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ON THE ARAKELOV GEOMETRY OF MODULI SPACES OF CURVES

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To Herb Clemens on his 60th birthday

Abstract

In this paper we compute the asymptotics of the natural metric on the line bundle over the moduli space $\mathcal{M}_g$ associated to the algebraic cycle $C - C^-$ in the jacobian $\operatorname{Jac} C$ of a smooth projective curve C of genus $g \geq 3$. The asymptotics are related to the structure of the mapping class group of a genus g surface.

1. Introduction

In this paper, we consider some problems in the Arakelov geometry of $\mathcal{M}_g$, the moduli space of smooth projective curves of genus g over $\mathbb{C}$. Specifically, we are interested in naturally metrized line bundles over $\mathcal{M}_g$ and their extensions to $\overline{\mathcal{M}}_g$, the Deligne–Mumford compactification of $\mathcal{M}_g$. These line bundles typically occur when computing the archimedean height of a curve. Here they arise in computations of the archimedean height of the algebraic cycle $C - C^-$ in the jacobian of a genus g curve C (cf. [8]).

As is well known, $\operatorname{Pic} \mathcal{M}_g$ is of rank 1 when $g \geq 3$. It is generated by the class λ of the determinant of the Hodge bundle $\mathcal{L} := \det \pi_* \Omega^1_{\mathcal{C}/\mathcal{M}_g}$, where $\mathcal{C}$ denotes the universal curve over $\mathcal{M}_g$. It has a natural metric which is induced from the standard one on $\pi_* \Omega^1_{\mathcal{C}/\mathcal{M}_g}$:

$$\|\omega\|^2 = \frac{i}{2} \int_C \omega \wedge \overline{\omega}, \qquad \omega \in H^0(C, \Omega^1).$$

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For any metrized line bundle $(\mathcal{N}, |\ |_{\mathcal{N}})$ over $\mathcal{M}_g$, there is some integer N such that $\mathcal{N} \cong \mathcal{L}^{\otimes N}$. This isomorphism is unique up to a constant function as there are no non-constant invertible functions on $\mathcal{M}_g$. Hence, any two metrized line bundles over $\mathcal{M}_g$ can be "compared"; there is a real valued function $f : \mathcal{M}_g \to \mathbb{R}$, unique up to a constant, so that

$$|\ |_{\mathcal{N}} = e^f |\ |_{\mathcal{L}^{\otimes N}}.$$

From this, it follows that, when $g \geq 3$, we have a group isomorphism

$$\left\{\begin{matrix}\text{isomorphism classes of metr-}\\ \text{ized line bundles over } \mathcal{M}_g\end{matrix}\right\} \cong \{(N, df) : N \in \mathbb{Z} \text{ and } f : \mathcal{M}_g \to \mathbb{R}\}$$

under which $(\mathcal{N}, |\ |_{\mathcal{N}})$ corresponds to (N, df). Thus, the study of interesting metrized line bundles over $\mathcal{M}_g$ is just the study of interesting real valued functions on $\mathcal{M}_g$ (mod constants).

Faltings [4] used such a comparison to define functions $\delta_g : \mathcal{M}_g \to \mathbb{R}$ for all $g \geq 1$. He did this by constructing a second metric on $\mathcal{L}$. Since both metrics are on the same line bundle, the function δ_g is a genuine function on $\mathcal{M}_g$, and not merely defined mod constants.

In this paper, we consider a third naturally metrized line bundle $(\mathcal{B}, |\ |_{\mathcal{B}})$ over $\mathcal{M}_g$ for all $g \geq 3$. As a line bundle, $\mathcal{B}$ is isomorphic to the $(8g+4)$th power of $\mathcal{L}$. The line bundle $\mathcal{B}$ is the *biextension line bundle* associated to the algebraic 1-cycle $C - C^-$ in the jacobian $\operatorname{Jac} C$ of a genus g curve C. We review the construction of such biextension bundles in Section 6. Denote the logarithm of the ratio of the metrics on $\mathcal{B}$ and $\mathcal{L}^{\otimes(8g+4)}$ by $\beta_g : \mathcal{M}_g \to \mathbb{R}$. It is well defined mod constants.

For two functions f_1 and f_2 defined on a punctured disk about 0, we write $f_1 \sim f_2$ as $t \to 0$ to mean $f_1(t) - f_2(t)$ remains bounded as $t \to 0$.

Theorem 1.1. *Suppose that $X \to \mathbb{D}$ is a proper family of stable curves of genus $g \geq 3$ over the unit disk. Suppose that X is smooth and that each X_t is smooth when $t \neq 0$.*

(i) *If X_0 is irreducible and has only one node, then*

$$\beta_g(X_t) \sim -g \log|t| - (4g+2) \log\log(1/|t|) \text{ as } t \to 0.$$

(ii) *If X_0 is reducible with one node and its components have genera h and $g-h$, then*

$$\beta_g(X_t) \sim -4h(g-h) \log|t| \text{ as } t \to 0.$$

These asymptotics may be compared with those of Faltings' δ-functions which were established by Jorgenson [**17**] and Wentworth [**29**]. In the notation of Theorem 1.1, their result is

(i) If X_0 irreducible with a single node,

$$3g\delta_g(X_t) \sim -(4g-1)\log|t| - 18\log\log(1/|t|) \text{ as } t \to 0.$$

(ii) If X_0 reducible with a single node and components of genera h and $g-h$

$$3g\delta_g(X_t) \sim -12h(g-h)\log|t| \text{ as } t \to 0.$$

The incommensurability of the asymptotics implies:

Corollary 1.2. *If $g \geq 3$, then the exact 1-forms $d\delta_g$ and $d\beta_g$ on $\mathcal{M}_g$ are linearly independent over $\mathbb{R}$.*

An explicit formula for β_3 is given in [**14**]. It involves the Siegel modular form χ_{18} and an integral. Although explicit, the integral does not seem to be computable at this time.

Motivated by the formula for β_3, and the asymptotics of β_g when $g \geq 3$, it seems natural to define

$$\beta_1 = -\log\|\Delta\| \text{ and } \beta_2 = -2\log\|\chi_{10,}\|$$

where

$$\Delta(q) = (2\pi)^{12} q \prod_{n\geq 0}(1-q^n)^{24}$$

is the cusp form of weight 12 in genus 1,

$$\chi_{10}(\Omega) = \prod_{\alpha \text{ even}} \vartheta_\alpha(0;\Omega)^2$$

is the Siegel modular form of weight 10 on $\mathfrak{h}_2$ (the Siegel upper half plane of rank 2), and $\| \ \|$ denotes the norm

$$\|F(\Omega)\| = |F(\Omega)|(\det \operatorname{Im}\Omega)^{k/2}, \quad \Omega \in \mathfrak{h}_g$$

of a modular form of weight k. Both β_1 and β_2 satisfy the asymptotic formulas in Theorem 1.1. It would be desirable to have a uniform construction of the β_g for all $g \geq 1$ that produces the β_1 and β_2 defined above.

Denote the Deligne–Mumford compactification of $\mathcal{M}_g$ by $\overline{\mathcal{M}}_g$ and the components of the boundary divisor $\overline{\mathcal{M}}_g - \mathcal{M}_g$ by Δ_h, where $0 \leq h \leq [g/2]$. Denote by λ the class of the determinant of the Hodge bundle $\mathcal{L}$ in the Picard group of the moduli stack $\operatorname{Pic}(\overline{\mathcal{M}}_g)$. Theorem 1.1 is a direct consequence of the following Theorem.

Theorem 1.3. *The biextension line bundle $\mathcal{B}$ over $\mathcal{M}_g$ extends naturally to a line bundle $\overline{\mathcal{B}}$ on $\overline{\mathcal{M}}_g$. The metric extends to a smooth metric on $\overline{\mathcal{B}}$ over $\overline{\mathcal{M}}_g - \Delta_0$ and extends continuously to the restriction of $\overline{\mathcal{B}}$ to any holomorphic arc meeting Δ_0 transversely. The first Chern class of the extended bundle is*

$$(8g+4)\lambda - g\delta_0 - 4\sum_{h=1}^{[g/2]} h(g-h)\delta_h \in \operatorname{Pic}(\overline{\mathcal{M}}_g).$$

At present, we do not know whether the metric extends continuously to all of $\overline{\mathcal{M}}_g$. This will probably have to wait until our understanding of limits of mixed Hodge structures improves. Work in this direction is being done by Pearlstein [**23**] and we hope that this question will be resolved in his work.

It is interesting to note that the extended line bundle $\overline{\mathcal{B}}$ is the line bundle shown by Moriwaki [**21**] to have non-negative degree on every complete curve in $\overline{\mathcal{M}}_g$ not contained in the boundary divisor Δ. This is no accident. In a subsequent paper, the first author will show that the curvature of $\overline{\mathcal{B}}$ on $\overline{\mathcal{M}}_g - \Delta_0$ is a non-negative 2-form which is locally L_1 on every holomorphic arc intersecting Δ_0 properly, thus giving an analytic proof of Moriwaki's inequality.

The principal tools used in the proof of Theorem 1.3 are abstract Hodge theory and results about the structure of mapping class and Torelli groups. These are linked by the algebraic cycle $C_x - C_x^-$ in the jacobian $\operatorname{Jac} C$ of C associated to a pointed curve (C, x). The main ingredient from Hodge theory is the theory of degenerations of (mixed) Hodge structures developed by Schmid [**26**], Steenbrink [**27**], Steenbrink and Zucker [**28**], and Guillen et al. [**7**]. It enters via a result of Dale Lear [**18**], which implies that the biextension line bundle $\mathcal{B}$ extends generically across Δ_0 and that $\beta_g(t)$ behaves like a rational multiple of $\log|t|$ near Δ_h when $h > 0$, and like a rational linear combination of $\log|t|$ and $\log\log(1/|t|)$ near Δ_0. Topology is used to compute these rational coefficients. The relation between the asymptotics of β_g near a boundary divisor and topology is analogous to the relation between the asymptotics of a meromorphic function $f(t)$ on the unit disk as $t \to 0$ and the monodromy of $\log f(t)$ when it is analytically continued around a small circle about the origin. The main topological tools originate from Dennis Johnson's work on the abelianization of the Torelli group [**16**].

The algebraic cycle $C_x - C_x^-$ (defined in Section 4) is a homologically trivial algebraic 1-cycle in the jacobian $\operatorname{Jac} C$ of C. It is the bridge which connects Johnson's work with abstract Hodge theory (cf. [**10**]). The associated normal function is a geometric realization of Johnson's representation of the abelianization of the Torelli group (cf. Remark 4.3).

Here, in more detail, is a summary of the proof of Theorem 1.3 and a guide to the structure of the paper. The first main task is to describe and construct the biextension line bundle $\mathcal{B} \to \mathcal{M}_g$. The first ingredient is the $\mathrm{Sp}(H)$-module $V = \Lambda^3 H/H$, where H is the first integral homology of a compact surface of genus g. Some elementary facts about this representation are reviewed in Section 3. The local system $\mathbb{V}$ over $\mathcal{M}_g$ corresponding to the representation V is naturally a variation of Hodge structure. The associated bundle of intermediate jacobians $\mathcal{J}(\mathbb{V})$ over $\mathcal{M}_g$ is a bundle of compact tori whose fibers have fundamental group isomorphic to V. The algebraic cycle $C_x - C_x^-$ gives rise to a section ν of $\mathcal{J}(\mathbb{V})$. This section is the *normal function* associated to $C - C^-$. These constructions are reviewed in Section 4.

