed curves in the Grassmannian, and this will allow us to compute its equations explicitly.

Let us begin with some preliminaries on symmetric polynomials following the first chapter of [17]. Let k be a ring of characteristic zero, and let $\mathcal{S}$ denote the subring of $k\{\{x_1, \dots\}\}$ consisting of symmetric polynomials. Given a decreasing sequence λ of natural numbers $\lambda_1 \geq \dots \geq \lambda_n$ define the associated Schur polynomial as:

$$\chi_\lambda(x) = \frac{det(x_j^{\lambda_i + n - i})}{det(x_j^{n-i})}.$$

It is also known that there exists a non-degenerated metric in $\mathcal{S}$ for which the Schur polynomials are an orthonormal basis. Denote by χ_λ^* the linear form $\mathcal{S} \to k$ defined by:

$$\chi_\lambda^*(\chi_\mu(x)) = \delta_{\lambda,\mu}.$$

Therefore, the identity morphism $Id : \mathcal{S} \to \mathcal{S}$ can be expressed as $\sum_\lambda \chi_\lambda(x)\chi_\lambda^*$. That is, for each element $p(x) \in \mathcal{S}$ one has that:

$$p(x) = \sum_\lambda \chi_\lambda(x)\chi_\lambda^*(p).$$

Remark 9.

1. Note that if k is a field, then one can express χ_λ^* in terms of differential operators. Explicitly, this is:

$$\chi_\lambda^* = \chi_\lambda(\widetilde{\partial}_x)|_{x=0},$$

where $\widetilde{\partial}_x = (\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2}, \dots)$. The operator $\sum_\lambda \chi_\lambda(x)\chi_\lambda(\widetilde{\partial}_x)|_{x=0}$ will be called Taylor expansion operator. Analogously, for more than one set of variables, for instance $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$, the identity can be written as:

$$Id = \sum_{\lambda,\mu} \chi_\lambda(x)\chi_\mu(y)\chi_\lambda(\widetilde{\partial}_x)\chi_\mu(\widetilde{\partial}_y)|_{x=y=0}.$$

2. Assume we have two sets of variables, as above. Observe that (applying the Taylor expansion operator to f):

$$\left(\sum_\lambda \chi_\lambda(y)\chi^*_\lambda(\widetilde{\partial}_x)\right) f(x)|_{x=0} = f(y).$$

   That is, this operator replaces x_i by y_i.

3. Similarly, by replacing x_i by $\frac{z^i}{i}$ and denoting by p_j the polynomials defined by $\exp(\sum_{i>0} x_i z^i) = \sum_{j\geq 0} p_j(x) z^j$, one has:

$$\text{coefficient of } z^j \text{ in } f(z) \ = p_j(\widetilde{\partial}_x)|_{x=0} f(x),$$

   (note that $p_j(\widetilde{\partial}_x)|_{x=0}(p_i(x)) = \delta_{ij}$).

4. Now, observe that:

$$P(\partial_y)|_{y=0} f(x+y) = P(\partial_x) f(x),$$

   where P is a polynomial, f is a function, and $x+y$ denotes $(x_1+y_1, x_2+y_2, \dots)$. It therefore follows that:

$$\text{coefficient of } z^j \text{ in } f(x+[z]) \ = \ p_j(\widetilde{\partial}_y)|_{y=0} f(x+y) \ = \ p_j(\widetilde{\partial}_x) f(x),$$

   where $[z]$ is $(z, \frac{z^2}{2}, \dots)$, and hence:

$$f(x+[z]) = \sum_j z^j p_j(\widetilde{\partial}_x) f(x).$$

   That is, the operator $\sum_j z^j p_j(\widetilde{\partial}_x)$ (also written as $\exp(\sum z^i \widetilde{\partial}_{x_i})$) replaces x_i by $x_i + \frac{z^i}{i}$. Finally, this operator relates the τ-function and the Baker-Akhiezer function by:

$$\psi_U(z,t) = \exp(-\sum t_i z^{-i}) \frac{\exp\left(\sum z^i \widetilde{\partial}_t\right) \tau_U(t)}{\tau_U(t)}.$$

We can now state the main result of this subsection in the form given in [10].

