Acta Math., 227 (2021), 103–113
DOI: 10.4310/ACTA.2021.v227.n1.a2

The number of closed ideals in $L(L_p)$

by

William B. Johnson

Texas A&M University
College Station, TX, U.S.A.

Gideon Schechtman

The Weizmann Institute of Science
Rehovot, Israel

1. Introduction

For a reasonably complete discussion of the history of constructing closed ideals in $L(L_p)$, see the introduction in [JPS]. Here, we just remark that in 1981, Bourgain, Rosenthal, and the second author [BRS] constructed $\aleph_1$ mutually non-isomorphic complemented subspaces of $L_p:=L_p(0,1)$ for $1<p\neq 2<\infty$, thereby producing (as noted in [P]) $\aleph_1$ different closed ideals in $L(L_p)$. (It is of course well known that the compact operators are the only closed ideal in $L(L_2)$.) At that time it was open whether, absent the continuum hypothesis, $L(L_p)$ contains a continuum of closed ideals. Recently, Schlumprecht and Zsák [SZ] built a continuum of closed ideals in $L_p:=L_p(0,1)$.

The main contribution of this paper is Theorem 4.1, in which we prove that $L(L_p)$, $1<p\neq 2<\infty$, has exactly $2^{2^{\aleph_0}}$ different closed ideals.

Recall the notions of small and large closed ideal in $L(X)$. An ideal is called *small* if it is contained in the ideal of strictly singular operators. Otherwise it is called *large*. The ideals built in [SZ] are all small, while the ones coming from infinite-dimensional complemented subspaces are clearly large. Our basic construction is designed to produce large ideals. Note that there are at most a continuum of mutually non-isomorphic complemented subspaces of L_p (as the density character of $L(L_p)$ and of the set of projections on L_p is the continuum). So necessarily we produce different kinds of ideals. Unfortunately, we do not produce any new complemented subspaces of L_p.

The new large ideals in $L(L_p)$ that we construct are "smallish" in the sense that, even though there are idempotents in the ideals whose ranges are isomorphic to ℓ_2 (see Remark 4.2), no operator in any of the ideals is an isomorphism on a copy of ℓ_p. The Kadec–Pełczyński dichotomy principle [KP] implies that every complemented subspace

W. B. J. was supported in part by NSF DMS-1900612. G. S. was supported in part by Israel Science Foundation.

of L_p that is not isomorphic to a Hilbert space contains a complemented subspace that is isomorphic to ℓ_p. Consequently, the range of any infinite-rank idempotent in any of the ideals that we construct in Theorem 4.1 (and, as we said, there are infinite-rank idempotents in the ideals) *must* be isomorphic to ℓ_2.

To put these new "smallish" large ideals into perspective within the Banach algebra $L(L_p)$, notice that it follows from the Kadec–Pełczyński dichotomy principle [KP] that there are exactly two different minimal large closed ideals in $L(L_p)$ when $2<p<\infty$, and thus also for $1<p<2$ (because an operator T in $L(L_p)$ is strictly singular if and only if T^* is strictly singular on L_q, $1/p+1/q=1$, by Weis' theorem [W]). The first of these is $\Gamma_{\ell_p}(L_p)$, the ideal of operators that factor through ℓ_p. This ideal is closed because an operator $T\colon X\to L_p$, $2<p<\infty$, factors through ℓ_p if and only if $I_{p,2}T$ is compact, where $I_{p,2}$ is the formal identity mapping from L_p into L_2; see [J]. One can prove using the Kadec–Pełczyński dichotomy principle [KP] that $I_{p,2}S$ is compact whenever S is a strictly singular operator on L_p, so the alternate characterization of $\Gamma_{\ell_p}(L_p)$ for $2<p<\infty$ also yields that $\Gamma_{\ell_p}(L_p)$ contains all strictly singular operators on L_p, $2<p<\infty$, and thus also for $1<p<2$ by [W].

The second minimal large closed ideal in $L(L_p)$ is the closure $\bar{\Gamma}_2(L_p)$ of the ideal $\Gamma_2(L_p)$ of operators on L_p that factor through a Hilbert space. Here the closure is needed; in fact, it is not hard to see that there are compact operators on L_p that do not factor through a Hilbert space.

We recall in passing that, as was noted in [JPS], the situation in $L(L_1)$ is nicer: $\Gamma_{\ell_1}(L_1)$ is the unique minimal closed large ideal in $L(L_1)$ and it contains all the strictly singular operators on L_1.

In Remark 4.3 we prove that the new large ideals we construct in $L(L_p)$ do not contain the strictly singular operators on L_p, and hence neither does $\bar{\Gamma}_2(L_p)$. All previously known large ideals in $L(L_p)$ other than $\bar{\Gamma}_2(L_p)$ do contain the strictly singular operators, and this is the first proof that $\bar{\Gamma}_2(L_p)$ does not. A byproduct of Remark 4.3, stated as Remark 4.4, is that $L(L_p)$ contains exactly $2^{2^{\aleph_0}}$ small closed ideals.