The biextension bundle $\mathcal{B} \to \mathcal{M}_g$ is the pullback $\nu^*\widehat{\mathcal{B}}$ of a metrized line bundle $\widehat{\mathcal{B}} \to \mathcal{J}(\mathbb{V})$. The construction of $\widehat{\mathcal{B}}$ and its metric uses abstract Hodge theory. In order to make the paper accessible to those not familiar with abstract Hodge theory, an introduction to, and characterization of, $\widehat{\mathcal{B}}$ and its metric is given in Section 6. The actual construction of $\widehat{\mathcal{B}}$ is reviewed, and its curvature computed, in Section 7. The description of the curvature used in both sections uses the construction, from foliation theory, of closed forms on flat torus bundles given in Section 5. This description allows us to apply a theorem Morita [**20**] to show that the first Chern class of $\mathcal{B}$ over $\mathcal{M}_g$ is $(8g+4)\lambda$.

The metrized biextension bundle $\widehat{\mathcal{B}}$ is easily seen to extend to the intermediate jacobian bundle $\mathcal{J}(\mathbb{V})$ over $\widetilde{\mathcal{M}}_g$, the moduli space of genus g curves of compact type. The normal function ν of $C_x - C_x^-$ also extends to $\widetilde{\mathcal{M}}_g$, which implies that $\mathcal{B}$ and its metric extend to $\widetilde{\mathcal{M}}_g$. A theorem of Lear [**18**], reviewed in Section 8, implies that $\mathcal{B}$ extends locally, as a metrized line bundle, across Δ_0. At this point, we can show that Theorem 1.3 implies Theorem 1.1.

The second main task is to compute the Chern class of the extended biextension line bundle $\overline{\mathcal{B}}$ over $\overline{\mathcal{M}}_g$. To do this, we consider the line bundle $\mathcal{B} \otimes \mathcal{L}^{\otimes(-(8g+4))}$ over $\mathcal{M}_g$. This is trivial, and the trivialization is unique up to a constant. In Section 9, we consider the representation of the genus g mapping class group induced by a trivializing section. Its restriction to the Torelli group T_g

$$\hat{\tau} : T_g \to \text{Heisenberg group of } (V, q),$$

where q denotes the $\mathrm{Sp}(H)$-invariant inner product that polarizes V, determines the Chern class of $\overline{\mathcal{B}}$. When $h > 0$, the conjugacy class of a small loop σ_h about Δ_h lies in the Torelli group and $\hat{\tau}(\sigma_h)$ lies in the center $\mathbb{Z}$ of the Heisenberg group of (V, q). The integer $\hat{\tau}(\sigma_h)$ is minus the coefficient of δ_h in the expression for $c_1(\mathcal{B})$ and determines the asymptotics of β_g near Δ_h. The Chern class of $\mathcal{B}$ over $\widetilde{\mathcal{M}}_g$ is computed in Section 11. The coefficient of Δ_0 is computed in Section 12 by restricting to the hyperelliptic locus, where $\overline{\mathcal{B}}$ is trivial, and comparing the computations of the previous section with a result of Cornalba and Harris [**3**].

Section 2 lists many of the conventions used in the paper, as well as a few basic results used later in the paper.

2. Preliminaries

All varieties in this paper will be over $\mathbb{C}$. The moduli space of n-pointed ($n \geq 0$) smooth projective curves will be denoted by $\mathcal{M}_g^n$ and its Deligne–Mumford compactification by $\overline{\mathcal{M}}_g^n$. We will be primarily interested in the cases when $n = 0$ and 1. When $n = 0$, we will write $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ instead of $\mathcal{M}_g^0$ and $\overline{\mathcal{M}}_g^0$.

The moduli space of principally polarized abelian varieties of dimension g will be denoted by $\mathcal{A}_g$. The level l covers of $\mathcal{M}_g$ and $\mathcal{A}_g$ will be denoted by $\mathcal{M}_g[l]$ and $\mathcal{A}_g[l]$.

All moduli spaces will be regarded as orbifolds, or more accurately, as stacks in the sense of Mumford [**22**]. By the fundamental group, we shall mean the orbifold fundamental group. In particular, when $g \geq 2$, the fundamental group of $\mathcal{M}_g^n$ is isomorphic to Γ_g^n, the mapping class group of an n-pointed, genus g surface. The fundamental group of $\mathcal{A}_g$ is isomorphic to $\mathrm{Sp}_g(\mathbb{Z})$.

The difference $\overline{\mathcal{M}}_g^n - \mathcal{M}_g^n$ is a divisor Δ which is smooth with normal crossings (provided we regard $\overline{\mathcal{M}}_g^n$ as an orbifold). When $n = 0$

$$\Delta = \bigcup_{h=0}^{[g/2]} \Delta_h.$$

The generic point of each Δ_h corresponds to a stable genus g curve with 1 node. The generic point of Δ_0 is irreducible and has normalization of genus $g-1$; the generic point of Δ_h when $h>0$ corresponds to a reducible curve with components of genera h and $g-h$.

We shall denote by $\widetilde{\mathcal{M}}_g$ the open subset $\overline{\mathcal{M}}_g - \Delta_0$. This is the moduli space of stable genus g curves of *compact type*. The period map $\mathcal{M}_g \to \mathcal{A}_g$ extends to $\widetilde{\mathcal{M}}_g$ and the extended period map $\widetilde{\mathcal{M}}_g \to \mathcal{A}_g$ is proper and has closed image.

The class of the determinant of the Hodge bundle in $\operatorname{Pic}\overline{\mathcal{M}}_g$ will be denoted by λ and the class of the divisor Δ_h by $\delta_h \in \operatorname{Pic}\overline{\mathcal{M}}_g$. Recall from [**1**] that $\operatorname{Pic}\overline{\mathcal{M}}_g$ is free with basis $\lambda, \delta_0, \delta_1, \dots, \delta_{[g/2]}$. From this, it follows that $\operatorname{Pic}\widetilde{\mathcal{M}}_g$ is also free with basis $\lambda, \delta_1, \dots, \delta_{[g/2]}$

The Torelli group T_g^n is the kernel of the natural map $\Gamma_g^n \to \mathrm{Sp}_g(\mathbb{Z})$ induced by the period map $\mathcal{M}_g^n \to \mathcal{A}_g$.

Denote the moduli space of smooth hyperelliptic curves of genus g by $\mathcal{H}_g$. We view it as an orbifold. The following lemma is surely well known, but we state and prove it for lack of a convenient reference.

Lemma 2.1. *For all $g \ge 1$ and all $n \ge 0$, the only invertible functions on $\mathcal{M}_g^n$, $\mathcal{H}_g$ and $\mathcal{A}_g$ are the constants. That is,*

$$H^0(\mathcal{M}_g^n, \mathcal{O}^\times) = H^0(\mathcal{H}_g, \mathcal{O}^\times) = H^0(\mathcal{A}_g, \mathcal{O}^\times) = \mathbb{C}^\times.$$

Proof. Suppose that X is a smooth variety. If f is a non-constant invertible function on X, then $d\log f/2\pi i$ is a non-trivial element of $H^1(X, \mathbb{Q})$. Each of the moduli spaces considered in this proposition has trivial first homology with rational coefficients. This is well known for $\mathcal{M}_g$ and $\mathcal{A}_g$. The corresponding statement for $\mathcal{H}_g$ follows as $\mathcal{H}_g$ is the moduli space of $(2g-2)$-tuples of distinct points in $\mathbb{P}^1$ modulo projective equivalence:

$$\mathcal{H}_g = \big((\mathbb{C} - \{0,1\})^{2g-1} - \Delta\big)/S_{2g+2},$$

where Δ denotes the fat diagonal. Now, $H_1(\mathbb{C} - \{0,1\})^{2g-1} - \Delta, \mathbb{Q})$ is the irreducible S_{2g+2}-module corresponding to the partition $[2g, 2]$ of $2g+2$ — cf. [**12**, (10.6)] — and thus does not contain a copy of the trivial representation of S_{2g+2}. (There, our space is denoted Y^2_{2g-1}, which is Y_1^{2g} by duality — see [**12**, (5.7)].) It follows that none of $\mathcal{M}_g$, $\mathcal{A}_g$ or $\mathcal{H}_g$ has a non-constant invertible function when $g \ge 1$. q.e.d.

Finally, some notation: If A is a $\mathbb{Z}$ (mixed) Hodge structure, we shall use $A_\mathbb{Z}$, $A_\mathbb{R}$, $A_\mathbb{C}$ to denote the integral lattice, the underlying real vector space, and the underlying complex vector space. If A has negative weight (typically -1), we shall set

$$J(A) = A_{\mathbb{C}}/(F^0 A + A_{\mathbb{Z}}).$$

When $A_{\mathbb{Z}}$ is torsion free, this is naturally isomorphic to $\mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Z}, A)$, where $\mathcal{H}$ denotes the category of mixed Hodge structures.

3. Some algebra

In this section, we dispense with some algebra essential when studying the algebraic cycle $C - C^-$ and in studying the Torelli group T_g. Suppose $g \geq 2$. Fix a lattice H of rank $2g$ with a unimodular symplectic form $(\ \cdot\ )$. This gives a natural identification of H with its dual. For example, we will take H to be $H_1(C, \mathbb{Z})$ where C is a compact Riemann surface with the intersection pairing, or more generally, $H_1(A, \mathbb{Z})$ where A is a principally polarized abelian variety. The dual of the intersection pairing may be viewed as an element θ of $\Lambda^2 H$. Set

$$V = \Lambda^3 H/(\theta \wedge H).$$

Denote the group of automorphisms of H that preserve the intersection pairing by $\mathrm{Sp}(H)$. By choosing a symplectic basis of H, we can identify it with $\mathrm{Sp}_g(\mathbb{Z})$. Note that V is an $\mathrm{Sp}(H)$-module.

Denote the intersection number of $x, y \in H$ by $x \cdot y$. This can be used to define an $\mathrm{Sp}(H)$-equivariant map

$$c : \Lambda^3 H \to H.$$

It takes $x \wedge y \wedge z$ to $(x \cdot y)z + (y \cdot z)x + (z \cdot x)y$. There is also the $\mathrm{Sp}(H)$-invariant map $H \to \Lambda^3 H$ that takes x to $x \wedge \theta$.

Proposition 3.1. *For all $x \in H$, $c(x \wedge \theta) = (g-1)x$. Consequently, if $g = 2$, then $c : \Lambda^3 H \to H$ is an isomorphism and $V = 0$.*

This shows that

$$\Lambda^3 H_{\mathbb{Q}} \cong V_{\mathbb{Q}} \oplus H_{\mathbb{Q}}$$

as $\mathrm{Sp}(H)$ modules. It is well known from the representation theory of the symplectic group that both $V_{\mathbb{Q}}$ and $H_{\mathbb{Q}}$ are irreducible as $\mathrm{Sp}(H)$-unique up to a constant multiple. Next, we give a formula for the primitive symplectic form $q : \Lambda^2 V \to \mathbb{Z}$.

