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Research Article

Algebraic Structures Derived from Foams

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Dedicated to Professor Akio Kawauchi on the occasion of his 60th birthday

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Abstract Foams are surfaces with branch lines at which three sheets merge. They have been used in the categorification of $\mathrm{sl}(3)$ quantum knot invariants and also in physics. The $2D$-TQFT of surfaces, on the other hand, is classified by means of commutative Frobenius algebras, where saddle points correspond to multiplication and comultiplication. In this paper, we explore algebraic operations that branch lines derive under TQFT. In particular, we investigate Lie bracket and bialgebra structures. Relations to the original Frobenius algebra structures are discussed both algebraically and diagrammatically.

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1 Introduction

Frobenius algebras have been used extensively in the study of categorification of the Jones polynomial [9], via 2-dimensional Topological Quantum Field Theory ($2D$-TQFT, [10]). For categorifications of other knot invariants, 2-dimensional complexes called foams have been used instead [8, 12]. Although $2D$-TQFT has been characterized [10] in terms of commutative Frobenius algebras, foams have not been algebraically characterized in terms of TQFT. Relations to Lie algebras, for example, have been suggested [8, 12] through their boundaries which are called webs and that are trivalent graphs. Foams have branch curves along which three sheets meet. Similar complexes appear as spines of 3-manifolds, and have been used for quantum invariants [4, 5, 7, 13].

Herein we study the types of algebraic operations that appear along the branch curves of foams in relation to $2D$-TQFT. Recall that a $2D$-TQFT is a functor from the category of 2-dimensional cobordisms to a category of R-modules (for some suitable ring R) that assigns an R-module to each connected component (circle) on the boundary of a surface, and an R-module homomorphism to a surface. In the case of a foam, we examine the associated algebraic operations that might be associated to branching circles in relation to the Frobenius algebra structure that occurs on the unbranched surfaces. Specifically, we identify and study Lie algebra and bialgebra structures in relation to branch curves and study their relations to the Frobenius algebra structure.

After reviewing necessary materials in Section 2, a Lie algebra structure for the branch curves is studied in Section 3, and comultiplications of bialgebras are examined in Section 4. The foam skein theory based on the bialgebra case is also defined in Section 4.

2 Preliminary

Algebraic structures we investigate include Frobenius algebras, Lie algebras, and bialgebras. We restrict to the following situations.

A *Lie algebra* is a module A over a unital commutative ring R with a binary operation $[\ ,\ ] : A \times A \to A$ that is bilinear and skew symmetric ($[x, y] = -[y, x]$ for $x, y \in A$) and satisfies the Jacobi identity ($[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for $x, y, z \in A$).

A *Frobenius algebra* is an algebra over R (that comes with associative linear multiplication $\mu : A \otimes A \to A$ and unit $\eta : R \to A$) with a nondegenerate form $\epsilon : A \to R$ that is associative ($\epsilon(\mu(\mu \otimes |_A)(x \otimes y \otimes z)) = \epsilon(\mu(|_A \otimes \mu)(x \otimes y \otimes z))$ for $x, y, z \in A$, where $|_A$ denotes the identity map on A). There is an induced coassociative comultiplication $\Delta : A \to A \otimes A$. See [10] for Frobenius algebras and [3] for diagrams for Frobenius algebras which we will use in this paper. A *bialgebra* is an algebra A over R with a comultiplication $\Delta : A \to A \otimes A$ that is an algebra homomorphism ($\Delta(xy) = \Delta(x)\Delta(y)$) and a counit $\epsilon : A \to R$ such that $(\epsilon \otimes |_A)\Delta = |_A = (|_A \otimes \epsilon)\Delta$. The following are typical examples.

Example 1. Let $A = A_N$ be the Frobenius algebra of truncated polynomial $A_N = R[X]/(X^N)$ for a commutative unital ring R and an integer $N > 1$, with counit (Frobenius form) ϵ determined by $\epsilon(x^{N-1}) = 1$ and $\epsilon(x^i) = 0$ for $i \neq N-1$. The comultiplication Δ is determined by $\Delta(1) = \sum_{i=0}^{N-1} X^i \otimes X^{N-1-i}$. Diagrammatically, this is represented by a "neck cutting" relation [2], which we call *a $\Delta(1)$-relation* to distinguish the specific relation given in [2] for $N = 2$. See the right of Figure 4 for a diagrammatic representation of the $\Delta(1)$-relation in this case.