Our construction and proof of Theorem 4.1 consist of two steps. In §2 we state and prove the technical Proposition 2.1. This easily yields Corollary 2.3, which gives a general criterion for a space with an unconditional basis to contain $2^{2^{\aleph_0}}$ different closed ideals. The criterion is in term of the existence of a special operator on the space.

In §3 we show that, for $1<q<2$, the space L_q contains a complemented subspace (this is Rosenthal's $\mathfrak{X}_q$ space, which has an unconditional basis) that admits an operator satisfying the criterion of Proposition 2.1. The construction here borrows a lot from a previous similar construction from [JS]. Duality and complementation then imply the main result.

2. The main proposition

There is a continuum of infinite subsets of the natural numbers $\mathbb{N}$, each two of which have only finite intersection. Denote some fixed such continuum by $\mathcal{C}$. For a finite-dimensional normed space E, we denote by $d(E)$ the Banach–Mazur distance (isomorphism constant) of E to a Euclidean space. Also, recall that, for an operator $T: X \to Y$ between two normed spaces, $\gamma_2(T)$ denotes its factorization constant through a Hilbert space:

$$\gamma_2(T) = \inf\{\|A\|\,\|B\| : T = AB, A: H \to Y, B: X \to H, H \text{ a Hilbert space}\}.$$

If T is of rank k, then $\gamma_2(T) \leqslant k^{1/2}\|T\|$ because every k-dimensional normed space is $k^{1/2}$-isomorphic to ℓ_2^k [T-J, Theorem 15.5]. Note that $d(E)$ is just $\gamma_2(I_E)$, where I_E is the identity operator on E.

PROPOSITION 2.1. *Let X be a Banach space with a 1-unconditional basis $\{e_i\}$, let Y be a Banach space, and let $T: X \to Y$ be an operator of norm at most 1 satisfying the following conditions:*

(a) *For some $\eta > 0$ and for every M, there is a finite-dimensional subspace E of X such that $d(E) > M$ and $\|Tx\| > \eta\|x\|$ for all $x \in E$.*

(b) *For some constant Γ and every m, there is an n such that every m-dimensional subspace E of $[e_i]_{i\geqslant n}$ satisfies $\gamma_2(T|_E) \leqslant \Gamma$.*

Then, there exist natural numbers $1 = p_1 < q_1 < p_2 < q_2 < \ldots$ such that, denoting

$$G_k := [e_i]_{i=p_k}^{q_k}$$

for each k, defining, for each $\alpha \in \mathcal{C}$, the operator $P_\alpha: X \to [G_k]_{k\in\alpha}$ to be the the natural basis projection, and setting $T_\alpha := TP_\alpha$, the following statement holds: If $\alpha_1, \ldots, \alpha_s \in \mathcal{C}$ (possibly with repetitions) and $\alpha \in \mathcal{C} \setminus \{\alpha_1, \ldots, \alpha_s\}$ then, for all $A_1, \ldots, A_s \in L(Y)$ and all $B_1, \ldots, B_s \in L(X)$,

$$\left\| T_\alpha - \sum_{i=1}^{s} A_i T_{\alpha_i} B_i \right\| \geqslant \frac{\eta}{2}. \tag{2.1}$$

Proof. Note first that we can strengthen condition (a) to include also that, given any n, one can chose the subspace E to also satisfy that it is contained in $[e_i]_{i>n}$. Now choose inductively $1 = p_1 < q_1 < p_2 < q_2 \ldots$ so that, for each k, $G_k = [e_i]_{i=p_k}^{q_k}$ contains a subspace E_k with $\|Tx\| > \eta\|x\|$ for all $x \in E_k$ and

$$d(E_k) \geqslant q_{k-1}$$

(as we will see, it is enough that $d(E_k)/q_{k-1}^{1/2} \to \infty$) and, if E is a subspace of $H_k = [G_l]_{l=p_{k+1}}^{\infty}$ with $\dim E \leqslant q_k$, then

$$\gamma_2(T|_E) < \Gamma.$$

Let now $P_\alpha: X \to [G_k]_{k\in\alpha}$ be the natural basis projection and set $T_\alpha := TP_\alpha$. Suppose that $\alpha_1, ..., \alpha_s \in \mathcal{C}$ (possibly with repetitions) and $\alpha \in \mathcal{C} \setminus \{\alpha_1, ..., \alpha_s\}$. Assume to the contrary that there are $A_1, ..., A_s \in L(Y)$ and $B_1, ..., B_s \in L(X)$ such that

$$\left\| T_\alpha - \sum_{i=1}^{s} A_i T_{\alpha_i} B_i \right\| < \frac{\eta}{2}. \tag{2.2}$$