Denote the image of $x \wedge y \wedge z$ in V by $\overline{x \wedge y \wedge z}$. It follows from the above that the function

$$j : V \to \Lambda^3 H$$

defined by

$$\overline{x \wedge y \wedge z} \to (g-1)x \wedge y \wedge z - \theta \wedge c(x \wedge y \wedge z)$$

is integral and $\mathrm{Sp}_g(\mathbb{Z})$ invariant. Denote the symplectic form on $\Lambda^3 H$ induced from that on H by $\langle\ ,\ \rangle$. That is,

$$\langle x_1 \wedge x_2 \wedge x_3, y_1 \wedge y_2 \wedge y_3 \rangle = \det(x_i \cdot y_j).$$

The symplectic form q on V is defined by

$$q(u,v) = \langle j(u), j(v) \rangle/(g-1).$$

An easy computation shows that

$$\langle x \wedge \theta, y \wedge \theta \rangle = (g-1)(x \cdot y).$$

Since $\theta \wedge H$ and $j(V)$ are orthogonal, it follows that q takes values in $\mathbb{Z}$.

Remark 3.2. The algebra above arises in the work of Johnson [**16**]. He constructed an $\mathrm{Sp}(H)$-invariant homomorphism

$$\tau_g^1 : H_1(T_g^1) \to \Lambda^3 H$$

from the abelianization of the Torelli group of a pointed, genus g surface (C, x), where H denotes $H_1(C, \mathbb{Z})$. There is a natural extension

$$1 \to \pi_1(C, x) \to T_g^1 \to T_g \to 1.$$

Abelianizing, we obtain the exact sequence

$$H \to H_1(T_g^1) \to H_1(T_g) \to 0.$$

The composite $H \to H_1(T_g^1) \to \Lambda^3 H$ is $x \mapsto x \wedge \theta$. From this, it follows that τ_g^1 induces a $\mathrm{Sp}(H)$-invariant homomorphism

$$\tau_g : H_1(T_g) \to V.$$

Fundamental theorems of Johnson assert that τ_g^1 and τ_g are isomorphisms mod 2-torsion.

4. The fundamental normal function

In this section, we recall the construction of the fundamental normal function over $\mathcal{M}_g$. This is a geometric (even motivic) realization of the Johnson homomorphism τ_g. More details can be found in [**10**] and [**11**, Section 7].

Suppose that A is a principally polarized abelian variety of dimension g. Let

$$V_{A,\mathbb{Z}} = \Lambda^3 H_1(A,\mathbb{Z})/H_1(A,\mathbb{Z}) = H_3(A,\mathbb{Z})/H_1(A,\mathbb{Z}).$$

Here, $H_1(A,\mathbb{Z})$ is imbedded in $\Lambda^3 H_1(A,\mathbb{Z})$ via the map that takes $x \in H_1(A,\mathbb{Z})$ to $x \wedge \theta$, where

$$\theta \in \Lambda^2 H_1(A,\mathbb{Z})$$

is the dual of the principal polarization. Note that when $g \leq 2$, $V_{A,\mathbb{Z}}$ is trivial — cf. Proposition 3.1.

Since the category of Hodge structures is abelian, and since wedging with θ is a morphism of Hodge structures, there is unique Hodge structure of weight -1 on V_A such that the quotient map

$$H_3(A,\mathbb{Z}(-1)) \to V_A$$

is a morphism of Hodge structures. From this, we can construct the compact complex torus

$$J(V_A) := V_{A,\mathbb{C}}/\left(F^0 V_A + V_{A,\mathbb{Z}}\right).$$

This can be thought of as the Griffiths intermediate jacobian associated to the primitive part of the third homology of A.

When A is the jacobian $\operatorname{Jac} C$ of a smooth projective curve C of genus g, we shall abuse notation and write V_C instead of $V_{\operatorname{Jac} C}$.

Harris' *Harmonic volume* [**15**, **24**] gives a canonical point $\nu(C)$ of $J(V_C)$ whose construction we now recall. Choose a point $x \in C$. This gives an imbedding

$$C \to \operatorname{Jac} C$$

by taking $y \in C$ to $[y] - [x]$. Denote the image of this map by C_x and the image of C_x under the involution $D \mapsto -D$ of $\operatorname{Jac} C$ by C_x^-. The cycle $C_x - C_x^-$ is homologous to 0, and therefore gives a point $\nu_x(C)$ in the Griffiths intermediate jacobian

$$J_1(\operatorname{Jac} C) := H_3(\operatorname{Jac} C,\mathbb{C})/\left(F^{-1} H_3(\operatorname{Jac} C) + H_3(\operatorname{Jac} C,\mathbb{Z})\right).$$

This compact complex torus maps to $J(V_C)$ as $H_3(\operatorname{Jac} C)$ is canonically isomorphic to $\Lambda^3 H_1(C,\mathbb{Z})$. The kernel of the quotient map $J(\operatorname{Jac} C) \to J(V_C)$ is naturally isomorphic to $\operatorname{Jac} C$. One has [**24**] that

$$\nu_x(C) - \nu_y(C) = -2([x]-[y]) \in \operatorname{Jac} C.$$

It follows that the image of $\nu_x(C)$ in $J(V_C)$ is independent of x. The common image of the $\nu_x(C)$ in $J(V_C)$ will be denoted by $\nu(C)$.

The abelian groups $V_{A,\mathbb{Z}}$ form a local system, or more accurately, a polarized variation of Hodge structure of weight -1 over $\mathcal{A}_g$. We shall denote it by $\mathbb{V}$. Taking the intermediate jacobian of each fiber gives a complex analytic bundle

$$\mathcal{J}(\mathbb{V}) \to \mathcal{A}_g$$

whose fiber over the moduli point $[A]$ of A is $J(V_A)$.

Remark 4.1. This bundle of tori is trivial (i.e., the fiber is 0) when $g \leq 3$.

Harmonic volume gives a lift of the period map $\mathcal{M}_g \to \mathcal{A}_g$:

$$\begin{tikzcd} & \mathcal{J}(\mathbb{V}) \arrow[d] \\ \mathcal{M}_g \arrow[ur, "\nu"] \arrow[r] & \mathcal{A}_g \end{tikzcd}$$

This lift is the *normal function* associated to the cycle $C - C^-$.

We shall denote $\overline{\mathcal{M}}_g - \Delta_0$ by $\widetilde{\mathcal{M}}_g$. This variety is the moduli space of genus g curves of compact type; that is, stable genus g curves whose dual graph is a tree, or equivalently, whose Pic^0 is an abelian variety. The period map extends to a proper map $\widetilde{\mathcal{M}}_g \to \mathcal{A}_g$.

Proposition 4.2. *The normal function ν extends to a normal function $\tilde{\nu} : \widetilde{\mathcal{M}}_g \to \mathcal{J}(\mathbb{V})$ that lifts the period mapping $\widetilde{\mathcal{M}}_g \to \mathcal{A}_g$.*

Proof. This follows immediately from [**10**, (7.1)] as the variation of Hodge structure $\mathbb{V}$ extends to a variation over $\widetilde{\mathcal{M}}_g$. q.e.d.

Remark 4.3. The fundamental group of $\mathcal{J}(\mathbb{V}) \to \mathcal{A}_g$ is naturally isomorphic to the semi-direct product $\mathrm{Sp}(H) \ltimes V$. The normal function ν induces a lift

$$\nu_* : \Gamma_g \to \mathrm{Sp}(H) \ltimes V$$

of the natural homomorphism $\rho : \Gamma_g \to \mathrm{Sp}(H)$ whose restriction to the Torelli group T_g is $2\tau_g$. For full details, see [**10**].

5. Flat torus bundles

In this section, we show how invariant cohomology classes on the fiber of a flat bundle of real tori can be lifted naturally to closed differential forms (and thus cohomology classes) on the total space of the torus bundle. We will apply this to construct to the torus bundle $\mathcal{J}(\mathbb{V}) \to \widetilde{\mathcal{M}}_g$ to construct the curvature form of a biextension bundle over $\mathcal{J}(\mathbb{V})$.

Suppose that $T \to B$ is a smooth family of compact tori over a connected manifold B. Fix a base point b_o of B and denote the fiber of it by F_o. We shall also assume that the family has a distinguished section, which we shall call the zero-section. The first result comes from foliation theory. It implies that the restriction mapping $H^k(T,\mathbb{R}) \to H^0(B, H^k(F_o), \mathbb{R})$ has a natural section.

Lemma 5.1. *For such a family of compact tori, there is a natural mapping*

$$s : H^0(B, H^k(F_o,\mathbb{R})) \to H^k(T,\mathbb{R}).$$

whose composition with the projection

$$H^k(T,\mathbb{R}) \to H^0(B, H^k(F_o,\mathbb{R}))$$

is the identity. Moreover, for each $u \in H^0(B, H^k(F_o))$, the extended class $\omega_u := s(u)$ has a natural representative $\tilde{\omega}_u$ whose restriction to the zero-section is trivial and to each fiber is translation invariant.

Proof. Denote the fiber of $T \to B$ over b by F_b. We have the local systems $\mathbb{V}_\mathbb{Z}$ and $\mathbb{V}_\mathbb{R}$ whose fibers over $b \in B$ are $H_1(F_b,\mathbb{Z})$ and $H_1(F_b,\mathbb{R})$. The quotient $\mathbb{V}_\mathbb{R}/\mathbb{V}_\mathbb{Z}$ is a family of tori over B which is naturally isomorphic to the family $T \to B$ via an isomorphism that preserves the zero-sections.

Denote the base point of B by b_o. The first homology of F_o, the fiber over b_o is a $\pi_1(B, b_o)$-module. Note that the group

$$\pi_1(B, b_o) \ltimes H_1(F_o,\mathbb{Z})$$

acts on $\widetilde{B} \times H_1(F_o,\mathbb{R})$ via the action

$$(\gamma, z) : (b, a) \mapsto (\gamma \cdot b, \gamma \cdot a + z).$$

Here $\widetilde{B}$ denotes a universal covering of B. The quotient

$$(\pi_1(B, b_o) \ltimes H_1(F_o,\mathbb{Z})) \backslash \left(\widetilde{B} \times H_1(F_o,\mathbb{R})\right)$$

is naturally isomorphic to $\mathbb{V}_\mathbb{R}/\mathbb{V}_\mathbb{Z}$ as a bundle over B.

Each $\pi_1(B, b_o)$-invariant cohomology class u in $H^k(F_o,\mathbb{R})$ is represented by a translation invariant k-form ω_u on $H_1(F_o,\mathbb{R})$. The pull-back of ω_u along the projection $p : \widetilde{B} \times H_1(F_o,\mathbb{R}) \to H_1(F_o,\mathbb{R})$ is a k-form invariant under the action of $\pi_1(B, b_o) \ltimes H_1(F_o, \mathbb{Z})$. It, therefore, descends to a closed form $\tilde{\omega}_u$ on $\mathbb{V}_\mathbb{R}/\mathbb{V}_\mathbb{Z}$ whose restriction to the fiber over b_o is ω_u.

The mapping s is defined by taking $s(u)$ to be the de Rham class of $\tilde{\omega}_u$. q.e.d.

