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On Classification of Real Hypersurfaces in a Complex Space Form with η-recurrent Shape Operator

Tee-How LOO

University of Malaya

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Abstract. In this paper, we classify real hypersurfaces in a non-flat complex space with η-recurrent shape operator.

1. Introduction

Let $M_n(c)$ be an n-dimensional complete and simply connected non-flat complex space form with complex structure J of constant holomorphic sectional curvature $4c$, i.e., it is either a complex projective space $\mathbf{C}P_n$ (for $c > 0$), or a complex hyperbolic space $\mathbf{C}H_n$ (for $c < 0$).

Suppose M is a connected real hypersurface in $M_n(c)$ and N is a unit normal vector field of M. We denote by $\Gamma(\mathcal{V})$ the module of all differentiable sections on the vector bundle $\mathcal{V}$ over M. Let $\xi = -JN$ be the structure vector field and A the shape operator on M. A *Hopf hypersurface* M in $M_n(c)$ is characterized by the condition that the structure vector field ξ is principal, i.e., $A\xi = \alpha\xi$, and it can be shown that this principal curvature α is a constant.

Typical examples of Hopf hypersurfaces are those with constant principal curvatures, nowadays, so-called real hypersurfaces of type A_1, A_2, B, C, D and E (resp. of type A_0, A_1, A_2 and B) in $\mathbf{C}P_n$ (resp. in $\mathbf{C}H_n$) (cf. [14], [12]). These real hypersurfaces can be expressed as tubes of constant radius over certain holomorphic or totally real submanifolds, and a self-tube in the ambient space (cf. [1], [2], [5]).

Other than these Hopf hypersurfaces, another example of real hypersurfaces in $M_n(c)$ are the class of ruled real hypersurfaces. *Ruled real hypersurfaces* in $M_n(c)$ are characterized by having a one-codimensional foliation whose leaves are complex totally geodesic hyperplanes in $M_n(c)$. The geometry of ruled real hypersurfaces in $M_n(c)$ was studied in [10].

The study of real hypersurfaces in a non-flat complex space form has been an active field in the past few decades. One of the first results is the non-existence of real hypersurfaces with parallel shape operator A, i.e., $\nabla A = 0$, where ∇ is the Levi-Civita connection of M.

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This fact is an immediate consequence of the Codazzi equation of such a real hypersurface. Motivated by this, Kimura and Maeda [6] studied the weaker notion of η-parallelism. The shape operator A is said to be *η-parallel* if it satisfies the following condition:

$$\langle (\nabla_X A)Y, Z\rangle = 0$$

for any X, Y and $Z \in \Gamma(\mathcal{D})$, where $\mathcal{D} := \mathrm{Span}\{\xi\}^{\perp}$, called the (*maximal*) *holomorphic distribution* on M. A number of results concerning real hypersurfaces with η-parallel shape operator have been obtained (cf. [6], [7], [8], [13]). In particular, a complete classification of real hypersurfaces in $M_n(c)$ with η-parallel shape operator was proved in [8] (cf. Theorem 4).

In another way to weaker the parallelism, Hamada [3] studied the recurrence of the shape operator of real hypersurfaces in $\mathbf{C}P_n$. The shape operator A is said to be recurrent if $\nabla A = A \otimes \omega$ for some 1-form ω in M. It was showed in [3] that the recurrence is also too strong to be satisfied by the shape operator of real hypersurfaces in $\mathbf{C}P_n$.

The shape operator A is said to be *η-recurrent* if there is a 1-form ω on M such that

$$\langle (\nabla_X A)Y, Z\rangle = \omega(X)\langle AX, Y\rangle$$

for any $X, Y, Z \in \Gamma(\mathcal{D})$. The η-parallelism and recurrence can be considered as special cases of η-recurrence. Hopf hypersurfaces in $M_n(c)$ with η-recurrent shape operator were classified in [4, 11].

THEOREM 1 ([4, 11]). *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 3$, $c \neq 0$. Then the shape operator A is η-recurrent if and only if M is locally congruent to one of the following spaces*:

(a) *For $c > 0$:*
- (A_1) *a tube over hyperplane $\mathbf{C}P_{n-1}$;*
- (A_2) *a tube over totally geodesic $\mathbf{C}P_k$, where $1 \leq k \leq n-2$;*
- (B) *a tube over complex quadric Q_{n-1}.*

(b) *For $c < 0$:*
- (A_0) *a horosphere*;
- (A_1) *a geodesic hypersphere or a tube over hyperplane $\mathbf{C}H_{n-1}$;*
- (A_2) *a tube over totally geodesic $\mathbf{C}H_k$, where $1 \leq k \leq n-2$;*
- (B) *a tube over totally real hyperbolic space $\mathbf{R}H^n$.*

The purpose of this paper is to improve the above theorem and classify real hypersurfaces in $M_n(c)$ with η-recurrent shape operator, i.e., we prove the following theorem.

