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Classifications of some classes of Zinbiel algebras

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Abstract

In this work, the nul-filiform and filiform Zinbiel algebras are described up to isomorphism. Moreover, the classification of complex Zinbiel algebras dimensions ≤ 3 is extended up to dimension 4.

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1 Introduction

One of the important objects of the modern theory of nonassociative algebras is Lie algebras. Active investigations in the theory of Lie algebras lead to the appearance of some generalizations of these algebras such as Mal'cev algebras, Lie superalgebras, binary Lie algebras, Leibniz algebras, and others.

In the present work, we consider algebras which are dual to Leibniz algebras. Recall that Leibniz algebras were introduced in [6] in the nineties of the last century. They are defined by the following identity:

$$\big[x,[y,z]\big] = \big[[x,y],z\big] - \big[[x,z],y\big].$$

J.-L. Loday in [5] studied categorical properties of Leibniz algebras and considered in this connection a new object – Zinbiel algebra (read Leibniz in the reverse order). Since the category of Zinbiel algebras is Koszul dual to the category of Leibniz algebras, sometimes they are also called dual Leibniz algebras [7].

In works [3, 4], some interesting properties of Zinbiel algebras were obtained. In particular, the nilpotency of an arbitrary complex finite-dimensional Zinbiel algebra was proved in [4].

For the examples of Zinbiel algebras, we refer to works [4, 5, 7].

Since description of all finite-dimensional complex Zinbiel algebras (which are nilpotent) is a boundless problem, it is natural to add certain restrictions for their investigation. One of such restrictions is the condition on the nilindex. Recall that in works [2, 8] some descriptions of nilpotent Leibniz algebras and Lie algebras are given.

At the beginning of the study of any class of algebras, it is important to describe up to isomorphism algebras of lower dimensions, because such description gives examples for to establish or reject certain conjectures. In this way, in [4], the classification of complex Zinbiel algebras of dimensions ≤ 3 is given. Applying some general results obtained for finite-dimensional Zinbiel algebras, we extend the classification of complex Zinbiel algebras up to dimension 4. It should be noted that this classification shows that associative algebras play the crucial role in the classification of four-dimensional algebras, which are defined by the multilinear identity of degree 3.

2 Classification of complex nul-filiform Zinbiel algebras

Definition 2.1. An algebra A over a field F is called Zinbiel algebra if for any $x, y, z \in A$ the identity

$$(x \circ y) \circ z = x \circ (y \circ z) + x \circ (z \circ y) \tag{2.1}$$

holds.

For a given Zinbiel algebra A, we define the following sequence:

$$A^1 = A, \quad A^{k+1} = A \circ A^k, \quad k \geq 1.$$

Definition 2.2. A Zinbiel algebra A is called nilpotent if there exists $s \in N$ such that $A^s = 0$. The minimal number s satisfying this property is called index of nilpotency or nilindex of the algebra A.

It is not difficult to see that the index of nilpotency of an arbitrary n-dimensional nilpotent algebra does not exceed the number $n + 1$.

Definition 2.3. An n-dimensional Zinbiel algebra A is called nul-filiform if $\dim A^i = (n + 1) - i$, $1 \leq i \leq n + 1$.

It is evident that the last definition is equivalent to the fact that algebra A has maximal index of nilpotency.

Theorem 2.4. *An arbitrary n-dimensional nul-filiform Zinbiel algebra is isomorphic to the following algebra:*

$$e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad \text{for } 2 \leq i + j \leq n, \tag{2.2}$$

where omitted products are equal to zero and $\{e_1, e_2, \ldots, e_n\}$ is a basis of the algebra, the symbols C^t_s are binomial coefficients defined as $C^t_s = \frac{s!}{t!(s-t!)}$.

Proof. Let A be an n-dimensional nul-filiform Zinbiel algebra and let $\{x_1, x_2, \ldots, x_n\}$ be a basis of the algebra A such that $x_1 \in A^1 \setminus A^2$, $x_2 \in A^2 \setminus A^3$, ..., $x_n \in A^n$. Since $x_2 \in A^2 \setminus A^3$, for some elements $b_{2,p}, c_{2,p}$ of algebra A, we have

$$x_2 = \sum b_{2,p} \circ c_{2,p} = \sum \alpha_{i,j} x_i \circ x_j = \alpha_{1,1} x_1 \circ x_1 + (*),$$

where $(*) \in A^3$, i.e., $x_2 = \alpha_{1,1} x_1 \circ x_1 + (*)$. Note that $\alpha_{1,1} x_1 \circ x_1 \neq 0$. Indeed, in the opposite case, $x_2 \in A^3$.