There are infinitely many $k \in \alpha \setminus \bigcup_{i=1}^{s} \alpha_i$. For each such k, let R_k be the basis projection onto $[G_l]_{l<k}$ and Q_k the basis projection onto $[G_l]_{l>k}$. Now, for any $i=1, ..., s$, we have $T_{\alpha_i} G_k = 0$ since $k \notin \alpha_i$, and $\dim(R_k B_i E_k) \leqslant q_{k-1}$ and $\dim(B_i E_k) \leqslant q_k$, so we get that, for each i,

$$\begin{aligned} \gamma_2(A_i T_{\alpha_i} B_i|_{E_k}) &\leqslant \gamma_2(A_i T_{\alpha_i} R_k B_i|_{E_k}) + \gamma_2(A_i T_{\alpha_i} Q_k B_i|_{E_k}) \\ &\leqslant q_{k-1}^{1/2} \|A_i\| \, \|B_i\| + \Gamma \|A_i\| \, \|B_i\|. \end{aligned}$$

Consequently,

$$\gamma_2\left(\sum_{i=1}^{s} A_i T_{\alpha_i} B_i|_{E_k} \right) \leqslant \left(\max_{i=1}^{s} \|A_i\| \, \|B_i\| \right) s (q_{k-1}^{1/2} + \Gamma). \tag{2.3}$$

On the other hand, since $\|x\| \geqslant \|T_\alpha x\| \geqslant \eta \|x\|$ for all $x \in E_k$, (2.2) implies that

$$\left(1 + \frac{\eta}{2}\right) \|x\| \geqslant \left\| \sum_{i=1}^{s} A_i T_{\alpha_i} B_i x \right\| \geqslant \frac{\eta \|x\|}{2}$$

for all $x \in E_k$. Since $d(E_k) \geqslant q_{k-1}$, we deduce that

$$\gamma_2\left(\sum_{i=1}^{s} A_i T_{\alpha_i} B_i|_{E_k} \right) \geqslant \frac{\eta}{2+\eta} q_{k-1}.$$

For k large enough, this contradicts (2.3). □

Remark 2.2. Observe that the only condition on T_α that was used to get the inequality (2.1) is that $\|x\| \geqslant \|T_\alpha x\| \geqslant \eta \|x\|$ for all x in E_k with $k \in \alpha$. Consequently, the proof of Corollary 2.3 below shows that any operator S in $L(X)$ for which there is $\eta > 0$ such that $\|Sx\| \geqslant \eta \|x\|$ for all x in E_k with $k \in \alpha$ cannot be in the closed ideal generated by $\{T_\beta : \beta \in \mathcal{C} \text{ with } \beta \neq \alpha\}$. In fact, from the proof of Proposition 2.1, only the inequality $\|Sx\| \geqslant \eta \|x\|$ for all x in H_k with $k \in \alpha$ and where H_k is isomorphic to E_k with isomorphism constant independent of k is sufficient to conclude that S is not in the closed ideal generated by $\{T_\beta : \beta \in \mathcal{C}, \text{ with } \beta \neq \alpha\}$. This observation will be used in Remark 4.3 at the end of this paper.

COROLLARY 2.3. *Let X be a Banach space with a 1-unconditional basis $\{e_i\}_i$ and assume there is an operator $T{:}\,X\to X$ of norm at most 1 satisfying* (a) *and* (b) *of Proposition 2.1. Then, $L(X)$ has exactly $2^{2^{\aleph_0}}$ different closed ideals.*

Proof. For any non-empty proper subset $\mathcal{A}$ of $\mathcal{C}$, let $\mathcal{I}_{\mathcal{A}}$ be the ideal generated by $\{T_\alpha\}_{\alpha\in\mathcal{A}}$; i.e., all operators of the form $\sum_{i=1}^s A_iT_{\alpha_i}B_i$ with $s\in\mathbb{N}$, $A_i,B_i\in L(X)$, $\alpha_i\in\mathcal{A}$, $i=1,\dots,s$. To avoid cumbersome notation, interpret $\mathcal{A}\subset\mathcal{C}$ to mean that $\mathcal{A}$ is a non-empty proper subset of $\mathcal{C}$.

Since we allow repetition of the T_{α_i}, it is easy to see that this really defines a (non-closed) ideal. Let $\mathcal{B}$ be a subset of $\mathcal{C}$ different from $\mathcal{A}$ and assume, without loss of generality, that $\mathcal{B}\not\subset\mathcal{A}$. Let $\alpha\in\mathcal{B}\setminus\mathcal{A}$. Then, by Proposition 2.1, $T_\alpha\notin\bar{\mathcal{I}}_{\mathcal{A}}$. Consequently, $\{\bar{\mathcal{I}}_{\mathcal{A}}\}_{\mathcal{A}\subset\mathcal{C}}$ are all different.