Theorem 5.4 (KP Equations). *The condition:*

$$\mathrm{Res}_{z=0}\, \psi_U(z,t)\cdot \psi_U^*(z,t')\frac{dz}{z^2} = 0$$

for a rational point $U \in \mathrm{Gr}(V)$ is equivalent to the infinite set of equations (indexed by a pair of Young diagrams λ_1, λ_2):

$$\Big(\sum p_{\beta_1}(\widetilde{\partial}_t)D_{\lambda_1,\alpha_1}(-\widetilde{\partial}_t)\cdot p_{\beta_2}(-\widetilde{\partial}_{t'})D_{\lambda_2,\alpha_2}(\widetilde{\partial}_{t'})\Big)\,|_{t=t'=0}\tau_U(t)\cdot\tau_U(t') \;=\; 0,$$

where the sum is taken over the 4-tuples $\{\alpha_1,\beta_1,\alpha_2,\beta_2\}$ of integers such that $-\alpha_1+\beta_1-\alpha_2+\beta_2 = 1$, and $D_{\lambda,\alpha} = \sum_\mu \chi_\mu$, where μ is a Young diagram such that $\lambda - \mu$ is a horizontal α-strip.

Proof. First, observe that Γ_- (and equivalently Γ_+) acts on $\mathrm{Gr}(V)$ by homotheties preserving the determinant sheaf. Then, by straightforward computation one shows that $\tau_U(t) = \sum_\lambda \Omega_\lambda(U)\chi_\lambda(t)$, the sum taken over the set of Young diagrams.

Now recall two basic facts about functions $f(x) \in k\{\{x_1,x_2,\dots\}\}$: the coefficient of z^β in $f([z])$ is precisely $p_\beta(\widetilde{\partial}_y)f(y)|_{y=0}$, where $[z] = (z, \frac{1}{2}z^2,\dots)$; and

$$\chi_\lambda(\widetilde{\partial}_y)|_{y=0}f(x+y) = \chi_\lambda(\widetilde{\partial}_x)f(x).$$

Thus

$$\text{coefficient of } z^\beta \text{ in } f(x+[z]) \;=\; p_\beta(\widetilde{\partial}_x)f(x).$$

Let us now begin, properly speaking, with the proof of the theorem. The residue condition is trivially equivalent to:

$$\text{coefficient of } z \text{ in } \exp(-\sum t_i z^{-i})\tau_U(t+z)\exp(\sum t'_i z^{-i})\tau_U(t'-z) \;=\; 0.$$

This coefficient is given by the sum:

$$\sum p_{\alpha_1}(-t)p_{\beta_1}(\widetilde{\partial}_t)\tau_U(t)\cdot p_{\alpha_2}(t')p_{\beta_2}(-\widetilde{\partial}_{t'})\tau_U(t')$$

over the 4-tuples $\{\alpha_1,\beta_1,\alpha_2,\beta_2\}$ of integers such that $-\alpha_1+\beta_1-\alpha_2+\beta_2 = 1$. But now a necessary and sufficient condition for an element $f(t,t') \in k\{\{t,t'\}\}$ to be zero is that $\chi_{\lambda_1}(\widetilde{\partial}_t)\chi_{\lambda_2}(\widetilde{\partial}_{t'})|_{t=t'=0}$ applied to f must be zero for all pairs of Young diagrams λ_1, λ_2. Simple computation together with formula **I**.5.16 of [17] gives the following infinite set of quadratic differential equations on τ_U:

$$\Big(\sum p_{\beta_1}(\widetilde{\partial}_t) D_{\lambda_1,\alpha_1}(-\widetilde{\partial}_t)\cdot p_{\beta_2}(-\widetilde{\partial}_{t'}) D_{\lambda_2,\alpha_2}(\widetilde{\partial}_{t'})\Big)\,|_{t=t'=0}\tau_U(t)\cdot\tau_U(t') \;=\; 0,$$

where $D_{\lambda,\alpha} = \sum_\mu \chi^*_\mu$, the sum being taken over the set of Young diagrams μ such that $\lambda-\mu$ is a horizontal α-strip. q.e.d.