A black dot in the figure labeled with a basis element of the Frobenius algebra A represents that the element is assigned to the factor of A corresponding to the component, and in terms of the map corresponding to the cobordism, it represents multiplication by the element.

In general, the $\Delta(1)$-relation is also described as follows (see [8,10]). For a commutative Frobenius algebra A over a unital ring R of finite rank and with a nondegenerate Frobenius form ϵ, there is a basis $\{x_i\}$ and a dual basis $\{y_i\}$, $i = 1, \ldots, n$, such that $\epsilon(x_i y_i) = \delta_{i,j}$, the Kronecker delta, and $x = \sum_i y_i \epsilon(x_i x)$. This situation is depicted in Figure 1, where the identity map $x \mapsto x$ corresponds to the cylindrical cobordism in the left of the figure, and the sum involving the Frobenius form ϵ is depicted in the right of the figure.

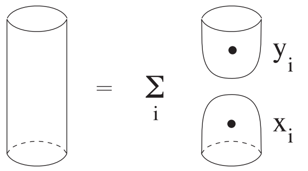

Figure 1: The $\Delta(1)$-relation.

Example 2. A Frobenius algebra structure on $A = \mathbb{Z}[a,b,c][X]/(X^3 - aX^2 - bX - c)$ is presented in [12] as follows. The multiplication and the unit are defined by those for polynomials; the Frobenius form (counit) ϵ is defined by $\epsilon(1) = \epsilon(X) = 0$, $\epsilon(X^2) = -1$. The comultiplication is accordingly computed as

$$\begin{aligned}
\Delta(1) &= -\big(1 \otimes X^2 + X \otimes X + X^2 \otimes 1\big) + a(1 \otimes X + X \otimes 1) + b(1 \otimes 1), \\
\Delta(X) &= -\big(X \otimes X^2 + X^2 \otimes X\big) + a(X \otimes X) - c(1 \otimes 1), \\
\Delta\big(X^2\big) &= -\big(X^2 \otimes X^2\big) - b(X \otimes X) - c(1 \otimes X + X \otimes 1).
\end{aligned}$$

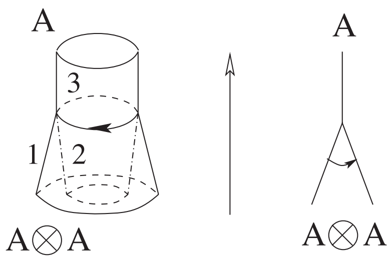

Figure 2: Operation on a branch circle.

We fix a $2D$-TQFT such that a connected circle corresponds to A. For TQFTs we refer to [10]. We follow definitions of foams in [8,12], except that facets of foams are decorated by basis elements of A, in a general way as in [6]. A foam without boundary is called closed.

We briefly summarize their definitions. $\mathbf{Foam}_A$ is the category of formal linear combinations over R of cobordisms of compact 2-dimensional complexes in 3-space with the following data. (1) Boundaries are planar graphs with trivalent rigid vertices. (2) For an interior point of a cobordism, the neighborhood of each point is homeomorphic to either Euclidean 2-space (a facet) or a *branch curve* where three facets of half planes meet. (3) Each facet is oriented, and the induced orientation on the branch curve is consistent among three facets that share the curve. (4) A cyclic order of facets are specified using the orientation of 3-space as depicted in Figure 2. (5) Each facet has a basis element of A assigned. (6) An annular cobordism as depicted in the left of Figure 1 is equivalent to the linear combination as depicted on the right. (7) Values $\theta(\alpha, \beta, \gamma) \in A$ of the theta foam, as depicted in Figure 3, are specified, for basis elements α, β, γ of A.

In [8,12], it was shown that the values in A of closed foams are well defined for values of the theta foams, as long as the *cyclic symmetry condition* $\theta(\alpha,\beta,\gamma)=\theta(\beta,\gamma,\alpha)=\theta(\gamma,\alpha,\beta)$ is satisfied.

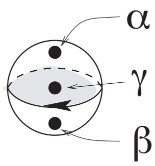

Figure 3: The theta foam.

By $2D$-TQFT for a chosen A, the two circles at the bottom left of Figure 2 are mapped to the factors of $A\otimes A$. For the cyclic order along the oriented branch circle as depicted, make a correspondence between the facets labeled 1, 2, 3, respectively, to the first, second, and the target factor of $A\otimes A\to A$. Thus the cobordism near a branch circle as depicted in the figure induces a linear map $A\otimes A\to A$ under the chosen TQFT and the values of theta foams. Denote this map by $\mathbf{m}:\mathbf{A}\otimes\mathbf{A}\to\mathbf{A}$. The goal of this paper is to investigate this map.