Since the density character of $L(X)$, for any separable X, is at most the continuum, it is easy to see that, for any separable space X, $L(X)$ has at most $2^{2^{\aleph_0}}$ different closed ideals. □

Remarks 2.4. (1) One can strengthen the conclusion of the corollary by getting an antichain of $2^{2^{\aleph_0}}$ closed ideals in $L(X)$; i.e., such a collection no two of whose members are included one in the other. For that one just uses a collection of $2^{2^{\aleph_0}}$ subsets of $\mathcal{C}$ no two of which are included one in the other.

(2) Similarly, one gets a collection of $2^{\aleph_0}$ different closed ideals in $L(X)$ that form a chain (by taking a chain of subsets of $\mathcal{C}$ of that cardinality). It is also easy to show by a density argument that, for any separable X, this is the maximal cardinality of any chain of closed ideals in $L(X)$.

(3) If Y is a Banach space that contains a complemented subspace X with the properties of Corollary 2.3, then clearly $L(Y)$ also has $2^{2^{\aleph_0}}$ different closed ideals (actually an antichain). The same is true also for any space isomorphic to such a Y.

(4) The simplest examples of spaces X that satisfy the hypotheses of Corollary 2.3 and thus $L(X)$ has $2^{2^{\aleph_0}}$ different closed ideals are $\big(\sum_i \ell_{r_i}^{n_i}\big)_2$ for $r_i\uparrow 2$ and n_i satisfying $n_i^{1/r_i-1/2}\to\infty$. It is not hard to show that the identity operator on such a space satisfies (a) and (b) of Proposition 2.1. Consequently, by (3), $L\big(\big(\sum_i \ell_{r_i}\big)_2\big)$ for $r_i\uparrow 2$ also has $2^{2^{\aleph_0}}$ different closed ideals. Interesting, but less natural, examples of separable spaces X with $L(X)$ having $2^{2^{\aleph_0}}$ different closed ideals were known before (see [M]). Unfortunately, $\big(\sum_i \ell_{r_i}^{n_i}\big)_2$ for $r_i\uparrow 2$ and $n_i^{1/r_i-1/2}\to\infty$ does not embed isomorphically as a complemented subspace into any L_p, $p<\infty$, so this example is not good for our purposes. Actually, at least for some sequences $\{(r_i,n_i)\}_i$ with the above properties, $\big(\sum_i \ell_{r_i}^{n_i}\big)_2$ does not even embed isomorphically into any L_p space, $p<\infty$. That this is true, for example, if each $(r,n)\in\{(r_i,n_i)\}$ repeats n times follows from [KS, Corollary 3.4].

In the next section, we show how to get complemented subspaces of the reflexive L_p spaces that satisfy the hypotheses of Corollary 2.3.

3. The operator T

In this section we prove that, for each $1<q<2$, there is a complemented subspace of L_q isomorphic to a space X with a 1-unconditional basis on which there is an operator of norm at most 1 with properties (a) and (b) of Proposition 2.1.

Recall that, for a sequence $u=\{u_j\}_{j=1}^{\infty}$ of positive real numbers and for $p>2$, the Banach space $\mathfrak{X}_{p,u}$ is the sequence space with norm

$$\|\{a_j\}_{j=1}^{\infty}\| = \max\left\{\left(\sum_{j=1}^{\infty}|a_j|^p\right)^{1/p}, \left(\sum_{j=1}^{\infty}|a_j u_j|^2\right)^{1/2}\right\}. \tag{3.1}$$

Rosenthal [R] proved that $\mathfrak{X}_{p,u}$ is isomorphic to a complemented subspace of L_p with the isomorphism constant and the complementation constant depending only on p. If u is such that, for all $\varepsilon>0$,

$$\sum_{u_j<\varepsilon} u_j^{2p/(p-2)} = \infty,$$

then one gets a space isomorphically different from ℓ_p, ℓ_2 and $\ell_p\oplus\ell_2$. However, for different u satisfying the condition above, the different $\mathfrak{X}_{p,u}$ spaces are mutually isomorphic. We denote by $\mathfrak{X}_p$ any of these spaces. Later, we shall need more properties of the spaces $\mathfrak{X}_{p,u}$, and of particular embeddings of them into L_p, but for now we only need the representation (3.1), and we think of $\mathfrak{X}_{p,u}$ as a subspace of $\ell_p\oplus_\infty\ell_2$.