Remark 10. Recalling Remark 3 one obtains four different ways to characterize the set of rational points of the infinite Grassmannian $\mathrm{Gr}(\mathbb{C}((z)))$ into the infinite dimensional projective space $\mathbb{P}\Omega(S)^*$; namely, the Plücker equations ([20]), the Bilinear Residue Identity (see Proposition 4.15 of [15], the KP hierarchy ([8]) and the Hirota's Bilinear equations ([8]). In the last two characterizations the differential operators introduced in Remark 9 are needed.

6. Characterization and equations of the moduli space of pointed curves in the Grassmannian

In this section, since our goal is to compute equations for the moduli of pointed curves, we study a slightly modified Krichever construction; namely, the application:

$$\begin{aligned} \{(C,p,\phi)\} &\to \mathrm{Gr}(V),\\ (C,p,\phi) &\mapsto \phi\left(H^0(C-p,\mathcal{O}_C)\right). \end{aligned}$$

Here and in the sequel $\mathrm{Gr}(V)$ will denote the infinite Grassmannian of the data $(V=k((z)),\mathcal{B},V_+=k[[z]])$ as in Example 1.

Let us introduce some more notation. Given a k-scheme S define:

$$\begin{aligned} \mathcal{O}_S[[z]] &= \varprojlim_n \mathcal{O}_S[z]/z^n,\\ \mathcal{O}_S((z)) &= \varinjlim_m z^{-m}\mathcal{O}_S[[z]]. \end{aligned}$$

Given a flat curve $\pi: C\to S$ and a section $\sigma: S\to C$ defining a Cartier divisor D denote:

$$\widehat{\mathcal{O}}_{C,D} \;=\; \varprojlim_n \mathcal{O}_C/\mathcal{O}_C(-n),$$

where $\mathcal{O}_C(-1)$ is the ideal sheaf of D. Observe that $\widehat{\mathcal{O}}_{C,D}$ is supported along $\sigma(S)$ and is then a sheaf of $\mathcal{O}_S$-algebras. We also define the following sheaf of $\mathcal{O}_S$-algebras:

$$\widehat{\Sigma}_{C,D} = \varinjlim_m \widehat{\mathcal{O}}_{C,D}(m).$$

Definition 6.1. Let S be a k-scheme. Define the functor $\widetilde{\mathcal{M}}^g_\infty$ over the category of k-schemes by:

$$S \rightsquigarrow \widetilde{\mathcal{M}}^g_\infty(S) = \{ \text{ families } (C, D, \phi) \text{ over } S \},$$

where these families satisfy:

1. $\pi : C \to S$ is a proper flat morphism, whose geometric fibres are integral curves of arithmetic genus g,
2. $\sigma : S \to C$ is a section of π, such that when considered as a Cartier Divisor D over C it is smooth, of relative degree 1, and flat over S. (We understand that $D \subset C$ is smooth over S, iff for every closed point $x \in D$ there exists an open neighborhood U of x in C such that the morphism $U \to S$ is smooth).
3. ϕ is an isomorphism of $\mathcal{O}_S$-algebras:

$$\widehat{\Sigma}_{C,D} \overset{\sim}{\to} \mathcal{O}_S((z)).$$

On the set $\widetilde{\mathcal{M}}^g_\infty(S)$ one can define an equivalence relation, $\sim$: (C, D, ϕ) and (C', D', ϕ') are said to be equivalent, if there exists an isomorphism $C \to C'$ (over S) such that the first family goes to the second under the induced morphisms.