Corollary 5.2. *With the same hypotheses as above, for all $r \geq 2$, the differentials*

$$d_r : E_r^{0,k} \to E_r^{r,k-r+1}$$

vanish in the Leray–Serre spectral sequence

$$E_2^{s,t} = H^s(B, H^t(F_o, \mathbb{R})) \implies H^{s+t}(T, \mathbb{R}).$$

Corollary 5.3. *If $H^1(B, H^1(F_o, \mathbb{R})) = 0$, then each $\pi_1(B, b_o)$-invariant element u of $H^2(F_o, \mathbb{R})$ extends to a unique element ω_u of $H^2(T, \mathbb{R})$ whose restriction to F_o is u and whose restriction to the zero-section is trivial. If u is a rational cohomology class, then so is ω_u.*

Proof. The invariant class u lifts to $\omega_u = s(u) \in H^2(T, \mathbb{R})$, which has the property that its restriction to the zero-section vanishes. The vanishing of $H^1(B, H^1(F_o, \mathbb{R}))$, Lemma 5.1 and the existence of a section (the zero-section) imply that the sequence

$$0 \to H^2(B) \to H^2(T) \to H^2(F_o)^{\pi_1(B, b_o)} \to 0$$

is exact with real, and therefore rational, coefficients. Restricting to the zero-section defines a splitting of the left-hand mapping. This gives a direct sum decomposition of $H^2(T)$ which is defined over $\mathbb{Q}$. The uniqueness and rationality statements follow. q.e.d.

Corollary 5.4. *For all $g \geq 1$, there is a unique element ϕ of $H^2(\mathcal{J}(\mathbb{V}), \mathbb{Q})$ whose restriction to the zero-section is trivial and whose restriction to each fiber is the class of the polarization q. It is characterized by these properties.*

Proof. When $g \leq 2$, the result is trivially true as the bundle $\mathcal{J}(\mathbb{V})$ has the trivial complex torus as fiber. When $g \geq 2$, we know that $H^1(\mathrm{Sp}_g(\mathbb{Z}), V)$ vanishes [**25**], and so one can apply the previous result. q.e.d.

We will see in Proposition 7.3 that 2ϕ is integral. The following result is due to Morita [**20**, (5.8)]. The precise statement below and a more direct proof can be found in [**13**].

Theorem 5.5 (Morita)**.** *For all $g \geq 1$, we have $\nu^*(2\phi) = (8g+4)\lambda$ in $H^2(\mathcal{M}_g, \mathbb{Z})$.*

6. Introduction to the biextension line bundle and its metric

This section is a non-technical introduction to the biextension bundle over $\mathcal{J}(\mathbb{V})$ and its metric. The metrized biextension bundle is characterized by its curvature form. The construction of the bundle using abstract Hodge theory is reviewed in Section 7.

Proposition 6.1. *There is at most one holomorphic line bundle $\widehat{\mathcal{B}}$ over $\mathcal{J}(\mathbb{V})$ whose first Chern class is 2ϕ and whose restriction to the zero-section is trivial.*

Proof. Two such bundles differ by a line bundle which is trivial on the zero-section and is topologically trivial on each fiber. The group of topologically trivial line bundles on $J(V_A)$ is $J(\check{V}_A)$ where

$$\check{V} = \mathrm{Hom}(V, \mathbb{Z}(1))$$

(cf. [**8**, (3.1.6)]). Thus, to show that two such bundles are isomorphic, it suffices to show that the bundle of intermediate jacobians associated to the variation $\check{\mathbb{V}}$ has no sections. It follows from [**10**, (9.2)] that all normal function sections of $\mathcal{J}(\check{\mathbb{V}})$ are torsion. The result therefore follows from the following algebraic fact. q.e.d.

Proposition 6.2. *For each prime p, there are no non-zero $\mathrm{Sp}_g(\mathbb{Z})$-invariant elements of $\mathrm{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}, \mathbb{F}_p)$, $\mathrm{Hom}_{\mathbb{Z}}(\Lambda^3 H_{\mathbb{Z}}, \mathbb{F}_p)$ or $\big(\Lambda^3 H_{\mathbb{Z}}\big) \otimes \mathbb{F}_p$.*

Proof. Note that $\mathrm{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}, \mathbb{F}_p) \cong \mathrm{Hom}_{\mathbb{F}_p}(V_{\mathbb{Z}} \otimes \mathbb{F}_p, \mathbb{F}_p)$. By the right exactness of tensor product, the natural mapping

$$\big(\Lambda^3 H_{\mathbb{Z}}\big) \otimes \mathbb{F}_p \to V_{\mathbb{Z}} \otimes \mathbb{F}_p$$

is surjective, from which it follows that $\mathrm{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}, \mathbb{F}_p)$ is a submodule of

$$\mathrm{Hom}_{\mathbb{F}_p}\big(\big(\Lambda^3 H_{\mathbb{Z}}\big) \otimes \mathbb{F}_p, \mathbb{F}_p\big).$$

When p is odd, $-I \in \mathrm{Sp}_g(\mathbb{Z})$ acts as -1 on $\mathrm{Hom}_{\mathbb{Z}}(\Lambda^3 H_{\mathbb{Z}}, \mathbb{F}_p)$ and $\big(\Lambda^3 H_{\mathbb{Z}}\big) \otimes \mathbb{F}_p$, which implies there are no invariants except possibly when

$p = 2$. Thus, it suffices to prove that

$$\left[\left(\Lambda^3 H_{\mathbb{Z}}\right) \otimes \mathbb{F}_2\right]^{\mathrm{Sp}_g(\mathbb{Z})} = 0 \text{ and } \mathrm{Hom}_{\mathrm{Sp}_g(\mathbb{Z})}\left(\left(\Lambda^3 H_{\mathbb{Z}}\right) \otimes \mathbb{F}_2, \mathbb{F}_2\right) = 0.$$

Choose a symplectic basis $a_1, \dots, a_g, b_1, \dots, b_g$ of $H_{\mathbb{Z}}$. Let H_j be the span of a_j and b_j. Then,

$$H_{\mathbb{Z}} = H_1 \oplus \cdots \oplus H_g.$$

The intersection of $\mathrm{Sp}_g(\mathbb{Z})$ with $\mathrm{GL}(H_j)$ is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$ and this gives an imbedding of $\mathrm{SL}_2(\mathbb{Z}) \times \cdots \times \mathrm{SL}_2(\mathbb{Z})$ (g factors) into $\mathrm{Sp}_g(\mathbb{Z})$.

Denote the mod 2 reduction of $H_{\mathbb{Z}}$ by W and of H_j by W_j. Set

$$U = \left(\Lambda^3 H_{\mathbb{Z}}\right) \otimes \mathbb{F}_2.$$

Since $\mathrm{SL}_2(\mathbb{F}_2)$ is isomorphic to S_3, the symmetric group on 3 letters, we have a diagonal imbedding of $G = S_3 \times \cdots \times S_3$ (g factors) into the image of $\mathrm{Sp}_g(\mathbb{Z})$ in $\mathrm{GL}(U)$.

Since $(\mathbb{F}_2)^2$ is a *simple* $SL_2(\mathbb{F}_2)$-module, it follows that each W_j is a simple G-module. This implies that if $i < j < k$, then $W_i \otimes W_j \otimes W_k$ is also a *simple* G-module. Since

$$\begin{aligned} U &\cong \bigoplus_{j \neq k} W_j \otimes \Lambda^2 W_k \oplus \bigoplus_{i<j<k} W_i \otimes W_j \otimes W_k \\ &\cong (W_1 \oplus \cdots \oplus W_g)^{g-1} \oplus \bigoplus_{i<j<k} W_i \otimes W_j \otimes W_k, \end{aligned}$$

it is also a direct sum of simple G-modules, none of which is trivial. It follows that $U^G = 0$ and $\mathrm{Hom}_G(U, \mathbb{F}_2) = 0$. The result follows. q.e.d.

Since the group of p-torsion sections of $\mathcal{J}(\check{\mathbb{V}})$ defined over $\mathcal{M}_g$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}, \mathbb{F}_p)^{\mathrm{Sp}_g(\mathbb{Z})}$, we have:

Corollary 6.3. *There are no non-trivial torsion sections of* $\mathcal{J}(\check{\mathbb{V}})$ *defined over* $\mathcal{M}_g$.

Since the only invertible functions on $\mathcal{A}_g$ are constants (Lemma 2.1), any two trivializations of a trivial line bundle over $\mathcal{A}_g$ differ by a constant. In particular, if the line bundle $\widehat{\mathcal{B}}$ over $\mathcal{J}(\mathbb{V})$ exists, the trivialization of its restriction to the zero-section is unique up to a constant. In particular, it makes sense to say that a metric on the restriction of $\widehat{\mathcal{B}}$ to the zero-section is constant.

In Section 7, we will construct the line bundle $\widehat{\mathcal{B}}$ over $\mathcal{J}(\mathbb{V})$ whose first Chern class is $2\phi \in H^2(\mathcal{J}(\mathbb{V}))$. We also construct a metric $|\ |_{\widehat{\mathcal{B}}}$ on it.

The metric has the property that the restriction of its curvature to each fiber is translation invariant and to the zero-section vanishes. Proposition 6.1 and the following result show that these properties characterize $\widehat{\mathcal{B}}$ and its metric.

Proposition 6.4. *If such a line bundle $\widehat{\mathcal{B}}$ over $\mathcal{J}(\mathbb{V})$ exists, then any two metrics on it that satisfy:*

(i) *the metric on the restriction of $\widehat{\mathcal{B}}$ to the zero-section is constant;*
(ii) *the curvature of the restriction of the metric to each fiber is translation invariant;*

are constant multiples of each other.

Proof. The second condition determines the metric on each fiber up to a constant multiple. The first condition then determines all the fiber metrics up to a common, constant multiple. q.e.d.

Remark 6.5. Note that since the bundle $\mathcal{J}(\mathbb{V})$ is trivial when $g = 2$, the line bundle $\widehat{\mathcal{B}}$ is also trivial in that case and has the trivial metric (cf. Remark 4.1).

Definition 6.6. The biextension line bundle $\mathcal{B} \to \widetilde{\mathcal{M}}_g$ is the pullback of the biextension line bundle $\widehat{\mathcal{B}} \to \mathcal{J}(\mathbb{V})$ along the fundamental normal function $\tilde{\nu} : \widetilde{\mathcal{M}}_g \to \mathcal{J}(\mathbb{V})$. Its metric $|\ |_{\mathcal{B}}$ is the pullback of $|\ |_{\widehat{\mathcal{B}}}$ along $\tilde{\nu}$.

We will use the same notation to denote the restriction of the biextension bundle to $\mathcal{M}_g$. It follows from Morita's Theorem (Theorem 5.5) that the first Chern class of the biextension bundle $\mathcal{B} \to \mathcal{M}_g$ is $(8g+4)\lambda$.

Note that it makes sense to talk about the constant metric on the trivial bundle over the hyperelliptic locus $\mathcal{H}_g$ as the only invertible functions on $\mathcal{H}_g$ are constants (cf. Lemma 2.1).