Let $\{e_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_p, and let $\{f_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_2. Let $v=\{v_j\}_{j=1}^{\infty}$ and $w=\{w_j\}_{j=1}^{\infty}$ be two positive real sequences such that $\delta_j=w_j/v_j\to 0$ as $j\to\infty$ and $\max_{j=1}^{\infty}\delta_j\leqslant 1$. Set

$$g_j^v = e_j + v_j f_j \in \ell_p\oplus_\infty\ell_2 \quad\text{and}\quad g_j^w = e_j + w_j f_j \in \ell_p\oplus_\infty\ell_2.$$

Then, $\{g_j^v\}_{j=1}^{\infty}$ is the unit vector basis of $\mathfrak{X}_{p,v}$ and $\{g_j^w\}_{j=1}^{\infty}$ is the unit vector basis of $\mathfrak{X}_{p,w}$. Define also

$$\Delta\colon \mathfrak{X}_{p,w} \longrightarrow \mathfrak{X}_{p,v}$$

by

$$\Delta g_j^w = \delta_j g_j^v.$$

Note that Δ is the restriction to $\mathfrak{X}_{p,w}$ of $K\in L(\ell_p\oplus_\infty\ell_2)$ defined by

$$K(e_j)=\delta_j e_j \quad\text{and}\quad K(f_j)=f_j.$$

Consequently, $\|\Delta\|\leqslant\|K\|=1$.

The following proposition follows immediately from the easily verified fact that $\|K|_{[e_j]_{j=m}^{\infty}}\|\to 0$ as $m\to\infty$.

PROPOSITION 3.1. *Given n, there exists an m such that, if E is an n-dimensional subspace of $[e_j]_{j=m}^{\infty}\oplus[f_j]_{j=1}^{\infty}\subset\ell_p\oplus_{\infty}\ell_2$, then $\gamma_2(K|_E)\leqslant 2$. In particular, if E is an n-dimensional subspace of $[g_j^w]_{j=m}^{\infty}\subset\mathfrak{X}_{p,w}$, then $\gamma_2(\Delta|_E)\leqslant 2$.*

Next, we define weights $\{v_j\}_j$ and $\{w_j\}_j$, with some additional properties. For that, we use different representations of the spaces $\mathfrak{X}_{p,u}$. It was proved in [R] that, if $\{X_j\}_{j=1}^{\infty}$ is a sequence of symmetric, each 3-valued, independent random variables all L_p-normalized, $2<p<\infty$, then $\{X_j\}_{j=1}^{\infty}$ is equivalent, in L_p, to $\{g_j^u\}_{j=1}^{\infty}$, the unit vector basis of $\mathfrak{X}_{p,u}$, where $u_j=\|X_j\|_2$. Defining $Y_j=X_j/\|X_j\|_q$, for $q=p/(p-1)$, $\{Y_j\}_{j=1}^{\infty}$ is equivalent, in L_q, to the basis $\{h_j^u\}_{j=1}^{\infty}$ of $\mathfrak{X}_{q,u}:=\mathfrak{X}_{p,u}^*$, that is dual to the unit vector basis of $\mathfrak{X}_{p,u}$.

Let us say already at this early stage that, for some appropriate weights $\{v_j\}_j$ and $\{w_j\}_j$, the operator T we are after will be of the form Δ^* followed by a norm-1 isomorphism from $\mathfrak{X}_{q,w}$ to $\mathfrak{X}_{q,v}$.

Recall that $P\colon L_p\to[X_j]_{j=1}^{\infty}$ defined by

$$Pf=\sum_{j=1}^{\infty}\biggl(\int_0^1 fY_j\biggr)X_j$$

defines a bounded projection onto $[X_j]_{j=1}^{\infty}$ (and P^* a bounded projection from L_q onto $[Y_j]_{j=1}^{\infty}$). The norms of the equivalences above and of the projections depend on p, but not on the particular weights u.

We now recall a construction from [JS, §4]. It was shown there that, given $1<q<2$, any sequence $\{\delta_i\}_{i=1}^{\infty}$ that decreases to zero, any sequence $\{r_i\}_{i=1}^{\infty}$ such that $q<r_i\uparrow 2$ fast enough and in particular satisfying $\delta_i^{q(2-r_i)/(2-q)}>\frac{1}{2}$, $i=1,2,\dots$, and for any sequence $\varepsilon_i\downarrow 0$, we can find two sequences $\{Y_i\}_i$ and $\{Z_i\}_i$ of symmetric, independent, 3-valued random variables, all normalized in L_q, with the following additional properties:

- Put $v_j=1/\|Y_j\|_2$ and $w_j=1/\|Z_j\|_2$. Then, there are disjoint finite subsets σ_i, $i=1,2,\dots$, of the integers such that $w_j=\delta_i v_j$ for $j\in\sigma_i$.
- There are independent random variables $\{\overline{Y}_i\}_i$ r_i-stable normalized in L_q, $\{\overline{Z}_i\}_i$ r_i-stable with $1\geqslant\|\overline{Z}_i\|_q\geqslant\frac{3}{4}$ for each i, and there are coefficients $\{a_j\}_j$ such that

$$\biggl\|\overline{Y}_i-\sum_{j\in\sigma_i}a_jY_j\biggr\|_q<\varepsilon_i \quad\text{and}\quad \biggl\|\overline{Z}_i-\sum_{j\in\sigma_i}\delta_ia_jZ_j\biggr\|_q<\varepsilon_i. \tag{3.2}$$