THEOREM 2. *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, $c \neq 0$. Then its shape operator A is η-recurrent if and only if M is locally congruent to to a ruled real hypersurface or one of the following spaces*:

(a) *For $c > 0$:*
- (A_1) *a tube over hyperplane $\mathbf{C}P_{n-1}$;*
- (A_2) *a tube over totally geodesic $\mathbf{C}P_k$, where $1 \leq k \leq n-2$;*

(B) *a tube over complex quadric* Q_{n-1}.

(b) *For* $c < 0$:

(A_0) *a horosphere*;

(A_1) *a geodesic hypersphere or a tube over hyperplane* $\mathbf{C}H_{n-1}$;

(A_2) *a tube over totally geodesic* $\mathbf{C}H_k$, *where* $1 \le k \le n-2$;

(B) *a tube over totally real hyperbolic space* $\mathbf{R}H^n$.

2. Preliminaries

In this section we shall recall some fundamental identities and known results in the theory of real hypersurfaces in a complex space form and fix some notations.

Let M be a connected real hypersurface isometrically immersed in $M_n(c)$, $n \geq 3$, N a unit normal vector field on M and $\langle , \rangle$ the Riemannian metric on M. We define a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η by

$$JX = \phi X + \eta(X)N, \quad JN = -\xi\,, \quad \eta(X) = \langle \xi, X\rangle$$

for any $X \in \Gamma(TM)$. Then we have

$$\phi^2 X = -X + \eta(X)\xi\,, \quad \phi\xi = 0\,, \quad \eta(\phi X) = 0\,, \quad \eta(\xi) = 1\,. \tag{1}$$

Denote by ∇ the Levi-Civita connection and A the shape operator on M. Then

$$(\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y\rangle\xi\,, \quad \nabla_X\xi = \phi AX \tag{2}$$

for any $X, Y \in \Gamma(TM)$.

Let R be the curvature tensor of M. Then the equations of Gauss and Codazzi are given respectively by

$$\begin{aligned} R(X,Y)Z = c\{&\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle \phi Y, Z\rangle\phi X - \langle \phi X, Z\rangle \phi Y \\ &-2\langle\phi X, Y\rangle\phi Z\} + \langle AY, Z\rangle AX - \langle AX, Z\rangle AY \end{aligned}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2\langle\phi X, Y\rangle\xi\}\,.$$

This following lemma is needed in the next section.

LEMMA 3 ([9]). *Let* M *be a non-Hopf real hypersurface in* $M_n(c)$, $n \geq 3$, $c \neq 0$. *Suppose* $A\xi = \alpha\xi + \beta U$ *and* $AU = \beta\xi + \gamma U$, *where* $\beta = ||\phi A\xi||$ (> 0) *and* $U = -\beta^{-1}\phi^2 A\xi$. *If there exists a unit vector field* $Z \perp \xi, U, \phi U$ *such that* $AZ = \lambda Z$ *and* $A\phi Z = \lambda\phi Z$, *then*

$$(\lambda - \gamma)(\lambda^2 - \alpha\lambda - c) - \beta^2\lambda = 0\,.$$

Finally, we state without proof the following result concerning real hypersurfaces in $M_n(c)$ with η-parallel shape operator.

THEOREM 4 ([8]). *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, $c \neq 0$. Then its shape operator A is η-parallel if and only if M is locally congruent to a ruled real hypersurface or one of the following spaces*:

(a) *For $c > 0$:*

(A_1) *a tube over hyperplane $\mathbf{C}P_{n-1}$;*

(A_2) *a tube over totally geodesic $\mathbf{C}P_k$, where $1 \leq k \leq n-2$;*

(B) *a tube over complex quadric Q_{n-1}.*

(b) *For $c < 0$:*

(A_0) *a horosphere;*

(A_1) *a geodesic hypersphere or a tube over hyperplane $\mathbf{C}H_{n-1}$;*

(A_2) *a tube over totally geodesic $\mathbf{C}H_k$, where $1 \leq k \leq n-2$;*

(B) *a tube over totally real hyperbolic space $\mathbf{R}H^n$.*

3. Proof of Theorem 2

Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, with η-recurrent shape operator, i.e.,

$$\langle (\nabla_X A)Y, Z\rangle = \omega(X)\langle AY, Z\rangle \tag{3}$$

for any $X, Y, Z \in \Gamma(\mathcal{D})$, where ω is a 1-form on M. By virtue of Theorem 1, we only need to consider the non-Hopf case. In this case, $\beta := ||\phi A\xi|| > 0$ and we may define a unit vector field $U := -\beta^{-1}\phi^2 A\xi$.