Similarly, for $x_3 \in A^3 \setminus A^4$, we have

$$x_3 = \sum a_{3,p} \circ \big(b_{3,p} \circ c_{3,p}\big) = \sum \alpha_{i,j,k} x_i \circ \big(x_j \circ x_k\big) = \alpha_{1,1,1} x_1 \circ \big(x_1 \circ x_1\big) + (**),$$

where $(**) \in A^4$ and $\alpha_{1,1,1} x_1 \circ (x_1 \circ x_1) \neq 0$ (otherwise $x_3 \in A^4$), i.e., $x_3 = \alpha_{1,1,1} x_1 \circ (x_1 \circ x_1) + (**)$. Continuing this process, we obtain that elements

$$e_1 := x_1, \quad e_2 := x_1 \circ x_1, \quad e_3 := x_1 \circ (x_1 \circ x_1), \quad \ldots, \quad e_n := (x_1 \circ \cdots \circ (x_1 \circ (x_1 \circ x_1)))$$

are distinct from zero. It is not difficult to check the linear independence of these elements. Hence, we can choose the elements $\{e_1, e_2, \dots, e_n\}$ as a basis of algebra A. We have by construction

$$e_1 \circ e_i = e_{i+1} \quad \text{for } 1 \le i \le n-1. \tag{2.3}$$

We prove equality (2.2) by induction on j for every i.

Using identities (2.1), (2.3), we can prove by induction the equality

$$e_i \circ e_1 = ie_{i+1} \quad \text{for } 1 \le i \le n-1,$$

i.e., equality (2.2) is true for $j = 1$ and every i.

Suppose that the equality is true for all $j \le k-1$ and every i.

Let us prove the equality (2.2) for $j = k$ and every i. Using the inductive hypothesis and the following chain of equalities:

$$\begin{aligned} e_i \circ e_k &= e_i \circ \big(e_1 \circ e_{k-1}\big) = \big(e_i \circ e_1\big) \circ e_{k-1} - e_i \circ \big(e_{k-1} \circ e_1\big) \\ &= ie_{i+1} \circ e_{k-1} - (k-1)e_i \circ e_k = iC_{i+k-1}^{k-1} e_{i+k} - (k-1)e_i \circ e_k, \end{aligned}$$

we obtain $ke_i \circ e_k = iC_{i+k-1}^{k-1} e_{i+k}$, i.e.,

$$e_i \circ e_k = \frac{i}{k} C_{i+k-1}^{k-1} e_{i+k} = \frac{i}{k} \frac{(i+k-1)!}{(k-1)!i!} e_{i+k} = \frac{(i+k-1)!}{k!(i-1)!} e_{i+k} = C_{i+k-1}^{k} e_{i+k}. \qquad \square$$

We denote the algebra from Theorem 2.4 as NF_n. It is not difficult to see that the n-dimensional Zinbiel algebra is one generated if and only if it is isomorphic to the algebra NF_n.

3 Classification of complex filiform Zinbiel algebras

In this section, we classify filiform Zinbiel algebras.

Definition 3.1. An n-dimensional Zinbiel algebra A is said to be filiform if $\dim A^i = n - i$, $2 \le i \le n$.

Definition 3.2. Given a filiform Zinbiel algebra A, put $A_i = A^i/A^{i+1}$, $1 \le i \le n-1$ and $\mathrm{grA} = A_1 \oplus A_2 \oplus \dots \oplus A_{n-1}$. Then $A_i \circ A_j \subseteq A_{i+j}$, $\dim A_1 = 2$, $\dim A_i = 1$ for $2 \le i \le n-1$ and we obtain the graded algebra grA. If an algebra B is isomorphic to grA, then we say that the algebra B is naturally graded.

In the following theorem, the classification of complex naturally graded filiform Zinbiel algebras is represented.