Proposition 6.7. *The restriction of the biextension line bundle to the hyperelliptic locus is the trivial bundle with a constant metric. Consequently, the restriction of the curvature of $(\mathcal{B}, |\ |_{\mathcal{B}})$ to $\mathcal{H}_g$ vanishes.*

Proof. Since the cycle $C_x - C_x^-$ vanishes identically when C is hyperelliptic and x is a Weierstrass point, it follows that the normal function ν vanishes identically on the hyperelliptic locus $\mathcal{H}_g$. The result follows as the biextension bundle and its metric are trivial on the zero-section. q.e.d.

7. Construction of the biextension bundle

In this section, we review the construction of the biextension line bundle $\widehat{\mathcal{B}} \to \mathcal{J}(\mathbb{V})$ and its metric $|\ \ |_{\widehat{\mathcal{B}}}$ (cf. [**8**], and [**9**, Section 8]).

We begin in the more general situation of a variation of Hodge structure $\mathbb{U}$ of weight -1 over a complex manifold X. Denote the dual variation $\mathrm{Hom}(\mathbb{U}, \mathbb{Z}(1))$ by $\check{\mathbb{U}}$; it is also of weight -1. The fibers of $\mathbb{U}$ and $\check{\mathbb{U}}$ over $x \in X$ will be denoted by U_x and $\check{U}_x$, respectively.

One has the corresponding bundles of intermediate jacobians $\mathcal{J}(\mathbb{U})$ and $\mathcal{J}(\check{\mathbb{U}})$ over X. Their fibers $J(U_x)$ and $J(\check{U}_x)$ over $x \in X$ are dual complex tori [**8**, (3.1.6)] and can be naturally identified with $\mathrm{Ext}_{\mathcal{H}}(\mathbb{Z}, U_x)$ and $\mathrm{Ext}_{\mathcal{H}}(U_x, \mathbb{Z}(1))$, respectively. For this reason, we shall denote the family of intermediate jacobians $\mathcal{J}(\check{\mathbb{U}})$ by $\check{\mathcal{J}}(\mathbb{U})$.

One can form the bundle

$$\mathcal{J}(\mathbb{U}) \times_X \check{\mathcal{J}}(\mathbb{U}) \to X.$$

There is a natural principal $\mathbb{C}^*$-bundle $\mathcal{B}(\mathbb{U})^*$ over this. Its fiber $\mathcal{B}(\mathbb{U})^*_x$ over $x \in X$ is the set of mixed Hodge structures B with

$$\mathrm{Gr}^W_k B = 0 \text{ if } k \neq 0, -1, -2$$

together with isomorphisms

$$\mathrm{Gr}^W_0 B \cong \mathbb{Z}, \ \ \mathrm{Gr}^W_{-1} B \cong U_x \quad \text{and} \quad \mathrm{Gr}^W_{-2} B \cong \mathbb{Z}(1).$$

The projection

$$\mathcal{B}(\mathbb{U})^* \to \mathcal{J}(\mathbb{U}) \times_X \check{\mathcal{J}}(\mathbb{U})$$

takes B to $(B/\mathbb{Z}(1), W_{-1}B)$. There is a natural action of $\mathbb{C}^*$ on the fibers which is described in [**8**, (3.2.2)]. This $\mathbb{C}^*$ can be naturally identified with $\mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Z}, \mathbb{Z}(1))$.

Denote the corresponding line bundle by $\mathcal{B}(\mathbb{U})$. It has a natural metric which is described in [**8**, (3.2.9)] and can be described briefly as follows: The moduli space of *real* mixed Hodge structures with weight graded quotients $\mathbb{R}, U_{x,\mathbb{R}}$ and $\mathbb{R}(1)$ is naturally isomorphic to $\mathbb{R}$. Taking B to the canonical real period of $B \otimes \mathbb{R}$ gives a function $\mu : \mathcal{B}(\mathbb{U})^* \to \mathbb{R}$. There is an explicit formula [**8**, (3.2.11)] for $\mu(B)$ in terms of the $\mathbb{Z}$-periods of B. The metric on $\mathcal{B}(\mathbb{U})^*$ is defined by

$$|B|_{\mathcal{B}} = \exp \mu(B).$$

Because of the natural isomorphism

$$H_1(J(U_x) \times J(\check{U}_x), \mathbb{Z}) \cong U_{x,\mathbb{Z}} \oplus \check{U}_{x,\mathbb{Z}},$$

we can canonically identify $H^2(J(U_x) \times J(\check{U}_x), \mathbb{C})$ with the set of skew symmetric bilinear mappings

$$\big(U_x \oplus \check{U}_x\big)^{\otimes 2} \to \mathbb{C}.$$

In particular, we have the two dimensional cohomology class Q defined by

$$Q : \big((u,\psi),(v,\phi)\big) \mapsto \frac{1}{2\pi i}(\psi(v) - \phi(u)),$$

which is integral and invariant under monodromy. It, therefore, determines a closed 2-form ω_Q on $\mathcal{J}(\mathbb{U}) \times_X \check{\mathcal{J}}(\mathbb{U})$ as in the proof of Proposition 5.1.

Proposition 7.1. *The first Chern class of the line bundle*

$$\mathcal{B}(\mathbb{U}) \to \mathcal{J}(\mathbb{U}) \times_X \check{\mathcal{J}}(\mathbb{U})$$

is the class ω_Q. It has the property that its restriction to each fiber is the class corresponding to Q and to the zero-section is trivial; these characterize it when $H^1(X,\mathbb{U})$ vanishes.

Proof. First note that the restriction of $\mathcal{B}(\mathbb{U})^*$ to the zero-section is trivial as it has the section

$$x \mapsto \mathbb{Z} \oplus V_x \oplus \mathbb{Z}(1).$$

It follows from the formula [**8**, (3.2.11)] that this section has constant length 1.

For the rest of the computations, we will use the notation and terminology of [**8**, Section 3]. We will consider the fiber of $\mathcal{B}(\mathbb{U})^*$ over $x \in X$. Set $U = U_x$ and $\check{U} = \check{U}_x$. Set

$$G_{U\,\mathbb{Z}} = \begin{pmatrix} 1 & U_{\mathbb{Z}} & \mathbb{Z}(1) \\ 0 & 1 & \check{U}_{\mathbb{Z}} \\ 0 & 0 & 1 \end{pmatrix}, \qquad G_U = \begin{pmatrix} 1 & U_{\mathbb{C}} & \mathbb{C} \\ 0 & 1 & \check{U}_{\mathbb{C}} \\ 0 & 0 & 1 \end{pmatrix}, \qquad F^0G_U = \begin{pmatrix} 1 & F^0U & 0 \\ 0 & 1 & F^0\check{U} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, the fiber of $\mathcal{B}(\mathbb{U})^*$ over $x \in X$ is

$$\mathcal{B}(\mathbb{U})^*_x := G_{U\,\mathbb{Z}} \backslash G_U / F^0G_U.$$

The natural projection of this to $J(U) \times J(\check{U})$ is induced by the obvious group homomorphism $G_U \to U \times \check{U}$. We can give local holomorphic framing s of $\mathcal{B}_x \to J(U) \times J(\check{U})$ by giving a global holomorphic section of

$$G_U/F^0G_U \to U/F^0U \times \check{V}/F^0\check{U}.$$

We do this by identifying U/F^0U with $\overline{F}^0U$ and $\check{U}/F^0\check{U}$ with $\overline{F}^0\check{V}$ and then defining

$$s(g,\gamma) = \begin{pmatrix} 1 & g & 0 \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} F^0 G_U,$$

where $g \in \overline{F}^0U$ and $\gamma \in \overline{F}^0\check{U}$. Using the formula [**8**, (3.2.11)] with

$$r = g + \overline{g}, \quad f = -\overline{g}, \quad \rho = \gamma + \overline{\gamma} \quad \text{and} \quad \psi = -\overline{\gamma}$$

one sees that $\mu(s(g,\gamma)) = -\operatorname{Re}\gamma(\overline{g})$. (Here and elsewhere in this proof, the conjugate $\hbar \in \check{U}$ of $f \in \check{U}$ is the function $\overline{f} : U \to \mathbb{C}$ defined by $\overline{f}(v) = \overline{f(\overline{v})}$ for all $v \in V$.) It follows that

$$\log |s|^2_{\mathcal{B}} = -2\operatorname{Re}\gamma(\overline{g}).$$

Choose a basis e_j of F^0U and a basis ϵ_j of $F^0\check{U}$ such that $\epsilon_j(\overline{e}_k) = 2\pi i\delta_{jk}$ all j and k. Since $F^0\check{U}$ is the annihilator of F^0U, $\epsilon_j(e_k) = 0$ for all j and k. Note that $\overline{\epsilon}_j(e_k) = -2\pi i \delta_{jk}$.

Now, if we write $g = \sum_j z_j\overline{e}_j$ and $\gamma = \sum_j u_j\overline{\epsilon}_j$, then

$$\log |s|^2_{\mathcal{B}} = -(\gamma + \overline{\gamma})(g + \overline{g}) = 2\pi i \sum_{j=1}^{n} \left(z_j\overline{u}_j - u_j\overline{z}_j\right).$$

The Chern class $c_1(\mathcal{B}(U)_x)$ is therefore represented by

$$\frac{1}{2\pi i}\,\partial\overline{\partial}\log|s|^2_{\mathcal{B}} = \sum_{j=1}^{n}\left(dz_j \wedge d\overline{u}_j - du_j \wedge d\overline{z}_j\right).$$

It remains to prove that $c_1(\mathcal{B}(\mathbb{U}))$ is given by Q. To see this, we use the identifications

$$U_{\mathbb{R}} \xrightarrow{\simeq} U_{\mathbb{C}}/F^0U \xleftarrow{\simeq} \overline{F}^0U \quad \text{and} \quad \check{U}_{\mathbb{R}} \xrightarrow{\simeq} \check{U}_{\mathbb{C}}/F^0\check{U} \xleftarrow{\simeq} \overline{F}^0\check{U}.$$

Set

$$f_j = e_j + \overline{e}_j, \quad Jf_j = i(e_j - \overline{e}_j), \quad \phi_j = \overline{\epsilon}_j - \epsilon_j, \quad J\phi_j = i(\overline{\epsilon}_j + \epsilon_j).$$

Then, the f_j and Jf_j form a basis of $U_{\mathbb{R}}$, and the ϕ_j and $J\phi_j$ form a basis of $\check{U}_{\mathbb{R}}$. Moreover, f_j corresponds to $\overline{e}_j$, ϕ_j corresponds to $\overline{\epsilon}_j$ and J to multiplication by i under the isomorphisms above.

Using the standard identification between $U_{\mathbb{R}}$ and the translation invariant vector fields on $U_{\mathbb{R}}/U_{\mathbb{Z}}$, we have

$$\langle dz_j, f_k\rangle = \langle d\overline{z}_j, f_k\rangle = \delta_{jk} \quad \text{and} \quad \langle dz_j, Jf_k\rangle = -\langle d\overline{z}_j, Jf_k\rangle = i\delta_{jk}.$$

Similarly,

$$\langle du_j, \phi_k \rangle = \langle d\overline{u}_j, \phi_k \rangle = \delta_{jk} \text{ and } \langle du_j, J\phi_k \rangle = -\langle d\overline{u}_j, J\phi_k \rangle = i\delta_{jk}.$$

The formula for the Chern class now follows by direct computation. The final statement follows directly from Corollary 5.3. q.e.d.