It suffices to prove that A is η-parallel or $\omega = 0$ according to Theorem 4. Suppose to the contrary that $\omega \neq 0$. Let W' be the vector field dual to ω and $b := ||\phi W'||$. Then $b > 0$ at some open subset G of M. Since we only study local geometric property, we may identify M with this open subset G and define a unit vector field $W = -b^{-1}\phi^2 W'$. Hence (3) can be rewritten as

$$\langle (\nabla_X A)Y, Z\rangle = b\langle X, W\rangle\langle AY, Z\rangle \tag{4}$$

for any $X, Y, Z \in \Gamma(\mathcal{D})$. It follow from the Codazzi equation, (4) and the fact $b > 0$ that

$$\langle X, W\rangle\langle AY, Z\rangle = \langle Y, W\rangle\langle AX, Z\rangle\,.$$

By putting $X = Z = W$ in the above equation, we obtain $\phi AW = \gamma\phi W$, where $\gamma = \langle AW, W\rangle$. Hence, after putting $X = W$ in the above equation, we have

$$\langle AY, Z\rangle = \gamma\langle Y, W\rangle\langle Z, W\rangle \tag{5}$$

for any $Y, Z \in \Gamma(\mathcal{D})$.

By (4) and (5), we see that $\gamma = 0$ is equivalent to $\omega = 0$. Hence, we get $\gamma \neq 0$. By differentiating covariantly both sides of the above equation in the direction of $X \in \Gamma(\mathcal{D})$; with the help of (1), (2) and 5, we have

$$\langle(\nabla_X A)Y, Z\rangle - \beta\langle Y, \phi AX\rangle\langle U, Z\rangle - \beta\langle Z, \phi AX\rangle\langle Y, U\rangle$$
$$= d\gamma(X)\langle Y, W\rangle\langle Z, W\rangle + \gamma\langle Y, \nabla_X W\rangle\langle Z, W\rangle + \gamma\langle Y, W\rangle\langle Z, \nabla_X W\rangle. \tag{6}$$

By using (4) and (5), the above equation becomes

$$\gamma b\langle X, W\rangle\langle Y, W\rangle\langle Z, W\rangle - \gamma\beta\langle Y, \phi W\rangle\langle X, W\rangle\langle U, Z\rangle - \gamma\beta\langle Z, \phi W\rangle\langle X, W\rangle\langle Y, U\rangle$$
$$= d\gamma(X)\langle Y, W\rangle\langle Z, W\rangle + \gamma\langle Y, \nabla_X W\rangle\langle Z, W\rangle + \gamma\langle Y, W\rangle\langle Z, \nabla_X W\rangle\,. \tag{7}$$

If we let $Y = Z = W$ in the above equation, then $\gamma b\langle X, W\rangle = d\gamma(X)$, for any $X \in \Gamma(\mathcal{D})$. With this fact, (7) reduces to

$$- \beta\langle Y, \phi W\rangle\langle X, W\rangle\langle U, Z\rangle - \beta\langle Z, \phi W\rangle\langle X, W\rangle\langle Y, U\rangle$$
$$= \langle Y, \nabla_X W\rangle\langle Z, W\rangle + \langle Y, W\rangle\langle Z, \nabla_X W\rangle\,. \tag{8}$$

Next, by letting $X = W$, $Y = Z = \phi W$ in (8), we have $\langle\phi W, U\rangle = 0$. Finally, after putting $X = W$ and $Z = \phi W$ in (8), yields $-\beta U = \langle\phi W, \nabla_W W\rangle W$. Since both U and W are unit vector fields, we may, without loss of generality, assume that $U = W$. This, together with (5), yields $AU = \beta\xi + \gamma U$ and $AZ = 0$, for any $Z \perp U, \xi$. According to Lemma 3, we can see that $\gamma = 0$. This contradicts the fact that $\gamma \neq 0$ and so the proof is completed.

The following result has been obtained in [7].

THEOREM 5 ([7]). *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, $c \neq 0$. Then M satisfies*

$$(\nabla_X A)Y = \{-c\langle\phi X, Y\rangle + \eta(AY)\langle X, \phi A\xi\rangle + \eta(AX)\langle Y, \phi A\xi\rangle$$
$$+ \varepsilon\langle(\phi A - A\phi)X, Y\rangle\}\xi$$

for any $X, Y \in \Gamma(\mathcal{D})$, where ε is a constant, if and only if M is locally congruent to one of the spaces stated in Theorem 2.

By Theorem 2 and Theorem 5, we can characterize the η-recurrence of A by an expression of the covariant derivative of A on the holomorphic distribution.

COROLLARY 6. *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, $c \neq 0$. Then the following are equivalent*:

1. *the shape operator A is η-recurrent*;
2. $(\nabla_X A)Y = \{-c\langle\phi X, Y\rangle + \eta(AY)\langle X, \phi A\xi\rangle + \eta(AX)\langle Y, \phi A\xi\rangle + \varepsilon\langle(\phi A - A\phi)X, Y\rangle\}\xi$, *for any $X, Y \in \Gamma(\mathcal{D})$, where ε is a constant*;
3. *M is locally congruent to one of the spaces stated in Theorem* 2.

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Present Address:
INSTITUTE OF MATHEMATICAL SCIENCES,
UNIVERSITY OF MALAYA,
50603 KUALA LUMPUR, MALAYSIA.
e-mail: looth@um.edu.my