Theorem 3.3. *An arbitrary n-dimensional $(n \ge 5)$ naturally graded complex filiform Zinbiel algebra is isomorphic to the following algebra:*

$$F_n : e_i \circ e_j = C_{i+j-1}^{j} e_{i+j}, \quad 2 \le i + j \le n-1,$$

where omitted products are equal to zero and $\{e_1, e_2, \dots, e_n\}$ is a basis of the algebra.

Proof. Let A be a Zinbiel algebra satisfying conditions of the theorem. Similar to the work [8], we choose a basis $\{e_1, e_2, \dots, e_n\}$ of the algebra A such that $A_1 = \langle e_1, e_n \rangle$, $A_j = \langle e_j \rangle$, $2 \le j \le n-1$ and $e_1 \circ e_i = e_{i+1}$ for $2 \le i \le n-2$.

Introduce the following notations:

$$e_1 \circ e_1 = \alpha_1 e_2, \quad e_1 \circ e_n = \alpha_2 e_2, \quad e_n \circ e_1 = \alpha_3 e_2, \quad e_n \circ e_n = \alpha_4 e_2.$$

Consider the possible cases.

Case 1. Let $(\alpha_1, \alpha_4) \neq (0,0)$. Then without loss of generality we can suppose $\alpha_1 \neq 0$. Change the basis as follows: $e_1' = e_1$, $e_i' = \alpha_1 e_i$, $2 \leq i \leq n-1$. We can suppose $\alpha_1 = 1$. Then the space spanned on the vectors $\{e_1, e_2, \dots, e_{n-1}\}$ forms a nul-filiform Zinbiel algebra of the dimension $n-1$. From the proof of Theorem 2.4, we can conclude $e_i \circ e_j = C_{i+j-1}^{j} e_{i+j}$, $2 \leq i+j \leq n-1$.

Let us show that the omitted products are equal to zero.

Apply identity (2.1) in the following multiplications:

$$\big(e_1 \circ e_n\big) \circ e_1 = e_1 \circ \big(e_n \circ e_1\big) + e_1 \circ \big(e_1 \circ e_n\big) \Longrightarrow 2\alpha_2 e_3 = \alpha_3 e_3 + \alpha_2 e_3,$$

i.e., $\alpha_2 = \alpha_3$;

$$\big(e_1 \circ e_1\big) \circ e_n = e_1 \circ \big(e_1 \circ e_n\big) + e_1 \circ \big(e_n \circ e_1\big) \Longrightarrow e_2 \circ e_n = \alpha_2 e_3 + \alpha_3 e_3,$$

hence, $e_2 \circ e_n = 2\alpha_2 e_3$;

$$\big(e_1 \circ e_n\big) \circ e_n = 2e_1 \circ \big(e_n \circ e_n\big) \Longrightarrow 2\alpha_2^2 e_3 = 2\alpha_4 e_3,$$

consequently, $\alpha_2^2 = \alpha_4$;

$$\big(e_n \circ e_1\big) \circ e_1 = 2e_n \circ \big(e_1 \circ e_1\big) \Longrightarrow 2\alpha_2 e_3 = 2e_n \circ e_2,$$

i.e., we have $e_n \circ e_2 = \alpha_2 e_3$.

Taking the change of the basic elements by the following way:

$$e_n' = -\alpha_2 e_1 + e_n, \quad e_i' = e_i \quad \text{for } 1 \leq i \leq n-1,$$

it is not difficult to see that $e_1' \circ e_n' = e_n' \circ e_1' = e_n' \circ e_n' = e_2' \circ e_n' = e_n' \circ e_2' = 0$. Moreover, the other products are not changed, i.e., we can suppose $\alpha_2 = \alpha_3 = \alpha_4 = 0$.

Using identity (2.1) and the method of mathematical induction, it is easy to prove

$$e_n \circ e_i = 0 \quad \text{for } 1 \leq i \leq n-1. \tag{3.1}$$

The equality

$$e_i \circ e_n = 0 \quad \text{for } 1 \leq i \leq n-1$$

can be proved by induction and using (2.1), (3.1).

Case 2. Let $(\alpha_1, \alpha_4) = (0,0)$. Then $(\alpha_2, \alpha_3) \neq (0,0)$. In the case of $\alpha_2 \neq -\alpha_3$, taking $e_1' = e_1 + e_n$, we obtain the conditions of Case 1. Therefore, we need to consider only the case of $\alpha_2 = -\alpha_3 \neq 0$. By the following change of basis:

$$e_1' = e_1, \quad e_n' = e_n, \quad e_i' = \alpha_2 e_i \quad \text{for } 2 \leq i \leq n-1,$$

we can suppose $\alpha_2 = 1$.