In terms of maps among tensor products of As, we use planar graphs regularly used in knot theory, as well as Frobenius algebras as in [3]. In particular, the Frobenius form (the counit) is depicted by a maximum, unit by a minimum, (co)multiplications by trivalent vertices. In this convention, diagrams are read from bottom to top, corresponding to the domain and range of maps. The map $\mathbf{m}$ corresponding to theta foams has a specified cyclic order, as indicated on the right of Figure 2. The map $\mathbf{m}$ is defined with this specific order, and the map with the opposite order, depicted by a diagram with the opposite arrow, represents the map $\mathbf{m}\circ\tau$, where $\tau:A\otimes A\to A\otimes A$ is the map induced from the transposition $\tau(x\otimes y)=y\otimes x$.

3 Lie algebras

In this section we show that there are infinitely many TQFTs under which Lie algebra structures are induced from the branch circle operation. Since our goal is to exhibit a Lie bracket, in this section we use the notation $[\ ,\ ]:A\times A\to A$, instead of $\mathbf{m}:\mathbf{A}\otimes\mathbf{A}\to\mathbf{A}$.

Proposition 3. *For any commutative unital ring R and a positive odd integer $N>1$, there exist a Frobenius algebra A over R and values of the theta foams in $\mathbf{Foam}_A$ such that the branch circle operation $\mathbf{m}$ induces a nontrivial Lie algebra structure on A.*

Proof. Let $A=R[X]/(X^N)$ for an odd integer $N>1$. For simplicity we denote $\theta(X^a,X^b,X^c)$ by $\theta(a,b,c)$ in this proof.

Let $N>3$. Define $\theta(a,b,c)=1$ if $a=0$, $b+c=N$ and $1<b<c$, as well as all cyclic permutations of such (a,b,c). Define $\theta(a,b,c)=-1$ if $a=0$, $b+c=N$ and $1<c<b$, as well as all cyclic permutations of such (a,b,c). Finally define $\theta(a,b,c)=0$ for all the other cases. For $N=3$, replace the conditions $1<b<c$ and $1<c<b$, respectively, by $b<c$ and $c<b$. We show that these theta foam values induce Lie brackets as desired.

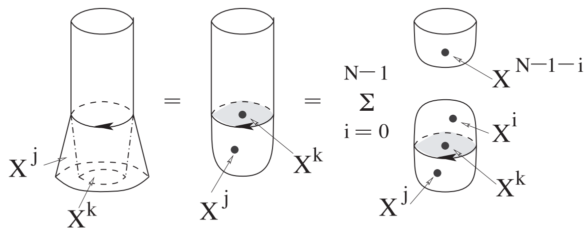

Figure 4: Evaluating bracket.

The operation $[X^j,X^k]$ is evaluated, using the $\Delta(1)$-relation, by

$$[X^j,X^k]=\sum_{i=0}^{N-1}\theta(X^i,X^j,X^k)X^{N-1-i}.$$

This calculation is depicted in Figure 4. Since $\theta(i,j,k) = 0$ unless $i + j + k = N$, we have $[X^j, X^k] = \theta(i,j,k)X^{N-1-i}$ where $i = N-(j+k)$ and $N-1-i = j+k-1$, so that $[X^j, X^k] = \theta(N-(j+k), j, k)X^{j+k-1}$. Note that if $j + k > N$, then the RHS is understood to be zero from the definition of θ. From the definition of θ by cyclic ordering, the skew symmetry of [,] is clear. We show the Jacobi identity

$$[X^j, [X^k, X^\ell]] + [X^\ell, [X^j, X^k]] + [X^k, [X^\ell, X^j]] = 0$$

case by case. First we compute

$$\begin{aligned}
[X^j, [X^k, X^\ell]] &= \theta\big(N-(k+\ell), k, \ell\big)\theta\big(N+1-(j+k+\ell), j, k+\ell-1\big),\\
[X^\ell, [X^j, X^k]] &= \theta\big(N-(j+k), j, k\big)\theta\big(N+1-(j+k+\ell), \ell, j+k-1\big),\\
[X^k, [X^\ell, X^j]] &= \theta\big(N-(\ell+j), \ell, j\big)\theta\big(N+1-(j+k+\ell), k, \ell+j-1\big),
\end{aligned}$$

hence it is sufficient to prove that the sum of the RHSs is zero.

Case 1. $j + k + \ell > N + 1$.