We now return to the case where $X = \mathcal{A}_g$ and $\mathbb{U}$ is the variation $\mathbb{V}$. For each $[A] \in \mathcal{A}_g$, the polarization q gives a MHS morphism $V_A \to \check{V}_A$; it takes v to $2\pi i\, q(v, \ \ )$. This induces a morphism $i_q : \mathcal{J}(\mathbb{V}) \to \check{\mathcal{J}}(\mathbb{V})$ over $\mathcal{A}_g$.

Definition 7.2. The biextension bundle $\widehat{\mathcal{B}} \to \mathcal{J}(\mathbb{V})$ is the pullback of the bundle

$$\mathcal{B}(\mathbb{V}) \to \mathcal{J}(\mathbb{V}) \times_{\mathcal{A}_g} \check{\mathcal{J}}(\mathbb{V})$$

along the map $(\mathrm{id}, i_q) : \mathcal{J}(\mathbb{V}) \to \mathcal{J}(\mathbb{V}) \times_{\mathcal{A}_g} \check{\mathcal{J}}(\mathbb{V})$. The metric $|\ \ |_{\widehat{\mathcal{B}}}$ is the pullback of $|\ \ |_{\mathcal{B}(\mathbb{V})}$.

Proposition 7.3. *The metric on the biextension bundle $\widehat{\mathcal{B}}$ is constant when restricted to the zero-section, has translation invariant curvature when restricted to any fiber $J(V_A)$. It has first Chern class $2\phi = 2\omega_q$. In particular, the class 2ϕ is integral.*

Proof. The assertions about the metric and its curvature follow immediately from Proposition 7.1. We will use Proposition 7.1 to compute the Chern class.

The class $c_1(\mathcal{B}(\mathbb{V}))$ corresponds to the bilinear form

$$Q : (V_A \oplus \check{V}_A)^{\otimes 2} \to \mathbb{C}$$

given by

$$Q : (u, \phi) \otimes (v, \psi) \mapsto (2\pi i)^{-1}(\phi(v) - \psi(u)).$$

Pulling this back along the map $(\mathrm{id}, i_q) : V_A \to V_A \oplus \check{V}_A$, we obtain the bilinear form

$$(u, v) \mapsto Q\left((u, q(u, \ \ )), (v, q(v, \ \ )\right) = 2q(u, v).$$

q.e.d.

Remark 7.4. One can give a moduli interpretation of the points of $\widehat{\mathcal{B}}^*$. Points of $\widehat{\mathcal{B}}^*_A$ are biextensions B with weight graded quotients identified with $\mathbb{Z}$, V_A and $\mathbb{Z}(1)$ that are essentially self dual in the following sense. If one pulls back the extension

$$\mathrm{Hom}_{\mathbb{Z}}(B/\mathbb{Z}(1), \mathbb{Z}(1)) \in \mathrm{Ext}^1_{\mathcal{H}}(\check{V}, \mathbb{Z}(1))$$

along the map $i_q : V \to \check{V}$ given by the polarization, then one obtains $W_{-1}B$.

The fiber $\widehat{\mathcal{B}}^*_A$ of $\widehat{\mathcal{B}}^*$ over $[A] \in \mathcal{A}_g$ is a locally homogeneous space. Let $G_{\mathbb{Z}}$ be the extension of $V_{\mathbb{Z}}$ by $\mathbb{Z}$ determined by the class $2\omega_q \in H^2(V_{\mathbb{Z}}, \mathbb{Z})$. So, as a set, $G_{\mathbb{Z}} = V_{\mathbb{Z}} \times \mathbb{Z}$. The multiplication is defined by

$$(u, n) \cdot (v, m) = (u + v, m + n + q(u, v)).$$

Define G to be the complexification of this group. (The underlying set is $V_{\mathbb{C}} \times \mathbb{C}$.) Let F^0G by the closed subgroup of G whose underlying set is $F^0V \times 0$. Then, it is not hard to see that

$$\widehat{\mathcal{B}}^*_A = G_{\mathbb{Z}} \backslash G / F^0 G.$$

In fact, the total spaces of $\mathcal{B}(\mathbb{V})^*$ and $\widehat{\mathcal{B}}^*$ are locally homogeneous varieties with fundamental groups $\mathrm{Sp}_g(\mathbb{Z}) \ltimes G_{U\,\mathbb{Z}}$ and $\mathrm{Sp}_g(\mathbb{Z}) \ltimes G_{\mathbb{Z}}$, respectively, as can be seen using [**2**, Section 4].

We conclude with a short computation we shall need. Define the commutator $[a, b]$ of two elements a and b of a group to be $aba^{-1}b^{-1}$.

Lemma 7.5. *The commutator of the two elements* (u, n) *and* (v, m) *of* $G_{\mathbb{Z}}$ *is* $\big(0, 2q(u, v)\big)$.

8. The reduction

In this section, we show that Theorem 1.1 is a direct consequence of Theorem 1.3. The main ingredient is a fundamental result of Dale Lear [**18**].

8.1. Lear's theorem. The following is a special case of [**18**, (6.2)] adapted to our present needs. Suppose that T is a Riemann surface and that $\mathcal{C} \to T$ is a semi-stable family of genus g curves over it whose generic fiber is smooth. This is classified by an orbifold mapping $f : T \to \overline{\mathcal{M}}_g$ whose image is not contained in Δ. Denote $f^{-1}(\widetilde{\mathcal{M}}_g)$ by T'.

Theorem 8.1 (Lear). *There exists an integer $N > 0$, which depends only on g, such that the line bundle $f^*\mathcal{B}^{\otimes N}$ over T' extends canonically to T as a holomorphic line bundle with continuous metric.*

For us, the significance of this result.

Corollary 8.2. *There is an integer $N > 0$ such that the biextension line bundle $\mathcal{B}^{\otimes N}$ over $\widetilde{\mathcal{M}}_g$ extends naturally to a line bundle on $\overline{\mathcal{M}}_g$. This extension is characterized by the requirement that the biextension metric extends continuously to a metric on the restriction of this line bundle to any disk transverse to Δ_0 at a smooth point.*

Proof. Lear's Theorem implies that the integer N and the restriction of the extended bundle $\overline{\mathcal{B}}$ to the disk $\mathbb{D}$ depend only on the local monodromy representation $\pi_1(\mathbb{D}^*) \to \mathrm{Sp}_g(\mathbb{Z}) \ltimes G_{\mathbb{Z}}$. This implies that the extension and the integer are independent of the choice of the disk, which implies that the extension exists. q.e.d.

Whether or not the metric on $\mathcal{B}$ over $\widetilde{\mathcal{M}}_g$ extends to a continuous metric on $\mathcal{B}$ over $\overline{\mathcal{M}}_g$ should follow from a good understanding of the singularities of period mappings of real variations of mixed Hodge structure in one and several variables. Pearlstein [**23**] has made significant progress on this problem and in doing so, has reproved Lear's Theorem [**23**, Section 5].

8.2. Theorem 1.3 implies Theorem 1.1. We shall regard the extended line bundle as an element of $(\operatorname{Pic}\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$. Denote the element that extends $\mathcal{B}$ by $\overline{\mathcal{B}}$. We have

$$c_1(\overline{\mathcal{B}}) = (8g+4)\lambda + \sum_{h=0}^{[g/2]} r_h \delta_h,$$

where each $r_h \in \mathbb{Q}$. We shall compute the coefficients r_h in subsequent sections.

An immediate consequence of the following result is that Theorem 1.3 implies Theorem 1.1.

Proposition 8.3. *With notation as in Theorem 1.1,*

$$\beta_g(t) \sim \begin{cases} r_0 \log|t| - (4g+2)\log\log\frac{1}{|t|}, & \text{when } h = 0,\\ r_h \log|t|, & \text{when } h > 0.\end{cases}$$

Proof. Let e be a meromorphic section of

$$\mathcal{N} := \overline{\mathcal{B}} \otimes \mathcal{L}^{\otimes(-(8g+4))}$$

that trivializes it over $\mathcal{M}_g$. It is unique up to a non-zero constant. (If not all the r_h are integers, raise this line bundle to a sufficiently high power to clear denominators and take Nth roots later.) Since

$$c_1(\mathcal{N}) = \sum_{h=0}^{[g/2]} r_h \delta_h,$$

e vanishes to order r_h along Δ_h. This line bundle is naturally metrized away from Δ_0 by the tensor product metric, which we denote by $|\ \ |_{\mathcal{N}}$. Since $|\ \ |_{\mathcal{N}}$ extends smoothly over Δ_h when $h > 0$, it follows that if $\mathbb{D}$ is a small disk with parameter t transverse to a smooth point of Δ_h which it intersects at $t = 0$ (as in Theorem 1.1), then

$$\beta(t) \sim \log |e(t)|_{\mathcal{N}} \sim \log |t^{r_h}| = r_h \log |t|$$

as $t \to 0$.

The situation is slightly more complicated when $h = 0$ as the metric on $\mathcal{L}$ does not extend (even continuously) across Δ_0. To proceed, we compute the behaviour of the metric $|\ \ |_{\mathcal{L}}$ on $\mathcal{L}$ near Δ_0. This is the source of the $\log\log(1/|t|)$ term.

Suppose that $\mathbb{D}$ is a small analytic disk in $\overline{\mathcal{M}}_g$ with parameter t that intersects Δ_0 transversally at a smooth point when $t = 0$. Suppose that s is a trivializing section of the restriction of $\mathcal{L}$ to $\mathbb{D}$.

Lemma 8.4. *As* $t \to 0$

$$\log |s(t)| \sim \frac{1}{2} \log\log \frac{1}{|t|}.$$

Proof of Lemma. Suppose that C is a smooth curve of genus g. We use the definition of the metric on $H^0(C, \Omega^1)$ given in the introduction. Suppose that $a_1, \dots, a_g, b_1, \dots, b_g$ is a symplectic basis of $H_1(C, \mathbb{Z})$ and that $w_1, \dots, w_g$ is the corresponding normalized basis of $H^0(C, \Omega^1)$. With respect to these data, the period matrix Ω of C is given by

$$\Omega_{jk} = \int_{b_j} w_k = \int_{b_k} w_j,$$

Thus,

$$w_j = a_j^* + \sum_m \Omega_{jm} b_m^*,$$

where $a_1^*, \dots, a_g^*, b_1^*, \dots, b_g^*$ is the dual basis of $H^1(C, \mathbb{Z})$. Since it is also symplectic,

$$(w_j, w_k) = \frac{i}{2} \int_C w_j \wedge \overline{w}_k = \operatorname{Im} \Omega_{jk}.$$

Standard linear algebra implies that

$$\|w_1 \wedge \cdots \wedge w_g\|^2 = \det \operatorname{Im} \Omega.$$

One can construct a trivializing section s over any disk transverse to Δ_0 at a smooth point as follows: Such a disk corresponds to a stable degeneration $\{C_t\}_{t \in \mathbb{D}}$ of smooth genus g curves to an irreducible stable curve with one node. By [**5**, p. 53], the normalized period matrix $\Omega(t)$ of C_t is, up to a bounded matrix-valued function of t,

$$\begin{pmatrix} \Omega_0 & v^T \\ v & \log t / 2\pi i \end{pmatrix},$$

where Ω_0 is the period matrix of the normalization of C_0 and $v \in \mathbb{C}^{g-1}$. (This can also be deduced from the work of Griffiths, as in [**6**].) It follows that

$$\det \operatorname{Im} \Omega(t) \sim \text{constant} \cdot \log \frac{1}{|t|} \quad \text{as} \quad t \to 0$$

and that

$$\log \|w_1(t) \wedge \cdots \wedge w_g(t)\| = \frac{1}{2} \log \det \operatorname{Im} \Omega(t) \sim \frac{1}{2} \log \log \frac{1}{|t|} \quad \text{as } t \to 0.$$

Here, $w_1(t), \dots, w_g(t)$ is a normalized basis of the abelian differentials on the curve C_t. Since $w_1(t) \wedge \cdots \wedge w_g(t)$ extends to a local framing of $\mathcal{L}$ on $\overline{\mathcal{M}}_g$, this establishes the result. q.e.d.