The products

$$\big(e_1 \circ e_n\big) \circ e_1 = e_1 \circ \big(e_n \circ e_1\big) + e_1 \circ \big(e_1 \circ e_n\big) \Longrightarrow e_2 \circ e_1 = 0,$$

$$0 = (e_1 \circ e_1) \circ e_2 = e_1 \circ (e_1 \circ e_2) + e_1 \circ (e_2 \circ e_1) = e_1 \circ e_3 = e_4$$

deduce the contradiction to the existence of algebra in this case. □

The following proposition allows to extract a "convenient" basis in an arbitrary complex filiform Zinbiel algebra. Such basis in the literature is called adapted [8].

Proposition 3.4. *There exists a basis $\{e_1, e_2, \dots, e_n\}$ in an arbitrary n-dimensional $(n \geq 5)$ complex filiform Zinbiel algebra such that the multiplication of the algebra has the following form:*

$$\begin{aligned} &e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad 2 \leq i+j \leq n-1, \\ &e_n \circ e_1 = \alpha e_{n-1}, \quad e_n \circ e_n = \beta e_{n-1}, \end{aligned} \tag{3.2}$$

where $\alpha, \beta \in \mathbb{C}$.

Proof. By Theorem 3.3 we have that any n-dimensional complex filiform Zinbiel algebra is isomorphic to the algebra of the form $F_n + \beta$, where

$$\begin{aligned} &\beta(e_1, e_i) = 0 \quad \text{for } 1 \leq i \leq n-1, \\ &\beta(e_n, e_n) \in \operatorname{lin}\{e_3, e_4, \dots, e_{n-1}\}, \\ &\beta(e_i, e_n), \beta(e_n, e_i) \in \operatorname{lin}\{e_{i+2}, e_{i+3}, \dots, e_{n-1}\} \quad \text{for } 1 \leq i \leq n-1, \\ &\beta(e_i, e_1) \in \operatorname{lin}\{e_{i+2}, e_{i+3}, \dots, e_{n-1}\} \quad \text{for } 2 \leq i \leq n-1, \\ &\beta(e_i, e_j) \in \operatorname{lin}\{e_{i+j+1}, e_{i+j+2}, \dots, e_{n-1}\} \quad \text{for } 2 \leq i, j \leq n-1. \end{aligned}$$

Similarly to the proof of Theorem 3.3, it is not difficult to establish that the multiplications

$$e_i \circ e_1, \quad e_i \circ e_j, \quad 1 \leq i, j \leq n-1$$

can be obtained from $e_1 \circ e_i = e_{i+1}$, $1 \leq i \leq n-2$, and identity (2.1).

By the similar process, we obtain

$$\beta(e_i, e_1) = 0 \quad \text{for } 1 \leq i \leq n-1, \quad \beta(e_i, e_j) = 0 \quad \text{for } 2 \leq i, j \leq n-1.$$

So, we have the products

$$e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad 2 \leq i+j \leq n-1.$$

Now define the products $e_i \circ e_n$ and $e_n \circ e_i$ for $1 \leq i \leq n$ and put

$$e_1 \circ e_n = \alpha_3 e_3 + \alpha_4 e_4 + \dots + \alpha_{n-1} e_{n-1}, \quad e_n \circ e_1 = \beta_3 e_3 + \beta_4 e_4 + \dots + \beta_{n-1} e_{n-1}.$$

Taking the change

$$e'_n = e_n - \alpha_3 e_2 - \alpha_4 e_3 - \dots - \alpha_{n-1} e_{n-2},$$

we can suppose $\alpha_3 = \alpha_4 = \dots = \alpha_{n-1} = 0$, i.e., $e_1 \circ e_n = 0$.

Identity (2.1) implies

$$(e_1 \circ e_n) \circ e_1 = e_1 \circ (e_n \circ e_1) + e_1 \circ (e_1 \circ e_n) \Longrightarrow e_1 \circ (\beta_3 e_3 + \beta_4 e_4 + \dots + \beta_{n-1} e_{n-1}) = 0.$$

Therefore, $\beta_3 e_4 + \beta_4 e_5 + \dots + \beta_{n-2} e_{n-1} = 0$ and $e_n \circ e_1 = \beta_{n-1} e_{n-1}$.