We may of course repeat each of the triplets $(r_i, \delta_i, \varepsilon_i)$ as many (finitely many) times as we wish. Thus we conclude that, given any sequence $\{\delta_i\}_{i=1}^\infty$ decreasing to zero, any sequence $\{r_i\}_{i=1}^\infty$ such that $q<r_i\uparrow 2$ and satisfying $\delta_i^{q(2-r_i)/(2-q)}>\frac{1}{2}$, $i=1,2,\ldots$, any sequence of integers n_i, and any sequence $\varepsilon_i\downarrow 0$, we can find two sequences $\{Y_i\}_i$ and $\{Z_i\}_i$ of symmetric, independent, 3-valued random variables, all normalized in L_q, with the following additional properties:

- Put $v_j=1/\|Y_j\|_2$ and $w_j=1/\|Z_j\|_2$. Then there are disjoint finite subsets $\sigma_{i,l}$, $i=1,2,\ldots$ and $l=1,\ldots n_i$, of the integers such that $w_j=\delta_i v_j$ for $j\in\sigma_{i,l}$.
- There exist independent random variables $\{\overline{Y}_{i,l}\}_{i,l}$ r_i-stable normalized in L_q, $\{\overline{Z}_{i,l}\}_{i,l}$ r_i-stable with $1\geqslant\|\overline{Z}_{i,l}\|_q\geqslant\frac{3}{4}$ for each i and l, and there exist coefficients $\{a_j\}_j$ such that

$$\Big\|\overline{Y}_{i,l}-\sum_{j\in\sigma_{i,l}} a_jY_j\Big\|_q<\varepsilon_i \quad\text{and}\quad \Big\|\overline{Z}_{i,l}-\sum_{j\in\sigma_{i,l}} \delta_i a_jZ_j\Big\|_q<\varepsilon_i. \tag{3.3}$$

Choosing the ε_i small enough, we may assume that $\{\sum_{j\in\sigma_{i,l}} a_jY_j\}_{l=1}^{n_i}$ is, in L_q, 2-equivalent to the unit vector basis of $\ell_{r_i}^{n_i}$, and similarly $\{\sum_{j\in\sigma_{i,l}} \delta_i a_jZ_j\}_{l=1}^{n_i}$ is, in L_q, 2-equivalent to the unit vector basis of $\ell_{r_i}^{n_i}$. Denoting by R the map that sends Y_j to δ_iZ_j for $j\in\sigma_{i,l}$, we get that this map satisfies that, for all i, there is a space E_i that is 2-isomorphic to $\ell_{r_i}^{n_i}$ such that $\|Rx\|\geqslant\frac{1}{4}\|x\|$ for all $x\in E_i$. Choosing the n_i large enough, we may also assume that, for all k,

$$n_i^{1/r_i-1/2}\to\infty \quad \text{as } i\to\infty.$$

Since $n_i^{1/r_i-1/2}$ is the distance of $\ell_{r_i}^{n_i}$ to a Hilbert space, we get that $d(E_i)\to\infty$.

We are now ready to state and prove the main proposition of this section.

PROPOSITION 3.2. *With the choice of $v=\{v_j\}_j$ and $w=\{w_j\}_j$ above, set $X=\mathfrak{X}_{q,v}$, let $\Delta^*:\mathfrak{X}_{q,v}\to\mathfrak{X}_{q,w}$ be the adjoint of Δ defined at the beginning of this section, and let S be a norm-1 isomorphism from $\mathfrak{X}_{q,w}$ onto $\mathfrak{X}_{q,v}$. Put $T=S\Delta^*$. Then, X and T satisfy the assumptions of Proposition 2.1.*

Proof. Since $T=ARB$ for isomorphisms A and B, the discussion above provides a proof of property (a). Property (b) follows by duality from Proposition 3.1. Indeed, fix m and n and let E be an m-dimensional subspace of $[h_i^v]_{i\geqslant n}$. We have that $\Delta^*(E)$ is a subspace of $[h_i^w]_{i\geqslant n}$, so there is a k-dimensional subspace F of $[g_i^w]_{i\geqslant n}$ that 2-norms E. Here, $k=k(m)$ depends only on m (and we used the 1-unconditionality of the bases). By Proposition 3.1, for some n depending only on k and thus only on m, $\gamma_2(\Delta|_F)\leqslant 2$. From this it is easy to get that $\gamma_2(\Delta^*|_E)\leqslant 4$. Consequently, this holds also for $T=S\Delta^*$. □

4. The main result and additional comments

THEOREM 4.1. *For every $1<p\neq 2<\infty$ the number of different closed ideals in $L(\mathfrak{X}_p)$ and in $L(L_p)$ is exactly $2^{2^{\aleph_0}}$. Moreover, each of these spaces contains an antichain of closed ideals of cardinality $2^{2^{\aleph_0}}$ and a chain of cardinality $2^{\aleph_0}$.*

Proof. For $\mathfrak{X}_q$, $1<q<2$, the theorem follows from Proposition 3.2 and Corollary 2.3. For $\mathfrak{X}_p$, $2<p<\infty$, it follows by simple duality. Since for $1<p\neq 2<\infty$ the space $\mathfrak{X}_p$ is isomorphic to a complemented subspace of L_p, it follows also for L_p.