In this case, the second factors of the RHS are zero, so that all terms are zero.

Case 2. $j + k + \ell \le N + 1$ and $k + \ell > N$.

The condition of this case implies that $j = 0$ and $k + \ell = N + 1$. Since $N + 1$ is even, k and ℓ have the same parity. The first factor $\theta(N-(k+\ell), k, \ell)$ is 0 since $N-(k+\ell) = -1$. (When the arguments of θ are out of range, then $\theta = 0$.) Suppose $k = \ell = (N+1)/2$. Then the second and the third terms are

$$\begin{aligned}
&\theta\big(N-(j+k), j, k\big)\theta\big(N+1-(j+k+\ell), \ell, j+k-1\big)\\
&\quad = \theta\big((N-1)/2, 0, (N+1)/2\big)\theta\big(0, (N+1)/2, (N-1)/2\big) = (-1)(-1) = 1,\\
&\theta\big(N-(\ell+j), \ell, j\big)\theta\big(N+1-(j+k+\ell), k, \ell+j-1\big)\\
&\quad = \theta\big((N-1)/2, (N+1)/2, 0\big)\theta\big(0, (N+1)/2, (N-1)/2\big) = (1)(-1) = -1,
\end{aligned}$$

as desired. Hence assume $k < \ell$ without loss of generality. For the second and third terms, we have

$$\begin{aligned}
&\theta\big(N-(j+k)j, k\big)\theta\big(N+1-(j+k+\ell), \ell, j+k-1\big)\\
&\quad = \theta(\ell-1, 0, k)\theta(0, \ell, k-1) = (1)(-1) = -1,\\
&\theta\big(N-(\ell+j), \ell, j\big)\theta\big(N+1-(j+k+\ell), k, \ell+j-1\big)\\
&\quad = \theta(k-1, \ell, 0)\theta(0, k, \ell-1) = (1)(1) = 1,
\end{aligned}$$

as desired.

Case 3. $j + k + \ell \le N + 1$ and $k + \ell \le N$.

First we check the case where two of j, k, ℓ are the same. Suppose $j = k$. Then the second term is zero, as $\theta(N-(j+k), j, k) = 0$. Furthermore, for the first and third terms, we have $\theta(N-(k+\ell), k, \ell) = -\theta(N-(k+\ell), \ell, j)$ and

$$\theta\big(N+1-(j+k+\ell), j, k+\ell-1\big) = \theta\big(N+1-(j+k+\ell), k, \ell+j-1\big),$$

as desired. The other cases ($k = \ell$, $j = \ell$) are checked similarly. Hence we may assume $j < k < \ell$.

Since θ vanishes unless one of the entries is 0, the first factors of the RHS are zero if $N > k + \ell$ and $0 < j < k < \ell$. Hence we may assume that $k + \ell = N$ or $j = 0$.

We continue to examine specific subcases. Suppose that $j = 0$ and $k + \ell < N$. The RHS becomes

$$\begin{aligned}
&\theta\big(N-(k+\ell), k, \ell\big)\theta\big(N+1-(k+\ell), 0, k+\ell-1\big)\\
&\quad + \theta(N-k, 0, k)\theta\big(N+1-(k+\ell), \ell, k-1\big)\\
&\quad + \theta(N-\ell, \ell, 0)\theta\big(N+1-(k+\ell), k, \ell-1\big).
\end{aligned}$$

If $j = 0$ and $k + \ell = N$, we have

$$\theta(0, k, \ell)\theta(1, 0, N-1) + \theta(\ell, 0, k)\theta(1, \ell, k-1) + \theta(k, \ell, 0)\theta(1, k, \ell-1).$$

If $1 < k$, then the sum is 0 since $\theta(1, 0, N-1) = 0$ and the arguments of the second and the third factors are all nonzero. If $k = 1$, then $\theta(0, 1, N-1) = \theta(\ell, 0, 1) = \theta(1, \ell, 0)$, so the sum is 0.

Now suppose that $k+\ell<N$. The first term is 0 and the second factors in the sum have arguments that are all non-zero unless $k=1$. If $k=1$, we have

$$\theta(N-1,0,1)\theta(N-\ell,\ell,0)+\theta(N-\ell,\ell,0)\theta(N-\ell,1,\ell-1)=0.$$

Finally, suppose that $j\neq 0$, so that $k+\ell=N$. The RHS becomes

$$\theta(0,N-\ell,\ell)\theta(1-j,j,N-1)+\theta(0,j,N-\ell)\theta(1-j,\ell,N-1)$$
$$+\,\theta\big(N-(\ell+j),\ell,j\big)\theta(1-j,N-\ell,\ell+j-1).$$

If $1<j$, then the first argument of all the second factors is negative, so the sum is 0. If $j=1$, then each term has a factor that is 0. □

Since the original motivation came from the foams in [8,12], we examine the Frobenius algebra in [12] closely. In this case, the multiplication that is induced by branch circles also satisfies the Jacobi identity.