Consider the product

$$\big(e_n \circ e_n\big) \circ e_1 = e_n \circ \big(e_n \circ e_1\big) + e_n \circ \big(e_1 \circ e_n\big) = 0$$

from which we obtain $e_n \circ e_n = \gamma e_{n-1}$ for some γ.

Similar to the proof of Theorem 3.3, we can obtain that $e_n \circ e_i = 0$ and $e_i \circ e_n = 0$ for $2 \leq i \leq n-1$, which complete the proof of the proposition. $\square$

The classification of complex filiform Zinbiel algebras is given in the following theorem.

Theorem 3.5. *An arbitrary n-dimensional ($n \geq 5$) complex filiform Zinbiel algebra is isomorphic to one of the following pairwise nonisomorphic algebras:*

$$\begin{aligned}
&F_n^1 : e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq n-1;\\
&F_n^2 : e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq n-1, \quad e_n \circ e_1 = e_{n-1};\\
&F_n^3 : e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq n-1, \quad e_n \circ e_n = e_{n-1}.
\end{aligned}$$

Proof. From Proposition 3.4, we have the multiplication (multiplication (3.2)) of n-dimensional complex filiform Zinbiel algebra, namely,

$$\begin{aligned}
&e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq n-1,\\
&e_n \circ e_1 = \alpha e_{n-1}, \quad e_n \circ e_n = \beta e_{n-1}.
\end{aligned}$$

Let us check the isomorphism inside this family of algebras. Consider the general change of the generators of the basic elements in the form

$$e_1' = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n, \quad e_n' = b_1 e_1 + b_2 e_2 + \cdots + b_n e_n,$$

where $a_1 \neq 0$ and $a_1 b_n - a_n b_1 \neq 0$. Then for the remainder elements of the new basis we have

$$e_2' = a_1^2 e_2 + h_3, \quad e_3' = a_1^3 e_3 + h_4, \quad \ldots, \quad e_{n-1}' = a_1^{n-1} e_{n-1},$$

where $h_i \in A^i$. The relations

$$e_i' \circ e_j' = C_{i+j-1}^j e_{i+j}', \quad 2 \leq i+j \leq n-1,$$

imply the following restrictions:

$$\begin{aligned}
&a_1 b_1 = 0,\\
&a_1 b_2 + 2 a_2 b_1 = 0,\\
&a_1 b_3 + C_3^2 a_2 b_2 + 3 a_3 b_1 = 0,\\
&\qquad\qquad\vdots\\
&a_1 b_{n-3} + C_{n-3}^{n-4} a_2 b_{n-4} + C_{n-3}^{n-5} a_3 b_{n-5} + \cdots + (n-3) a_{n-3} b_1 = 0,\\
&a_1 b_{n-2} + C_{n-2}^{n-3} a_2 b_{n-3} + C_{n-2}^{n-4} a_3 b_{n-4} + \cdots + (n-2) a_{n-2} b_1 + \alpha a_n b_1 + \beta a_n b_n = 0.
\end{aligned}$$

From these restrictions, we get

$$b_1 = b_2 = \cdots = b_{n-3} = 0, \quad b_{n-2} = -\frac{a_n b_n}{a_1} \beta.$$

Consider the product

$$e'_n \circ e'_1 = \Big(-\frac{a_n b_n}{a_1}\beta e_{n-2} + b_{n-1}e_{n-1} + b_n e_n \Big) \circ \big(a_1 e_1 + a_2 e_2 + \cdots + a_n e_n\big)$$
$$= -(n-2)\beta a_n b_n e_{n-1} + \alpha a_1 b_n e_{n-1} + \beta a_n b_n e_{n-1} = \big(\alpha a_1 b_n - (n-3)a_n b_n\big)e_{n-1}.$$