Definition 6.2. The moduli functor of pointed curves of genus g, $\mathcal{M}^g_\infty$, is the functor over the category of k-schemes defined by the sheafication of the functor:

$$S \rightsquigarrow \widetilde{\mathcal{M}}^g_\infty(S)/\sim,$$

(the superindex g will be left out to denote the union over all $g \geq 0$).

Proposition 6.3. *The sheaf* $\varinjlim_m \pi_* \mathcal{O}_{C,D}(m)$ *is an* S*-valued point of* $\operatorname{Gr}^{1-g}(\widehat{\Sigma}_{C,D})$ *for all* $(C, D, \phi) \in \mathcal{M}^g_\infty$.

Proof. Consider the following exact sequence:

$$0 \to \mathcal{O}_{C,D}(-n) \to \mathcal{O}_{C,D}(m) \to \mathcal{O}_{C,D}(m)/\mathcal{O}_{C,D}(-n) \to 0$$

for $n, m \geq 0$. Take now π_* and recall that:

$$\pi_*\left(\mathcal{O}_{C,D}(m)/\mathcal{O}_{C,D}(-n)\right) \overset{\sim}{\to} \widehat{\mathcal{O}}_{C,D}(m)/\widehat{\mathcal{O}}_{C,D}(-n),$$

since it is concentrated on $\sigma(S)$. One then has the following long exact sequence:

$$\begin{aligned} 0 \to & \pi_*\mathcal{O}_{C,D}(-n) \to \pi_*\mathcal{O}_{C,D}(m) \\ \to & \widehat{\mathcal{O}}_{C,D}(m)/\widehat{\mathcal{O}}_{C,D}(-n) \to R^1\pi_*\mathcal{O}_{C,D}(-n), \end{aligned}$$

and hence:

$$\begin{aligned} 0 \to & \pi_*\mathcal{O}_{C,D}(-n) \to \varinjlim_m \pi_*\mathcal{O}_{C,D}(m) \\ \to & \widehat{\Sigma}_{C,D}/\widehat{\mathcal{O}}_{C,D}(-n) \to R^1\pi_*\mathcal{O}_{C,D}(-n). \end{aligned}$$

Recalling now from [1] a Grauert type theorem that holds in this case: there exists $\{U_i\}$ a covering of S and $\{m_i\}$ integers such that $\pi_*\mathcal{O}_C(m_i)_{U_i}$ is locally free of finite type, and $R^1\pi_*\mathcal{O}_C(m_i)_{U_i} = 0$; the result follows. q.e.d.

Observe now that ϕ induces an isomorphism of infnite Grassmannians:

$$\mathrm{Gr}^{1-g}(\widehat{\Sigma}_{C,D}) \overset{\sim}{\to} \mathrm{Gr}^{1-g}(V)$$

in a natural way since there exists an integer r such that:

$$z^{n+r}\mathcal{O}_S[[z]] \subseteq \phi(\widehat{\mathcal{O}}_{C,D}(n)) \subseteq z^{n-r}\mathcal{O}_S[[z]] \qquad \forall n.$$

Summing up:

$$\phi\left(\varinjlim_m \pi_*\mathcal{O}_{C,D}(m)\right)$$

is a S-valued point of $\mathrm{Gr}^{1-g}(V)$ for all $(C, D, \phi) \in \mathcal{M}^g_\infty$. In other words, we have defined a morphism of functors:

$$\begin{aligned} K : \mathcal{M}_\infty &\longrightarrow \mathrm{Gr}(V), \\ (C, D, \phi) &\longmapsto \phi\left(\varinjlim_n \pi_*\mathcal{O}_C(n)\right), \end{aligned}$$

which will be called the **Krichever morphism**.(Note that K considered for the Spec($\mathbb{C}$)-valued points is the usual Krichever map, see [13], [21], [24]).