We can now compute the asymptotics of β near Δ_0. Continuing with the notation from above, we can write the section $e(t)$ of the restriction of $\mathcal{N}$ to the disk $\mathbb{D}$ in the form

$$e = t^{r_0} s_B \otimes s_L^{-(8g+4)}$$

where s_B is a trivializing section of the restriction of $\overline{\mathcal{B}}$ to $\mathbb{D}$ and s_L is a trivializing section of the restriction of $\mathcal{L}$ restricted to $\mathbb{D}$. Since the metric on $\mathcal{B}$ extends to a continuous metric on the restriction of $\overline{\mathcal{B}}$ to $\mathbb{D}$, Lemma 8.4 implies

$$\begin{aligned} \beta(t) &= r_0 \log |t| + \log |s_B(t)|_{\mathcal{B}} - (8g+4) \log |s(t)|_{\mathcal{L}} \\ &\sim r_0 \log |t| - (4g+2) \log \log (1/|t|) \end{aligned}$$

as $t \to 0$. q.e.d.

9. A fundamental representation

As remarked in the introduction, the coefficients r_h are computed by a monodromy computation. In order to do this, we need to construct the appropriate global representation of the mapping class group. This, we do in this section.

The basic idea behind computing the coefficients r_h when $h > 0$ is the following: Suppose that X is a complex manifold and that D is a divisor in X with irreducible components $D_1, \ldots, D_r$. Suppose that $\mathcal{E}$ is a holomorphic line bundle over X and that e is a meromorphic section of $\mathcal{E}$ that is holomorphic over $X - D$ and trivializes it there. Suppose that γ_j is the boundary of a small holomorphic disk $\mathbb{D}$ in X that is transverse to D_j. The restriction of the $\mathbb{C}^*$ bundle $\mathcal{E}^*$ to $\mathbb{D}$ is trivial, and thus has fundamental group $\mathbb{Z}$, where the generator corresponds to a loop in one fiber that encircles the zero-section positively.

Lemma 9.1. *The loop $e_*(\gamma_j)$ equals n_j times the positive generator of $\pi_1(\mathcal{E}|_{\mathbb{D}}^*)$ if and only if e has order n_j along D_j.*

In our case, we take $\mathcal{E}$ to be the line bundle $\mathcal{B} \otimes \mathcal{L}^{\otimes(-(8g+4))}$ over $\widetilde{\mathcal{M}}_g$. In this case, we consider the global representation as the structure of the mapping class group which will help us show that $r_h = -4h(g-h)$.

It is helpful to view the section e as a lift of the normal function ν as in the following diagram:

$$\begin{tikzcd}
 & \left(\widehat{\mathcal{B}} \otimes \mathcal{L}^{\otimes(-(8g+4))}\right)^*. \arrow[d] \\
 & \mathcal{J}(\mathbb{V}) \arrow[d] \\
\mathcal{M}_g \arrow[uur, "e"] \arrow[ur, "\nu"] \arrow[r] & \mathcal{A}_g
\end{tikzcd}$$

The framing e, therefore, induces a representation

$$\hat{\rho} : \Gamma_g \to \pi_1\big(\big(\widehat{\mathcal{B}} \otimes \mathcal{L}^{\otimes(8g+4)}\big)^*, *\big)$$

of the mapping class group.

The projection induces a natural homomorphism

$$\pi_1\big(\big(\widehat{\mathcal{B}} \otimes \mathcal{L}^{\otimes(8g+4)}\big)^*, *\big) \to \pi_1(\mathcal{A}_g, *) \cong \mathrm{Sp}_g(\mathbb{Z}).$$

Proposition 9.2. *The fundamental group of* $\big(\widehat{\mathcal{B}} \otimes \mathcal{L}^{\otimes(8g+4)}\big)^*$ *is an extension*

$$1 \to G_{\mathbb{Z}} \to \pi_1\big(\big(\widehat{\mathcal{B}} \otimes \mathcal{L}^{\otimes(8g+4)}\big)^*, *\big) \to \mathrm{Sp}_g(\mathbb{Z}) \to 1.$$

Moreover, the composition of the representation $\hat{\rho}$ *with the projection to* $\mathrm{Sp}_g(\mathbb{Z})$ *is the natural homomorphism* $\Gamma_g \to \mathrm{Sp}_g(\mathbb{Z})$.

Proof. The kernel is the fundamental group of the pullback of $\big(\widehat{\mathcal{B}} \otimes \mathcal{L}^{\otimes(8g+4)}\big)^*$ to $\mathfrak{h}_g$, the Siegel upper half space. Since the pullback of $\mathcal{L}$ to $\mathfrak{h}_g$ is trivial, this has the same homotopy type as the pullback of $\widehat{\mathcal{B}}^*$ to $\mathfrak{h}_g$. Since $\mathfrak{h}_g$ is contractible, this has the homotopy type of the fiber $\widehat{\mathcal{B}}^*_A$ of $\widehat{\mathcal{B}}^*$ over $[A]$. Since

$$\widehat{\mathcal{B}}^*_A = G_{\mathbb{Z}} \backslash G_{\mathbb{C}} / F^0 G$$

and since $G_{\mathbb{C}}/F^0G$ is contractible, the first assertion follows. The second assertion is clear. q.e.d.

As an immediate consequence, we see that the restriction of $\hat{\rho}$ to the Torelli group T_g gives a homomorphism

$$\hat{\tau} : T_g \to G_{\mathbb{Z}}. \tag{1}$$

The composition of this homomorphism with the natural homomorphism $G_{\mathbb{Z}} \to V_{\mathbb{Z}}$ is twice the Johnson homomorphism [**9**, (6.3)].

10. Topological preliminaries

Before computing r_h when $h > 0$, we need to clarify our conventions. Although the final answer is independent of them, signs in intermediate results do. If α and β are paths in a topological space, then $\alpha\beta$ is the path obtained by following α and then β. With this convention, if X is a contractible space and Γ is a group that acts properly discontinuously and freely on X, then for each choice of a base point x of X, there is a natural group isomorphism between $\pi_1(\Gamma\backslash X, x)$ and Γ. The group element $g \in \Gamma$ corresponds to the homotopy class of the loop in $\Gamma\backslash X$ that is the homotopy class of the projection of a path in X that goes from x to gx.

Now, suppose that Γ acts on a topological space S on the left. Then, Γ acts on $X \times S$ on the left. The quotient is a flat S bundle over $\Gamma\backslash X$.

Proposition 10.1. *The monodromy of the bundle* $\Gamma\backslash(X \times X)$ *around a loop in* $\Gamma\backslash X$ *corresponding to* $g \in \Gamma$ *is* $g^{-1} \in \mathrm{Aut}\, S$.

Proof. The point $(x, s) \in X \times S$ is parallel transported to (gx, s) along the path corresponding to g. But this point is identified with $(x, g^{-1}u)$ under the group action. The result follows. q.e.d.

Now, suppose that C is a smooth projective curve and that S is the underlying topological space. Since $\mathcal{M}_g$ is the quotient of Teichmüller space on the left by the mapping class group, there is a natural isomorphism between the mapping class group Γ_g of S and $\pi_1(\mathcal{M}_g, [C])$. Applying the above in the case where $\Gamma = \mathrm{Diff}^+ S$, the group of orientation preserving diffeomorphisms of S, we obtain.

Corollary 10.2. *The isotopy class of the geometric monodromy of the universal curve about a loop in $\mathcal{M}_g$ based at $[C]$ that represents $\phi \in \Gamma_g$ is ϕ^{-1}.*

11. Coefficient of Δ_h when $h > 0$

Suppose that $g \geq 3$. Fix a stable curve C_0 with one node whose moduli point lies in Δ_h. There is a stable family of curves $\{C_t\}_{t \in \mathbb{D}}$ over the disk whose central fiber is C_0 and where each other fiber is smooth. Choose a base point t_o in the punctured disk. We shall use the oriented surface S underlying C_{t_o} as our reference surface. We can identify $\pi_1(\mathcal{M}_g, [C_{t_o}])$ with the mapping class group Γ_g of S. The geometric monodromy of this family is a Dehn twist $\sigma_h : S \to S$ about a simple closed curve that separates S into two bounded surfaces, one of genus h, the other of genus $g - h$.

Note that σ_h lies in the Torelli group T_g and, moreover, that it lies in the kernel of the Johnson homomorphism $T_g \to V_{\mathbb{Z}}$. Consequently, its image under the homomorphism $\hat{\tau} : T_g \to G_{\mathbb{Z}}$, constructed in Section 9, lies in the center $\mathbb{Z}$ of $G_{\mathbb{Z}}$.

Proposition 11.1. *If $g \geq 3$, then $\hat{\tau}(\sigma_h) = 4h(g - h)$.*

This computation is equivalent (up to a scaling constant) to a result of Morita [**19**, Theorem 5.7] and will be proved below. Combining it with Proposition 10.2 completes the computation of r_h when $h > 0$.

Corollary 11.2. *If $h > 0$, then $r_h = -4h(g - h)$.*

Proof of Proposition 11.1. Let W be a genus h subsurface of the reference surface S such that σ_h is the Dehn twist about the boundary of W. Choose a base point $* \in \partial W$. The boundary of W (with the induced orientation) determines an element c in $\pi_1(W,*)$. One can choose generators $a_1,\dots,a_h,b_1,\dots,b_h$ of $\pi_1(W,*)$ whose homology classes comprise a symplectic basis of $H_1(W,\partial W)$ and such that

$$\prod_{j=1}^{h}[a_j,b_j]=c^{-1}.$$

We can regard W as the real oriented blow up of the compact surface W' obtained by identifying ∂W to a point P. We shall use P as a base point of W'. Let F be the unit tangent bundle of W'. The base point $*$ of W corresponds to a point in the fiber of F over P that we shall also denote by $*$ and shall use as a base point for F.