The statements about chains and antichains are a consequence of the remarks at the end of §2. □

Remark 4.2. As is stated in the introduction, the new ideals in $L(L_p)$ and $L(\mathfrak{X}_p)$, $1<p\neq 2<\infty$, constructed in Theorem 4.1 are all large, and in fact contain projections whose ranges are isomorphic to ℓ_2.

Proof. First, we observe that it is enough to show that, for each $\alpha\in\mathcal{C}$, the operator T_α on X (recall that X is isomorphic to $\mathfrak{X}_q$, where $1<q<2$), isomorphically preserves a copy of ℓ_2. Here, T is the operator produced in Proposition 3.2, and T_α is defined in the statement of Proposition 2.1. Indeed, since any subspace of L_q, $1<q<2$, that is isomorphic to ℓ_2 contains a further infinite-dimensional subspace that is complemented in L_q (this fact was probably first observed by Pełczyński; see [JS, p. 1106] for a proof), this will show that the identity on ℓ_2 factors through T_α, and hence there is a projection in the ideal generated by T_α whose range is isomorphic to ℓ_2. This will give Remark 4.2 for $L(\mathfrak{X}_p)$ when $1<p<2$, and the case of $L(\mathfrak{X}_p)$ for $2<p<\infty$ follows by duality. The statement for $L(L_p)$, $1<p\neq 2<\infty$, is then immediate.

To show that T_α isomorphically preserves a copy of ℓ_2, note that the space $\mathfrak{X}_{q,v}$ we built contains a modular space [LT, Definition 4.d.1] $\ell_{\{r_i\}}$ with $r_i\uparrow 2$ on which T_α is an isomorphism and thus (by passing to a subsequence of the sequence r_i that tends quickly to 2), also contains an isomorph of ℓ_2 on which T_α is an isomorphism. □

Remark 4.3. The large ideals in $L(L_q)$ and $L(\mathfrak{X}_q)$ constructed in Theorem 4.1 do not contain the ideal of strictly singular operators.

Sketch of proof. By [W] and how we constructed the ideals in $L(L_q)$ from the ideals in $L(\mathfrak{X}_q)$, it is enough to consider the ideals constructed in $L(\mathfrak{X}_q)$ for $1<q<2$. Let T be the operator and X be the space isomorphic to $\mathfrak{X}_q$ that are defined in Proposition 3.2 and which satisfy the assumptions of Proposition 2.1. Let $\{T_\alpha{:}\alpha\in\mathcal{C}\}$ be the corresponding operators on X given by Proposition 3.2. As in the proof of Corollary 2.3, for $\mathcal{A}$ a (always non-empty, proper) subset of $\mathcal{C}$, let $\mathcal{I}_\mathcal{A}$ be the ideal in $L(X)$ generated by $\mathcal{A}$.

Given $\mathcal{A}\subset\mathcal{C}$, take any $\alpha\in\mathcal{C}$ that is not in $\mathcal{A}$. We know that T_α is not in $\overline{\mathcal{I}}_{\mathcal{A}}$, but we want a strictly singular operator that is not in $\overline{\mathcal{I}}_{\mathcal{A}}$ and T_α is not strictly singular. Let $Y:=\big(\sum_{k=1}^{\infty} G_k\big)_q$, where the G_k are the block subspaces of X defined in the proof of Proposition 2.1. The G_k are contractively complemented in X, and X is isomorphic to a complemented subspace of L_q, hence Y is isomorphic to a complemented subspace of ℓ_q (and thus to ℓ_q by Pełczyński's well-known theorem, but we do not need this), which in turn is isomorphic to a complemented subspace of X. Define $U: Y\to X$ by making U the identity on each G_k and extending by linearity and continuity. This is ok because L_q has type q and $(G_k)_k$ is a monotonely unconditional Schauder decomposition for a subspace of X, hence the decomposition (G_k) has an upper q-estimate (even with constant 1). Let E_k be the subspace of G_k defined in the proof of Proposition 2.1. The operator $T_\alpha U$ is strictly singular and $\|T_\alpha U x\|\geqslant\eta\|x\|$ for all x in E_k with k in α. Since Y is isomorphic to a complemented subspace of X, we also get a strictly singular operator $S: X\to X$ and subspaces H_k of Y, with H_k isomorphic to E_k (with isomorphism constant independent of k), such that $\|Sx\|\geqslant\eta\|x\|$ for all $x\in H_k$, with $k\in\alpha$. By Remark 2.2, this is enough to yield that S is not in the closed ideal in $L(X)$ generated by $\{T_\beta:\beta\in\mathcal{C}$ with $\beta\neq\alpha\}$. □

Remark 4.4. $L(L_q)$ and $L(\mathfrak{X}_q)$, $1<q\neq 2<\infty$, both contain exactly $2^{2^{\aleph_0}}$ closed small ideals.