On the other hand,

$$e'_n \circ e'_1 = \alpha' e'_{n-1} = \alpha' a_1^{n-1} e_{n-1}.$$

Comparing the coefficients at the basic element e_{n-1}, we obtain

$$\alpha a_1 b_n - (n-3)\beta a_n b_n = \alpha' a_1^{n-1}.$$

Consider the product

$$e'_n \circ e'_n = \Big(-\frac{a_n b_n}{a_1}\beta e_{n-2} + b_{n-1}e_{n-1} + b_n e_n \Big) \circ \Big(-\frac{a_n b_n}{a_1}\beta e_{n-2} + b_{n-1}e_{n-1} + b_n e_n \Big)$$
$$= b_n^2 \beta e_{n-1}.$$

On the other hand,

$$e'_n \circ e'_n = \beta' e'_{n-1} = \beta' a_1^{n-1} e_{n-1}.$$

Comparing the coefficients, we have

$$b_n^2 \beta = \beta' a_1^{n-1}.$$

Now consider the following cases.

Case 1. Let $\beta = 0$. Then $\beta' = 0$ and $\alpha b_n = \alpha' a_1^{n-2}$. If $\alpha = 0$, then $\alpha' = 0$ and we have algebra F_n^1. If $\alpha \neq 0$, then taking $b_n = \frac{a_1^{n-2}}{\alpha}$ we get $\alpha' = 1$, i.e., the algebra F_n^2 is obtained.

Case 2. Let $\beta \neq 0$. Then putting $b_n = \sqrt{\frac{a_1^{n-1}}{\beta}}$, $a_n = \frac{\alpha a_1}{(n-3)\beta}$ we get $\beta' = 1$, $\alpha' = 0$ and algebra F_n^3 is obtained.

Note that the obtained algebras are not pairwise isomorphic. □

Comparing the description of complex filiform Leibniz algebras [2] and the result of Theorem 3.5, we can note how the class of complex filiform Zinbiel algebras is "thinner". So, although Zinbiel algebras and Leibniz algebras are Koszul dual, they are quantitatively strongly distinguished even in the class of filiform algebras.

It is known that for any variety of algebras $\mathbb{Z}_2$-graded algebras of that variety can be defined, which is called superalgebras. In the same way, we define a notion of Zinbiel superalgebra $A = A_0 \oplus A_1$ by the following identity:

$$(x \circ y) \circ z = x \circ (y \circ z) + (-1)^{\alpha\beta} x \circ (z \circ y),$$

where $y \in A_\alpha$, $z \in A_\beta$ and $\alpha, \beta \in \mathbb{Z}_2$.

It should be noted that the proof on the nilpotency of finite-dimensional complex Zinbiel algebras can be extended to the proof of solvability of the finite-dimensional complex Zinbiel superalgebras.

Moreover, from the obtained classification of complex filiform Zinbiel algebras, we give the following conjecture.

Conjecture 3.6. *Finite-dimensional complex $\mathbb{Z}_2$-graded Zinbiel algebra (Zinbiel superalgebra) is nilpotent.*

4 Classification of four-dimensional complex Zinbiel algebras

Since an arbitrary finite-dimensional complex Zinbiel algebra is nilpotent, then for an arbitrary four-dimensional Zinbiel algebra A the condition $A^5 = 0$ holds.

Take into account that the direct sum of nilpotent Zinbiel algebras is nilpotent; further we will not consider split algebras case.

Theorem 4.1. *An arbitrary four-dimensional complex nonsplit Zinbiel algebra is isomorphic to the one of the following pairwise nonisomorphic algebras:*