Proposition 4. *Let $A=\mathbb{Z}[a,b,c][X]/(X^3-aX^2-bX-c)$ with Frobenius structure defined as in Example 2 from [12]. The branch curve operation $[\ ,\ ]$ is skew-symmetric and satisfies the Jacobi identity:*

$$\big[U,[V,W]\big]+\big[V,[W,U]\big]+\big[W,[U,V]\big]=0$$

for any $U,V,W\in A$.

Proof. This is confirmed by calculations. From the axioms of A and the theta foam values that are given in [12]:

$$\theta\big(1,X,X^2\big)=\theta\big(X^2,1,X\big)=\theta\big(X,X^2,1\big)=1=-\theta\big(1,X^2,X\big)=-\theta\big(X,1,X^2\big)=-\theta\big(X^2,X,1\big),$$

while $\theta=0$ for any other arguments, we compute using the $\Delta(1)$ relation for Example 2:

$$[1,X]=-1,\quad \big[1,X^2\big]=X-a,\quad \big[X,X^2\big]=-X^2+aX+b.$$

Then one computes

$$\big[1,\big[X,X^2\big]\big]=-X,\quad \big[X,\big[X^2,1\big]\big]=a,\quad \big[X^2,[1,X]\big]=X-a,$$

as desired. In general, we consider cyclic permutations of X^j, X^k, and X^ℓ in the expression $[X^j,[X^k,X^\ell]]$. Since the bracket is skew-symmetric, we need only to consider the cases in which j, k, and ℓ are distinct. The remaining case follows by skew-symmetry. □

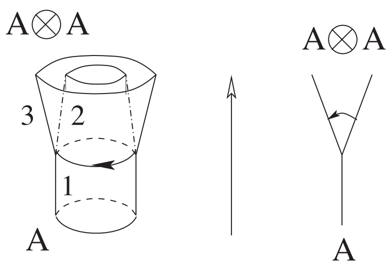

Figure 5: Upside-down operation.

We define the operation $\mathbf{\Delta}:\mathbf{A}\to\mathbf{A}\otimes\mathbf{A}$ that is associated to the left of Figure 5, a diagram that is up-side down of Figure 2, in which one circle branches into two from bottom to top. A cyclic order is specified in the figure. If we specify the ordered tensor factors assigned to each sheet by A_i, $i=1,2,3$, then the operation is defined as $\mathbf{\Delta}:\mathbf{A_1}\to\mathbf{A_3}\otimes\mathbf{A_2}$. A planar diagram representing this operation is depicted in the right of the figure. Imitating Sweedler notation $\Delta(u)=\sum u_{(1)}\otimes u_{(2)}$ for comultiplication, we denote $\mathbf{\Delta(u)}=\sum\mathbf{u}_{((\mathbf{1}))}\otimes\mathbf{u}_{((\mathbf{2}))}$. Lemma 5 relates this operation to the unit map, and diagrammatic formulations are given in Figure 6.

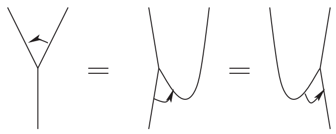

Figure 6: $\mathbf{\Delta}$ can be defined from left or right.

Lemma 5. *Let $A = \mathbb{Z}[a,b,c][X]/(X^3 - aX^2 - bX - c)$, with $\Delta(1)$-condition defined as in Example 2. The map $\mathbf{\Delta}$ is computed as follows:*

$$\begin{aligned}
\mathbf{\Delta}(\mathbf{1}) &= 1 \otimes X - X \otimes 1, \\
\mathbf{\Delta}(\mathbf{X}) &= a(1 \otimes X - X \otimes 1) - \big(1 \otimes X^2 - X^2 \otimes 1\big), \\
\mathbf{\Delta}\big(\mathbf{X^2}\big) &= \big(a^2 + b\big)(1 \otimes X - X \otimes 1) - a\big(1 \otimes X^2 - X^2 \otimes 1\big) + \big(X \otimes X^2 - X^2 \otimes X\big).
\end{aligned}$$

Direct calculations show the following.