Now let us state the following characterization of the image of K, which is well known in the complex case:

Theorem 6.4. *A point $U \in \mathrm{Gr}(V)(S)$ lies in the image of the Krichever morphism, if and only if $\mathcal{O}_S \subset U$ and $U \cdot U \subseteq U$, where $\cdot$ is the product of V.*

Proof. Suppose we have such a point U of $\mathrm{Gr}(V)(S)$. Since $\mathcal{M}_\infty$ and $\mathrm{Gr}(V)$ are sheaves, we can assume that S is affine, $S = \mathrm{Spec}(B)$. Now, $U \subset B((z))$ is a sub-B-algebra and has a natural filtration U_n by the degree of z^{-1}. Let $\mathcal{U}$ denote the associated graded algebra. It is not difficult to prove that $C = \mathrm{Proj}_B(\mathcal{U})$ is a curve over B.

Let I be the ideal sheaf generated by the elements $a \in \mathcal{U}$ such that the homogeneous localization $\mathcal{U}^0_{(a)}$ is isomorphic to $\mathcal{U}$. Since I is locally principal, it defines a section $\sigma : S \to C$.

Finally, from the inclusion $U \subset \mathcal{O}_S((z))$ one easily deduces an isomorphism $\phi : \widehat{\Sigma}_{C,D} \xrightarrow{\sim} \mathcal{O}_S((z))$.

An easy calculation shows that this construction and the Krichever morphism are the inverse of each other. q.e.d.

Theorem 6.5. *$\mathcal{M}_\infty$ is representable by a closed subscheme of* $\mathrm{Gr}(V)$.

Proof. By the preceding characterization of $\mathcal{M}_\infty$, it will suffice to recall the following result of [1]:

Let A be a k-algebra, and V a sheaf of A-modules over the category of A-algebras. Let M, M' be two quasi-coherent subsheaves of V, such that V/M' is isomorphic to an inverse limit of finite type free A-modules. There then exists an ideal $I \subseteq A$ such that every morphism $f : A \to B$ with the property $M_B \subseteq M'_B$ (as subsheaves of V_B) factorizes through A/I, where the subindex B denotes the canonically induced sheaf of B-modules over the category of B-algebras.

Assuming this result, and since $\hat{V}_S/L$ is isomorphic to an inverse limit of finite type A-modules for $L \in \mathrm{Gr}(V)(S)$, one deduces that the conditions of 6.4 are closed.

Let us now prove the claim. By the hypothesis one has that $(V/M')_B \xrightarrow{\sim} V_B/M'_B$ for all morphisms $A \to B$. Let f be a morphism $A \to B$. Then the canonical inclusion $j : (M + M')/M' \to V/M'$ gives:

$$j^i_B : ((M + M')/M')_B \to (V/M')_B \xrightarrow{\sim} \varprojlim_i L_i \to L_i,$$

where L_i are finite type free B-modules. Thus $M_B \subseteq M'_B$ if and only if j^i_B is identically zero for all i.

Recall that given a sub-A-module $\bar{M}$ of a finite type free A-module $L = A^n$, the inclusion morphism $\bar{j} : \bar{M} \to L$ assigns to each $\bar{m} \in \bar{M}$ a set of coordinates $(\bar{j}_1(\bar{m}), \dots, \bar{j}_n(\bar{m}))$, and hence $\bar{j}_B$ is identically zero if and only if $f : A \to B$ factorizes through the ideal generated by $\{j_i(\bar{m}) \mid \bar{m} \in \bar{M}, i = 1, \dots, n\}$. This concludes the proof. q.e.d.

Our aim, now, is to give explicit characterizations of the set of rational points U satisfying the conditions of Theorem 6.4.