$$A_1 : e_1 \circ e_1 = e_2,\ e_1 \circ e_2 = e_3,\ e_2 \circ e_1 = 2e_3,\ e_1 \circ e_3 = e_4,\ e_2 \circ e_2 = 3e_4,\ e_3 \circ e_1 = 3e_4;$$
$$A_2 : e_1 \circ e_1 = e_3,\ e_1 \circ e_2 = e_4,\ e_1 \circ e_3 = e_4,\ e_3 \circ e_1 = 2e_4;$$
$$A_3 : e_1 \circ e_1 = e_3,\ e_1 \circ e_3 = e_4,\ e_2 \circ e_2 = e_4,\ e_3 \circ e_1 = 2e_4;$$
$$A_4 : e_1 \circ e_2 = e_3,\ e_1 \circ e_3 = e_4,\ e_2 \circ e_1 = -e_3;$$
$$A_5 : e_1 \circ e_2 = e_3,\ e_1 \circ e_3 = e_4,\ e_2 \circ e_1 = -e_3,\ e_2 \circ e_2 = e_4;$$
$$A_6 : e_1 \circ e_1 = e_4,\ e_1 \circ e_2 = e_3,\ e_2 \circ e_1 = -e_3,\ e_2 \circ e_2 = -2e_3 + e_4;$$
$$A_7 : e_1 \circ e_2 = e_3,\ e_2 \circ e_1 = e_4,\ e_2 \circ e_2 = -e_3;$$
$$A_8(\alpha) : e_1 \circ e_1 = e_3,\ e_1 \circ e_2 = e_4,\ e_2 \circ e_1 = -\alpha e_3, e_2 \circ e_2 = -e_4,\ \alpha \in \mathbb{C};$$
$$A_9(\alpha) : e_1 \circ e_1 = e_4,\ e_1 \circ e_2 = \alpha e_4,\ e_2 \circ e_1 = -\alpha e_4,\ e_2 \circ e_2 = e_4,\ e_3 \circ e_3 = e_4,\ \alpha \in \mathbb{C};$$
$$A_{10} : e_1 \circ e_2 = e_4,\ e_1 \circ e_3 = e_4,\ e_2 \circ e_1 = -e_4,\ e_2 \circ e_2 = e_4,\ e_3 \circ e_1 = e_4;$$
$$A_{11} : e_1 \circ e_1 = e_4,\ e_1 \circ e_2 = e_4,\ e_2 \circ e_1 = -e_4,\ e_3 \circ e_3 = e_4;$$
$$A_{12} : e_1 \circ e_2 = e_3,\ e_2 \circ e_1 = e_4;$$
$$A_{13} : e_1 \circ e_2 = e_3,\ e_2 \circ e_1 = -e_3,\ e_2 \circ e_2 = e_4;$$
$$A_{14} : e_2 \circ e_1 = e_4,\ e_2 \circ e_2 = e_3;$$
$$A_{15}(\alpha) : e_1 \circ e_2 = e_4,\ e_2 \circ e_2 = e_3,\ e_2 \circ e_1 = \frac{1+\alpha}{1-\alpha} e_4,\ \alpha \in \mathbb{C} \setminus \{1\};$$
$$A_{16} : e_1 \circ e_2 = e_4,\ e_2 \circ e_1 = -e_4,\ e_3 \circ e_3 = e_4.$$

Proof. Note that the result of [1, Proposition 3.1] also holds for Zinbiel algebras. Therefore, we have the following possible cases for invariant sequence $(\dim A^2, \dim A^3, \dim A^4)$:

$$(3, 2, 1),\ (2, 1, 0),\ (2, 0, 0),\ (1, 0, 0),\ (0, 0, 0).$$

Obviously, a Zinbiel algebra with the condition $(3, 2, 1)$ is nul-filiform. Using Theorem 2.4, we obtain the algebra A_1.

Consider an algebra with the invariant sequence $(2, 1, 0)$ (this algebra is filiform).

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of algebra A satisfying the conditions $A^2 = \{e_3, e_4\}$, $A^3 = \{e_4\}$. Then we can suppose that

$$e_1 \circ e_1 = \alpha_1 e_3 + \alpha_2 e_4, \quad e_1 \circ e_2 = \alpha_3 e_3 + \alpha_4 e_4, \quad e_2 \circ e_1 = \alpha_5 e_3 + \alpha_6 e_4,$$
$$e_2 \circ e_2 = \alpha_7 e_3 + \alpha_8 e_4, \quad e_1 \circ e_3 = e_4, \quad e_2 \circ e_3 = \alpha_9 e_4,$$

where $(\alpha_1, \alpha_3, \alpha_5, \alpha_7) \neq (0, 0, 0, 0)$.

Case 1. Let $(\alpha_1, \alpha_7) \neq (0, 0)$. Then by arguments analogous to the arguments in the proofs of Theorems 3.3 and 3.5 we obtain the following algebras:

$$e_1 \circ e_1 = e_3, \quad e_1 \circ e_3 = e_4, \quad e_3 \circ e_1 = 2e_4;$$

$$e_1\circ e_1=e_3,\quad e_1\circ e_2=e_4,\quad e_1\circ e_3=e_4,\quad e_3\circ e_1=2e_4;$$
$$e_1\circ e_1=e_3,\quad e_1\circ e_3=e_4,\quad e_2\circ e_2=e_4,\quad e_3\circ e_1=2e_4.$$

Note that the algebra defined by multiplication

$$e_1\circ e_1=e_3,\quad e_1\circ e_3=e_4,\quad e_3\circ e_1=2e_4$$

is split. So, in this case, we have the algebras A_2, A_3.