Sketch of proof. Again, it is enough to deal with the case of $L(\mathfrak{X}_q)$ with $1<q<2$. Let X and T be as in Remark 4.3. For $\mathcal{A}\subset\mathcal{C}$, let $\mathcal{J}_{\mathcal{A}}$ be the ideal in $L(X)$ generated by $\{T_\alpha UP:\alpha\in\mathcal{A}\}$, where P is any fixed projection from X onto a subspace isomorphic to Y (we identify Y with that subspace). All $\overline{\mathcal{J}}_{\mathcal{A}}$ are small ideals, and clearly $\mathcal{J}_{\mathcal{A}}$ is contained in the ideal $I_{\mathcal{A}}$ generated by $\{T_\alpha:\alpha\in\mathcal{A}\}$. But in Remark 4.3 we saw that $T_\alpha UP$ is not contained in $\bar{I}_{\mathcal{A}}$ when $\alpha\notin\mathcal{A}$, so $\overline{\mathcal{J}}_{\mathcal{A}}\neq\overline{\mathcal{J}}_{\mathcal{B}}$ when $A\neq B$. □

Added in proof

The two authors and Chris Phillips observed recently that, for any Banach space X, two different closed ideals in $L(X)$ are also not isomorphic as Banach algebras; i.e., are not homomorphic by an homomorphism that is continuous in both directions. It follows that there are $2^{2^{\aleph_0}}$ closed ideals in $L(L_p(0,1))$, $1<p\neq 2<\infty$, that are mutually non-isomorphic as Banach algebras.

References

[BRS] Bourgain, J., Rosenthal, H. P. & Schechtman, G., An ordinal L^p-index for Banach spaces, with application to complemented subspaces of L^p. *Ann. of Math.*, 114 (1981), 193–228.

[J] Johnson, W. B., Operators into L_p which factor through 1_p. *J. London Math. Soc.*, 14 (1976), 333–339.

[JPS] Johnson, W. B., Pisier, G. & Schechtman, G., Ideals in $L(L_1)$. *Math. Ann.*, 376 (2020), 693–705.

[JS] Johnson, W. B. & Schechtman, G., Multiplication operators on $L(L_p)$ and l_p-strictly singular operators. *J. Eur. Math. Soc.* (*JEMS*), 10 (2008), 1105–1119.

[KP] Kadec, M. I. & Pełczyński, A., Bases, lacunary sequences and complemented subspaces in the spaces L_p. *Studia Math.*, 21 (1961/62), 161–176.

[KS] Kwapień, S. a. & Schütt, C., Some combinatorial and probabilistic inequalities and their application to Banach space theory. II. *Studia Math.*, 95 (1989), 141–154.

[LT] Lindenstrauss, J. & Tzafriri, L., *Classical Banach Spaces.* I. Ergebnisse der Mathematik und ihrer Grenzgebiete, 92. Springer, Berlin–New York, 1977.

[M] Mankiewicz, P., A superreflexive Banach space X with $L(X)$ admitting a homomorphism onto the Banach algebra $C(\beta\mathbf{N})$. *Israel J. Math.*, 65 (1989), 1–16.

[P] Pietsch, A., *Operator Ideals.* Mathematische Monographien, 16. VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[R] Rosenthal, H. P., On the subspaces of L^p $(p>2)$ spanned by sequences of independent random variables. *Israel J. Math.*, 8 (1970), 273–303.

[SZ] Schlumprecht, T. & Zsák, A., The algebra of bounded linear operators on $\ell_p \oplus \ell_q$ has infinitely many closed ideals. *J. Reine Angew. Math.*, 735 (2018), 225–247.

[T-J] Tomczak-Jaegermann, N., *Banach–Mazur Distances and Finite-Dimensional Operator Ideals.* Pitman Monographs and Surveys in Pure and Applied Mathematics, 38. Longman Scientific & Technical, Harlow; John Wiley & Sons, New York, 1989.

[W] Weis, L., On perturbations of Fredholm operators in $L_p(\mu)$-spaces. *Proc. Amer. Math. Soc.*, 67 (1977), 287–292.

William B. Johnson
Department of Mathematics
Texas A&M University
College Station, TX 77843-3368
U.S.A.
johnson@math.tamu.edu

Gideon Schechtman
Department of Mathematics
Weizmann Institute of Science
Rehovot, 76100
Israel
gideon@weizmann.ac.il

Received March 26, 2020