Lemma 6. *For the map $\mathbf{\Delta}$ in Lemma 5, the equalities*

$$\mathbf{\Delta}(\mathbf{V}) = \sum \big[\mathbf{V}, \mathbf{1_{(1)}}\big] \otimes \mathbf{1_{(2)}} = \sum \mathbf{1_{(1)}} \otimes \big[\mathbf{1_{(2)}}, \mathbf{V}\big]$$

hold for any $V \in A$.

The diagram for this relation is depicted in Figure 6. Other relations that follow are depicted in Figure 7.

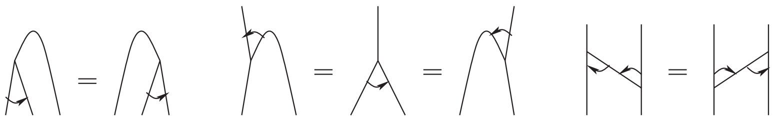

Figure 7: Other symmetric relations.

The following relations hold for maps in Frobenius algebras and maps associated to branch circles. Here we used the notation $\mathbf{m}$ instead of [,] to formulate in tensor products. The equalities are diagrammatically represented in Figure 8.

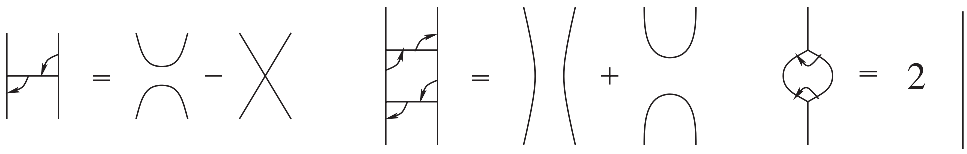

Figure 8: Web skein relations.

Proposition 7. *For $A = \mathbb{Z}[a,b,c]/(X^3 - aX^2 - bX - c)$ with θ values as above, the map $\mathbf{\Delta} : \mathbf{A} \to \mathbf{A} \otimes \mathbf{A}$ satisfies the following identities:*

$$\begin{gathered}
(\mathbf{m} \otimes |)(| \otimes \mathbf{\Delta}) = \mathit{\Delta}(1)(\epsilon\mu) - \tau, \\
\big((\mathbf{m} \otimes |)(| \otimes \mathbf{\Delta})\big)^{\mathbf{2}} = | + \mathit{\Delta}(1)(\epsilon\mu), \\
\mathbf{m}\mathbf{\Delta} = 2|.
\end{gathered}$$

Proof. The first and the third equalities are verified by calculations on basis elements. The second relation is diagrammatically computed as in Figure 9. Note that the value of $\epsilon\mu\mathit{\Delta}(1)$ is 3. □

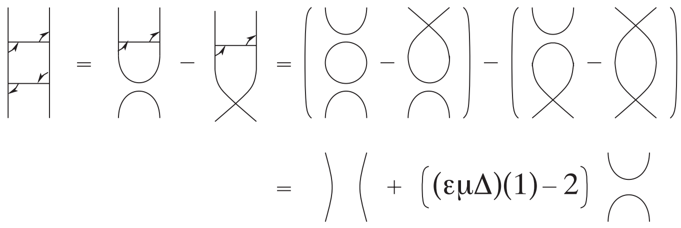

Figure 9: Proof of the skein relation.

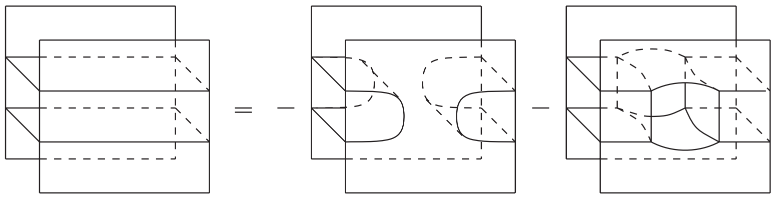

Figure 10: A surface skein relation in [8,12].

Remark 8. The skein relations stated in Proposition 7, as planar diagrams (instead of surface skein relation), coincide with those described in [8] as a description of Kuperberg's invariant [11], with the choice of $q = 1$.

Thus, the operation at branch curve of the foam used to categorify the quantum $\mathrm{sl}(3)$ invariant satisfies the skein relations at the classical limit of the invariant.

Remark 9. The second relation in Proposition 7 is related to the local surface skein relation in [8,12] as follows. The local relation in [8,12] is depicted in Figure 10. Notice the negative signs, as well as resemblance to our relation.