Case 2. Let $(\alpha_1,\alpha_7)=(0,0)$. Then $(\alpha_3,\alpha_5)\neq(0,0)$. If $\alpha_3\neq-\alpha_5$, then taking $e_1'=Ae_1+e_2$, where $A\neq-\alpha_9$, we have Case 1. It remains to consider the case $\alpha_3=-\alpha_5$. Denote $e_3'=\alpha_3e_3+\alpha_4e_4$. Then we can write

$$e_1\circ e_1=\alpha_2e_4,\quad e_1\circ e_2=e_3,\quad e_2\circ e_1=-e_3+\alpha_6e_4,$$
$$e_2\circ e_2=\alpha_8e_4,\quad e_1\circ e_3=e_4,\quad e_2\circ e_3=\alpha_9e_4.$$

Consider the products

$$\begin{aligned}(e_1\circ e_2)\circ e_1&=e_1\circ(e_2\circ e_1)+e_1\circ(e_1\circ e_2)=e_1\circ(-e_3+\alpha_6e_4)+e_1\circ e_3\\&=0\Longrightarrow e_3\circ e_1=0,\\(e_1\circ e_2)\circ e_2&=2e_1\circ(e_2\circ e_2)=0\Longrightarrow e_3\circ e_2=0.\end{aligned}$$

If we replace basic elements as follows:

$$e_1'=e_1,\quad e_2'=e_2-\alpha_9e_1,\quad e_3'=e_3-\alpha_2\alpha_9e_4,\quad e_4'=e_4,$$

we obtain $\alpha_9=0$, i.e., the multiplication in the algebra has the following form:

$$e_1\circ e_1=\alpha e_4,\quad e_1\circ e_2=e_3,\quad e_2\circ e_1=-e_3+\beta e_4,\quad e_2\circ e_2=\gamma e_4,\quad e_1\circ e_3=e_4$$

(omitted products are equal to zero).

Let us check the isomorphism inside this family.

Consider the general change of generators of the basic elements:

$$e_1'=a_1e_1+a_2e_2+a_3e_3,\quad e_2'=b_1e_1+b_2e_2+b_3e_3$$

where $a_1b_2-a_2b_1\neq0$. Expressing basic elements $\{e_3',e_4'\}$ via basic elements $\{e_1,e_2,e_3,e_4\}$ and analyzing the relations of the family in new basis, we obtain the following restrictions:

$$a_1^2\alpha+a_1a_2\beta+a_2^2\gamma+a_1a_3=\alpha'a_1^2b_2,\quad a_1b_2\beta+2a_2b_2\gamma+a_1b_3=\beta'a_1^2b_2,$$
$$b_2\gamma=\gamma'a_1^2,\quad b_1=0.$$

Consider the following cases.

Case 2.1. Let $\gamma=0$. Then $\gamma'=0$ and

$$a_1\alpha+a_2\beta+a_3=\alpha'a_1b_2,\quad b_2+b_3=\beta'a_1b_2.$$

Taking $a_3=-a_1\alpha-a_2\beta$ and $b_3=-b_2$, we obtain $\alpha'=\beta'=0$, i.e., we have the algebra A_4.

Case 2.2. Let $\gamma\neq0$. Then putting

$$b_2=\frac{a_1^2}{\gamma},\quad a_3=-\frac{a_1^2\alpha+a_1a_2\beta+a_2^2\gamma}{a_1},\quad b_3=\frac{a_1b_2\beta+2a_2b_2\gamma}{a_1},$$

we get $\gamma'=1$, $\alpha'=\beta'=0$, i.e., the algebra A_5 is obtained.

Note that algebras with the conditions $(2,0,0)$, $(1,0,0)$, $(0,0,0)$ are associative algebras. Therefore, we can use the classification of four-dimensional algebras of Leibniz [1], i.e., choose algebras with the condition $A^3 = 0$. $\square$

Note that the problem of classification of complex five-dimensional Zinbiel algebras is open and the solution of this problem is equivalent to the classification of such associative algebras, which is still not obtained.

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