Theorem 6.6. *Let S be a Young diagram of virtual cardinal n, and let U be a rational point of $F_{A_S} \subset \mathrm{Gr}^n(V)$. The following three conditions are equivalent:*

1. *$k \subset U$ and $U \cdot U \subseteq U$,*
2. *$U \cdot U = U$,*
3. *$0 \notin S$ and $\mathrm{Res}_{z=0}\, \psi_U(z,t)\psi_U(z,t')\psi_U^*(z,t'')z^{n-3}dz = 0$.*

Proof. 1 $\implies$ 2 is trivial.

For 2 $\implies$ 1, one has only to check that $k \subset U$. But recall that the element u of $U - \{0\} \subset k((z)) - \{0\}$ of the highest order is unique (up to a non-zero scalar) and should therefore satisfy $u = \lambda \cdot u \cdot u$; hence $u = \lambda^{-1} \in k - \{0\}$.

1 $\implies$ 3 First, note that $k \subset U$ implies $0 \notin S$. It is now clear by 5.2 that the first condition leads to the third.

3 $\implies$ 1 If the residue condition is satisfied, then $U^\perp \subseteq (U \cdot U)^\perp$ and therefore $U \cdot U \subseteq U$, as desired. Now, since $0 \notin S$, an element $u \in U$ of the highest order must have non-negative order, and because of $u \cdot u \in U$, one concludes that $u = \lambda \in k - \{0\}$. q.e.d.

Proposition 6.7. *Let S be a Young diagram. A necessary and sufficient condition for the existence of a rational point $U \in F_{A_S}$ such that $U \cdot U = U$ is that $0 \notin S$ and $\mathbb{Z} - S$ should be closed under addition.*

Proof. Obvious. q.e.d.

This condition will be called the Weierstrass gap property (WGP). Let us denote by $\mathrm{Gr}_W(V)$ the open subscheme of $\mathrm{Gr}(V)$ consisting of the union of the open subsets F_{A_S} such that S satisfies WGP. Then one has:

Corollary 6.8. *The subset*

$$\{U \in \mathrm{Gr}^n(V) \mid k \subset U \text{ and } U \cdot U \subseteq U\}$$

is given by one of the following equivalent conditions:

1. $\mathrm{Gr}_W(V) \cap \Big\{U \in \mathrm{Gr}(V) \mid \mathrm{Res}_{z=0}\, \psi_U(z,t)\psi_U(z,t')\psi_U^*(z,t'')z^{n-3}dz = 0\Big\}$,

2. $\begin{cases} \mathrm{Res}_{z=0}\psi_U^*(z,t)\dfrac{dz}{z^{n+1}} \;=\; 0, \\ \mathrm{Res}_{z=0}\psi_U(z,t)\psi_U(z,t')\psi_U^*(z,t'')z^{n-3}dz \;=\; 0. \end{cases}$

Proof. The first one is obvious. For the second one we only need to show that the condition $k \subset U$ is equivalent to $\mathrm{Res}_{z=0}\, \psi_U^*(z,t)\frac{dz}{z^{n+1}} = 0$. However, from the proof of Theorem 5.3 and formula 4.11 it is easily deduced that $\mathrm{Res}_{z=0}\, f(z) \cdot \psi_U^*(z,t)\frac{dz}{z^{n+1}} = 0$ if and only if $f(z) \in U$.

q.e.d.

Remark 11. Assume that $U \in F_{A_S} \subset \mathrm{Gr}^n(V)\mathrm{Spec}(k))$, S being the sequence associated to a Young diagram, is a point that lies on the image of the Krichever morphism; that is, there exists $(C, p, \phi) \in \mathcal{M}_\infty^g$ such that $K(C, p, \phi) = U$. Note that from the above construction one has an isomorphism $H^0(C - p, \mathcal{O}_C) \tilde{\to} U$ and hence:

- $n = 1 - g$,
- the arithmetic genus of C equals $\#(\mathbb{Z}_{<0} \cap S)$,
- the set of Weierstrass gaps of C at p is precisely $\mathbb{Z}_{<0} \cap S$.