Choose a vector field ξ on W' whose only zero is at $P\in W'$. This induces a map $\xi:(W,*)\to(F,*)$ and, therefore, a homomorphism

$$\xi_*:\pi_1(W,*)\to\pi_1(F,*).$$

Denote by f the element of $\pi_1(F,*)$ that corresponds to going once around the fiber over P in the positive (i.e., counterclockwise) direction. It follows from the Hopf Index Theorem that $\xi_*(c)=f^{2h-2}$. Consequently,

$$\xi_*\left(\prod_{j=1}^{h}[a_j,b_j]\right)=f^{2-2h}.$$

Denote by $\mathrm{Aut}_0\,\pi_1(W,*)$ the group of automorphisms ϕ of $\pi_1(W,*)$ such that $\phi(c)=c$. There is a natural homomorphism

$$\chi:\pi_1(F,*)\to\mathrm{Aut}_0\,\pi_1(W,*)$$

that covers the natural left action of $\pi_1(W')$ on itself by conjugation. Note that σ_h can be viewed as an element of $\mathrm{Aut}_0\,\pi_1(W,*)$. In fact,

$$\sigma_h:x\mapsto cxc^{-1}=\chi(f^{-1})(x).$$

It follows that

$$\sigma_h^{2h-2}=\chi(f^{2-2h})=\chi\circ\xi_*\left(\prod_{j=1}^{h}[a_j,b_j]\right)\in\mathrm{Aut}_0\,\pi_1(W,*).$$

Observe that the image of $\pi_1(W,*)$ in $\mathrm{Aut}_0\,\pi_1(W,*)$ is naturally a subgroup of Γ_g as each automorphism coming from $\pi_1(W,*)$ is induced by a diffeomorphism of W that fixes ∂W pointwise. In fact, the image lies in the Torelli group. Let

$$\phi : \pi_1(W,*) \to T_g$$

be the composite. Then, since $G_{\mathbb{Z}}$ is a two-step nilpotent group

$$\begin{aligned}\hat{\tau}(\sigma_h) &= \frac{1}{2h-2}\,\hat{\tau}\left(\prod_{j=1}^{h}[\phi(a_j),\phi(b_j)]\right)\\ &= \frac{1}{2h-2}\sum_{j=1}^{h}[\hat{\tau}\circ\phi(a_j),\hat{\tau}\circ\phi(b_j)] \in G_{\mathbb{Z}}\end{aligned}$$

and the value of this expression depends only on the images of the $\phi(a_j)$ and $\phi(b_j)$ under the Johnson homomorphism $T_g \to V$ (cf. Lemma 7.5).

In order to compute the image of σ_h in $G_{\mathbb{Z}}$, we shall need the following result. Denote the homology class of $x \in \pi_1(W,*)$ by $[x]$. Let $w' = \sum_{j=1}^{h}[a_j]\wedge[b_j]$. We view this as an element of $\Lambda^2 H_1(W)$.

Lemma 11.3. *The image of $x \in \pi_1(W,*)$ under the composite*

$$\pi_1(W,*) \xrightarrow{\phi} T_g \xrightarrow{\text{Johnson}} V,$$

where ϕ is the composite

$$\pi_1(W,*) \longrightarrow \pi_1(C,*) \xrightarrow{\text{inner}} T_g,$$

is the image of $[x]\wedge w' \in \Lambda^3 H_1(W)$ in $V = \Lambda^3 H_1(S)/H_1(S)$ under the map induced by the natural inclusion $H_1(W) \hookrightarrow H_1(S)$.

Proof. Denote the surface obtained from W by collapsing the boundary to a point by $\overline{W}$. Observe that the quotient mapping induces an isomorphism on H_1. Set

$$\Gamma_{W,\partial W} = \pi_0 \operatorname{Diff}^+(W,\partial W) \quad\text{and}\quad \Gamma_{\overline{W},*} = \pi_0 \operatorname{Diff}^+(\overline{W},*),$$

where $*$ is the image of ∂W in $\overline{W}$, be the mapping class groups of $(W,\partial W)$ and $(\overline{W},*)$, respectively. Denote the associated Torelli groups by $T_{W,\partial W}$ and $T_{\overline{W},*}$. In Harer's notation, these are isomorphic to $T_{h,1}$ and T_h^1, respectively. There are Johnson homomorphisms

$$T_{W,\partial W} \to \Lambda^3 H_1(W) \quad\text{and}\quad T_{\overline{W},*} \to \Lambda^3 H_1(\overline{W}), \tag{2}$$

which are compatible in the sense that the first is the composition of the second with the natural quotient mapping $T_{W,\partial W} \to T_{\overline{W},*}$ once one identifies $H_1(W)$ with $H_1(\overline{W})$. The homomorphisms (2) both induce isomorphisms on H_1 mod torsion.

The first step is to prove that the composite

$$H_1(\pi_1(\overline{W},*)) \to H_1(T_{\overline{W},*}) \to \Lambda^3 H_1(\overline{W})$$

takes $[x]$ to $[x]\wedge\omega'$. This follows directly from the geometric definition of the Johnson homomorphism that is given in [**16**] and which is explained in detail in [**10**, Section 3]. Indeed, the mapping torus $M \to S^1$ that corresponds to conjugation by x is trivial as the diffeomorphism ϕ of C which induces it is isotopic to the identity when the base point is ignored. The section of base points takes $t \in S^1$, which we regard as $[0,1]/0 \sim 1$, to $\gamma(1-t)$, where γ is any loop representing x. The map from M to $\operatorname{Jac} C$ then takes $(u,t) \in C \times S^1$ to $u - \gamma(1-t)$. This has homology class $[x] \wedge \omega'$.

By the compatibility of the Johnson homomorphisms for $(W, \partial W)$ and $(\overline{W}, *)$, it follows that

$$H_1(T_{W,\partial W}) \to \Lambda^3 H_1(W)$$

takes $[x]$ to $[x] \wedge \omega'$.

The result now follows as the inclusion of W into S induces a homomorphism $T_{W,\partial W} \to T_g$ such that the diagram

$$\begin{array}{ccc} T_{W,\partial W} & \longrightarrow & T_g \\ \big\downarrow & & \big\downarrow \\ \Lambda^3 H_1(W) & \longrightarrow & V \end{array}$$

commutes, where the bottom arrow is induced by the induced map $H_1(W) \to H_1(S)$. q.e.d.

Since the composite of $\hat{\tau} : T_g \to G_{\mathbb{Z}}$ with the natural homomorphism $G_{\mathbb{Z}} \to V$ is twice the Johnson homomorphism, it follows from Lemma 7.5 that

$$\hat{\tau}(\sigma_h) = \frac{8}{2h-2} \sum_{j=1}^{h} q\big([a_j] \wedge w', [b_j] \wedge w'\big) \in \mathbb{Z}.$$

We will abuse notation and not distinguish between an element of $\pi_1(W,*)$ and its homology class. We can extend the symplectic basis $a_1, \dots, a_h, b_1, \dots, b_h$ of $H_1(W)$ to a symplectic basis $a_1, \dots, a_g, b_1, \dots, b_g$

of $H_1(S)$. Set

$$w = \sum_{j=1}^{g} a_j \wedge b_j$$

and $w'' = w - w'$. Denote $H_1(W)$ by H' and its orthogonal complement in $H_1(S)$ by H''. Recall from Section 3 that if $u, v \in V$, then $q(u,v) = \langle j(u), j(v)\rangle/(g-1)$. A short computation shows that if $x \in H'$, then

$$j(x \wedge w') = (g-1)\, x \wedge w' - (h-1)\, x \wedge w = (g-h)\, x \wedge w' - (h-1)\, x \wedge w''.$$

Since $H' \wedge w''$ is orthogonal to $H'' \wedge w'$ in $\Lambda^3 H_1(S)$, we have

$$\begin{aligned}
&\sum_{j=1}^{h} q(a_j \wedge w', b_j \wedge w') \\
&\quad = \frac{1}{g-1} \sum_{j=1}^{h} \big((g-h)^2 \langle a_j \wedge w', b_j \wedge w'\rangle + (h-1)^2 \langle a_j \wedge w'', b_j \wedge w''\rangle\big) \\
&\quad = \frac{1}{g-1} \sum_{j=1}^{h} \big((g-h)^2 (h-1) + (h-1)^2 (g-h)\big) \\
&\quad = \frac{1}{g-1}\, h(h-1)(g-h)(g-1) \\
&\quad = h(h-1)(g-h).
\end{aligned}$$

Thus, $\hat{\tau}(\sigma_h) = 8h(g-h)(h-1)/(2h-2) = 4h(g-h)$. q.e.d.

Corollary 11.4. *The first Chern class of the biextension bundle $\mathcal{B}$ over $\widetilde{\mathcal{M}}_g$ is*

$$c_1(\mathcal{B}) = (8g+4)\lambda - 4 \sum_{h=1}^{[g/2]} h(g-h)\delta_h.$$

12. The coefficient of Δ_0

Our final task is to determine the coefficient of δ_0. In principle, we could do this using the representation defined in Section 9. However, this appears to be quite complicated, and it seems easier to do it directly by restriction to the hyperelliptic locus, where we know that $\overline{\mathcal{B}}$ is trivial, and using a known linear equivalence to show $r_0 = -g$.

The set of points of $\overline{\mathcal{M}}_g$ corresponding to stable hyperelliptic curves forms a subvariety $\overline{\mathcal{H}}_g$. When $h > 0$, the intersection $\Delta_j \cap \overline{\mathcal{H}}_g$ is an irreducible divisor. We shall denote its class by δ_j. When $h = 0$,

$$\Delta_0 \cap \overline{\mathcal{H}}_g = \Xi_0 \cup \bigcup_{h=1}^{[(g-1)/2]} \Xi_h,$$

where Ξ_h, $1 \leq h \leq (g-1)/2$ is the closure of the locus in $\overline{\mathcal{H}}_g$ consisting of the stable hyperelliptic curves that are the union of two smooth projective curves, one of genus h and another of genus $g-h-1$, which intersect transversely in two points. The divisor Ξ_0 consists of the closure of the locus of irreducible hyperelliptic curves whose normalization is of genus $g-1$. Each Ξ_h is a divisor in $\overline{\mathcal{H}}_g$. Denote the class of Ξ_h in $\operatorname{Pic} \overline{\mathcal{H}}_g$ by ξ_h.

Cornalba and Harris [**3**, Proposition 4.7] prove that the classes ξ_h, $0 \leq h \leq (g-1)/2$ and $\delta_h, 1 \leq h \leq g/2$ are linearly independent in $\operatorname{Pic} \overline{\mathcal{H}}_g$ and that in $\operatorname{Pic} \overline{\mathcal{H}}_g$, one has

$$(8g+4)\lambda = g\xi_0 + \sum_{j=1}^{[(g-1)/2]} (j+1)(g-j)\xi_j + 4\sum_{h=1}^{[g/2]} h(g-h)\delta_h.$$

Proposition 6.7 implies that $\nu^*\phi$ vanishes on $\overline{\mathcal{H}}_g$. Thus, we have

$$(8g+4)\lambda = -r_0\xi_0 + 4\sum_{h=1}^{[g/2]} h(g-h)\delta_h + \sum_{j\geq 1} c_j\xi_j$$

in $H^2(\overline{\mathcal{H}}_g, \mathbb{Q})$ and thus in $\operatorname{Pic} \overline{\mathcal{H}}_g$. Since, when $j > 0$, Ξ_j is contained in the codimension 2 stratum of $\overline{\mathcal{M}}_g$, it follows that $r_0 = -g$, as claimed. This completes the proof of Theorem 1.3.

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