Consider the surface obtained from the planar diagram corresponding to the left-most diagram of Figure 9 by taking the product with the circle S^1. Such a surface represents the map $((\mathbf{m}\otimes|)(|\otimes\mathbf{\Delta}))^{\mathbf{2}}$ under the TQFT. Locally, the surface looks like the left of Figure 10, in which the product of the diagram in Figure 9 with an interval is depicted. The horizontal direction of Figure 10 is the direction of the interval. After performing the holes' relations depicted in Figure 10 locally, move the holes of each term along the S^1 factor to the other side. In Figure 11, this process is depicted for the first term in the RHS of Figure 10. The middle vertical hole between the two sheets in the left figure in the front part is widened along the S^1 factor to the back side as shown in the middle figure. Then one obtains a tube connecting two vertical sheets. Then perform the *bamboo cutting relation*, as depicted in the right of Figure 11, that is computed by applying the $\Delta(1)$-relation three times. The bamboo cutting relation removes a cylinder with two copies of properly immersed parallel disks, and caps off the boundary circles with disks. In this case, one computes that it is the negative of the original bamboo segment. These negative signs cancel, and we obtain our equation. Similarly, the process for the second term is depicted in Figure 12. In this case, a horizontal hole between vertical sheets is made in the front part of the S^1 factor as in the left of the figure, and is then widened along the S^1 factor. We obtain two upper and lower sheets connected by a tube in the back side as shown in the middle of the figure. Then the bamboo cutting relation is performed to obtain the negative of the two sheets as depicted in the right of the figure. Thus, our relation follows from theirs, or algebraically as we have shown.

We also point out that the second and the third relations in Proposition 7 for $A = \mathbb{Z}[a,b,c]/(X^3 - aX^2 - bX - c)$ have interpretations in $\mathbf{Foam}_A$. One simply takes the product of these diagrams with S^1 to obtain foams, and the equalities hold in $\mathbf{Foam}_A$. The first equality, however, is not realized in $\mathbf{Foam}_A$, as the intersection of surfaces is not allowed in $\mathbf{Foam}_A$.

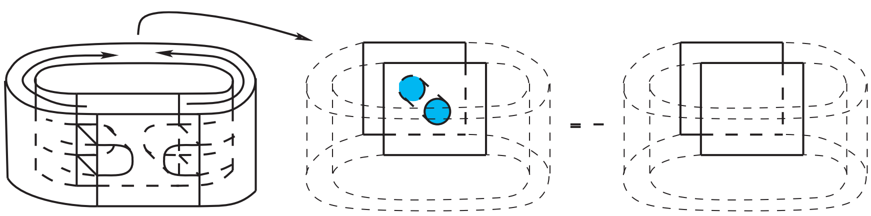

Figure 11: Cutting a bamboo segment for the first term.

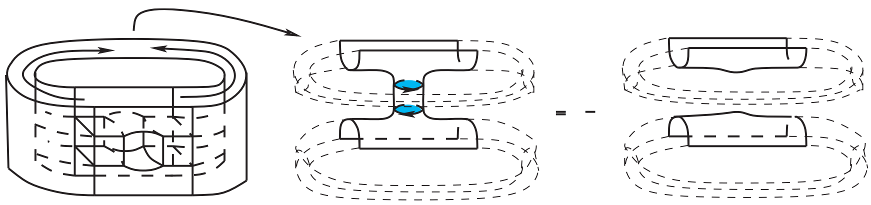

Figure 12: Cutting a bamboo segment for the second term.

4 Bialgebras

In this section we investigate functors whose image of branch curves induces bialgebra structure for group algebras. Let G be a group. Let $A = R[G]$ be the group ring with a commutative unital ring R. It is well known that A has a commutative Hopf algebra structure defined as follows (see, e.g., [10]). Define $\mathbf{\Delta} : \mathbf{A} \to \mathbf{A} \otimes \mathbf{A}$ by linearly extending $\mathbf{\Delta}(\mathbf{x}) = \mathbf{x} \otimes \mathbf{x}$. (This is different from the comultiplication as a Frobenius algebra $\Delta(x) = \sum_{x=yz} y \otimes z$.) The unit map is defined as the same as the Frobenius unit map $\eta(1) = 1_G$, where 1_G is the identity element of G. (The counit map as a Frobenius algebra is defined by $\epsilon(1_G) = 1$ and $\epsilon(x) = 0$ for $x \neq 1_G$.) The following shows that there is a strong requirement for group algebras to give bialgebra structures through branch curves.