Theorem 6.9. *The condition:*

$$\mathrm{Res}_{z=0}\, \psi_U(z,t)\psi_U(z,t')\psi_U^*(z,t'')\frac{dz}{z^{2+g}} \;=\; 0$$

for a rational point $U \in \mathrm{Gr}^{1-g}(V)$ is equivalent to the infinite set of equations:

$$P(\lambda_1,\lambda_2,\lambda_3)\Big|_{\substack{t=0\\ t'=0\\ t''=0}} \left(\tau_U(t)\cdot\tau_U(t')\cdot\tau_U(t'')\right) \;=\; 0$$

for every three Young diagrams $\lambda_1, \lambda_2, \lambda_3$, where $P(\lambda_1, \lambda_2, \lambda_3)$ is the differential operator defined by:

$$\sum p_{\beta_1}(\widetilde{\partial}_t)D_{\lambda_1,\alpha_1}(-\widetilde{\partial}_t)\cdot p_{\beta_2}(\widetilde{\partial}_{t'})D_{\lambda_2,\alpha_2}(-\widetilde{\partial}_{t'})\cdot p_{\beta_3}(-\widetilde{\partial}_{t''})D_{\lambda_3,\alpha_3}(\widetilde{\partial}_{t''}),$$

where the sum is taken over the 6-tuples $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3\}$ of integers such that $-\alpha_1 + \beta_1 - \alpha_2 + \beta_2 - \alpha_3 + \beta_3 = 1 + g$.

Remark 12. The meaning of the Residue Identity

$$Res_{z=0}\psi\psi\psi^* z^{-(g+2)}dz = 0,$$

where ψ is the Baker function of (C, p, z), is the following: for all sections $s_1, s_2 \in U = H^0(C - p, \mathcal{O}_C)$, $\omega \in U^\perp = H^0(C - p, \Omega_C)$ the differential $s_1 \cdot s_2 \cdot \omega$ has residue zero at p. Or, what amounts to the same: let D_i be the divisor of poles of s_i $(i = 1, 2)$ and D^* that of ω, then $D_1 + D_2 + D^* = K + m \cdot p$ for some nonnegative integer m and some canonical divisor K.

These differential equations are the equations of the moduli of curves (the image of the functor K) in the infinite Grassmannian. A very important fact is that a theta function of a Jacobian variety satisfies these differential equations, which are not obtained from the KP equations. Moreover we have:

Corollary 6.10. *A formal series $\tau(t) \in k\{\{t_1, t_2, \dots\}\}$ is the τ-function associated with a rational point of $\mathcal{M}^g_\infty \subset \mathrm{Gr}^{1-g}(V)$ (and may therefore be written in terms of the theta function of a Jacobian variety) if and only if it satisfies the following set of equations:*

1. *the KP equations (given in Theorem 5.4),*
2. *the p.d.e.'s given in Theorem 6.9,*
3. *the p.d.e.'s:*

$$\left(\sum_{-\alpha+\beta=1-g} p_\beta(-\widetilde{\partial}_t) D_{\lambda,\alpha}(\widetilde{\partial}_t) \right) |_{t=0} \tau_U(t) = 0$$

for all Young diagrams λ.

Proof. Note that the third condition is $\mathrm{Res}_{z=0}\, \psi^*_U(z, t) \frac{dz}{z^{2-g}} = 0$ but given in terms of partial differential equations. q.e.d.

Remark 13. These technics have been used in [19] for the study of the moduli space of Prym varieties and for generalizing the characterizations of Jacobians given by Mulase ([18]) and Shiota ([22]).

Remark 14. An open problem now is to re–state Corollary 6.8 as a characterization for a pseudodifferential operator to come from algebro-geometric data.

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UNIVERSIDAD DE SALAMANCA, SPAIN