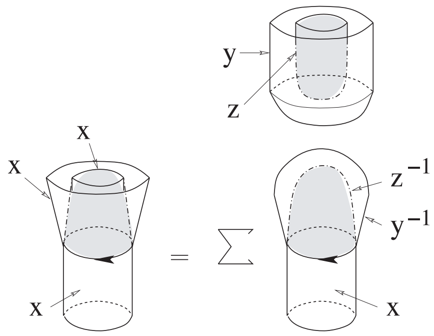

Figure 13: Comultiplication by theta foams.

Proposition 10. *Let G be an abelian group. For any unital ring R, the branch circle operation $\mathbf{m}$ induces a bialgebra structure on A if and only if every nonidentity element of G has order 2.*

Proof. The $\Delta(1)$-relation is written as $\Delta(1) = \sum_{y \in G} y \otimes y^{-1}$, and reducing $\mathbf{\Delta}$ into the theta foam is depicted in Figure 13. For $\mathbf{\Delta}(\mathbf{x}) = \mathbf{x} \otimes \mathbf{x}$ to hold in the figure, we have $y = z = x$, and the value of the theta foam being $\theta(x, y^{-1}, z^{-1}) = 1$ for $y = z = x$ and 0 otherwise.

For θ to satisfy the cyclic symmetry, this condition is satisfied if and only if $x^{-1} = x$ (x having order 2) for any $x \in G$, and in this case, the theta foam values are determined by $\theta(x, x, x) = 1$ for any $x \in G$ and 0 otherwise. $\square$

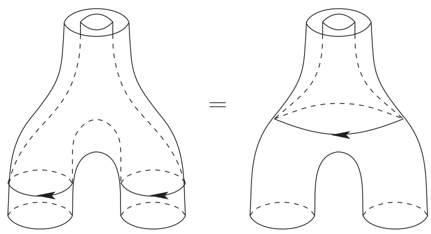

Figure 14: The compatibility condition of a bialgebra.

Remark 11. The condition of a bialgebra that the comultiplication is an algebra homomorphism (also called a compatibility condition) $\mathbf{\Delta}(\mathbf{ab}) = \mathbf{\Delta}(\mathbf{a})\mathbf{\Delta}(\mathbf{b})$ for $a, b \in A$ is represented by surfaces in Figure 14.

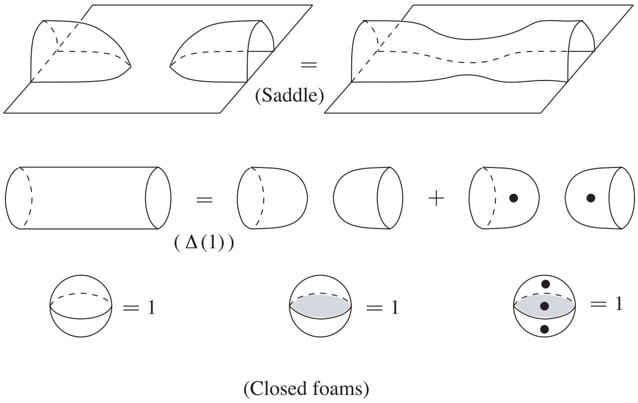

Figure 15: Surface skein relations.

Remark 12. In [1], skein modules for 3-manifolds based on embedded surfaces modulo the surface skein relations described in [2] were defined and studied. Surface skein modules were generalized in [6] using general commutative Frobenius algebras. Such notions are directly generalized to foams, with various skein relations at hand. Skein modules for $\mathrm{sl}(3)$ foams are analogously defined using the local skein relations given in [8, 12], for example.

Here we propose local skein relations based on the foams in Proposition 10 with the bialgebra on branch curves for the group ring $A = \mathbb{Z}[x]/(x^2 - 1)$. Considering that the move characteristic to bialgebras is the compatibility condition as depicted in Figure 14, we take a local change that happens at the saddle point of this move, as depicted in the top of Figure 15 (labeled as saddle) as a local surface skein relation. Other relations in Figure 15 are those coming from Frobenius algebra structure and the theta foam values as before. In particular, a dot represents the element $x \in A$, and the values of closed foams are in $\mathbb{Z}$.

Thus the skein module $\mathbf{F}(M)$ in this case can be defined to be the free module generated by the isotopy classes of foams in a given 3-manifold M modulo the local surface skein relations in Figure 15. Although computations of this skein module in general are out of the scope of this paper, it seems interesting to look into relations between these skein modules foams and the topology of 3-manifolds.

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