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Automorphisms of the Torelli Complex for the One-holed Genus Two Surface

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Abstract. Let S be a connected, compact and orientable surface of genus two having exactly one boundary component. We describe any automorphism of the Torelli complex of S, and describe any automorphism of the Torelli group of S. More generally, we study superinjective maps from the Torelli complex of S into itself, and show that any finite index subgroup of the Torelli group of S is co-Hopfian.

1. Introduction

Let $S = S_{g,p}$ be a connected, compact and orientable surface of genus g with p boundary components. Let $\mathrm{Mod}^*(S)$ be the *extended mapping class group* of S, i.e., the group of isotopy classes of homeomorphisms from S onto itself, where isotopy may move points of the boundary of S. When $p \leq 1$, the *Torelli group* of S, denoted by $\mathcal{I}(S)$, is defined as the subgroup of $\mathrm{Mod}^*(S)$ consisting of all elements acting on the homology group $H_1(S, \mathbf{Z})$ trivially. As a consequence of [3], [4], [5] and [12], if $g \geq 3$ and $p \leq 1$, then any isomorphism between any two finite index subgroups of $\mathcal{I}(S)$ is the conjugation by an element of $\mathrm{Mod}^*(S)$. One purpose of this paper is to show the same conclusion when $g = 2$ and $p = 1$. A key step in the proof of these results is to describe any automorphism of the Torelli complex $\mathcal{T}(S)$ of S, which is a simplicial complex on which $\mathrm{Mod}^*(S)$ naturally acts. The Torelli complex (with a certain marking) of a closed surface was first introduced by Farb-Ivanov [5]. Our computation of automorphisms of $\mathcal{T}(S_{2,1})$ is more delicate than those in the other cases. One difficulty stems from lowness of the dimension of $\mathcal{T}(S_{2,1})$. In fact, $\mathcal{T}(S_{2,1})$ is of dimension 1, and for any surface S dealt with in the cited references, $\mathcal{T}(S)$ is of dimension at least 2. A large part of this paper is dedicated to understanding which kind of simple cycles appears in $\mathcal{T}(S_{2,1})$. This is a problem peculiar to our case. See also Remark 1.4 for difficulty in our case.

Let us introduce terminology and notation to define the Torelli complex. A simple closed curve in S is called *essential* in S if it is neither homotopic to a single point of S nor isotopic to a boundary component of S. Let $V(S)$ denote the set of isotopy classes of essential simple

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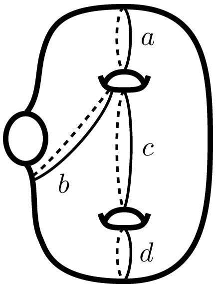


FIGURE 1. The pair $\{a, b\}$ is a BP. Any other pair of the four curves, a, b, c and d, is not a BP.

closed curves in S. For $\alpha, \beta \in V(S)$, we define $i(\alpha, \beta)$ to be the *geometric intersection number* of α and β, i.e., the minimal cardinality of $A \cap B$ among representatives A and B of α and β, respectively. Let $\Sigma(S)$ denote the set of non-empty finite subsets σ of $V(S)$ with $i(\alpha, \beta) = 0$ for any $\alpha, \beta \in \sigma$. We extend i to the symmetric function on the square of $V(S) \sqcup \Sigma(S)$ with $i(\alpha, \sigma) = \sum_{\beta \in \sigma} i(\alpha, \beta)$ and $i(\sigma, \tau) = \sum_{\beta \in \sigma, \gamma \in \tau} i(\beta, \gamma)$ for any $\alpha \in V(S)$ and $\sigma, \tau \in \Sigma(S)$.

An essential simple closed curve a in S is called *separating* in S if $S \setminus a$ is not connected. Otherwise, a is called *non-separating* in S. These properties depend only on the isotopy class of a. Let $V_s(S)$ be the subset of $V(S)$ consisting of all elements whose representatives are separating in S. We mean by a *bounding pair (BP)* in S a pair of essential simple closed curves in S, $\{a, b\}$, such that

- a and b are disjoint and non-isotopic;
- each of a and b is non-separating in S; and
- the surface obtained by cutting S along $a \cup b$ is not connected

(see Figure 1). These conditions depend only on the isotopy classes of a and b. Let $V_{bp}(S)$ be the subset of $\Sigma(S)$ consisting of all elements corresponding to a BP in S.

DEFINITION 1.1. The *Torelli complex* $\mathcal{T}(S)$ of S is defined as the abstract simplicial complex so that the set of vertices of $\mathcal{T}(S)$ is the disjoint union $V_s(S) \sqcup V_{bp}(S)$, and a non-empty finite subset σ of $V_s(S) \sqcup V_{bp}(S)$ is a simplex of $\mathcal{T}(S)$ if and only if we have $i(\alpha, \beta) = 0$ for any $\alpha, \beta \in \sigma$.

For $\alpha \in V(S)$, let $t_\alpha \in \mathrm{Mod}^*(S)$ denote the *(left) Dehn twist* about α. We note that if $p \leq 1$, then the Torelli group $\mathcal{I}(S)$ contains t_α and $t_\beta t_\gamma^{-1}$ for any $\alpha \in V_s(S)$ and any $\{\beta, \gamma\} \in V_{bp}(S)$, and is generated by all elements of these forms as discussed by Johnson [10]. This fact is a motivation for introducing the Torelli complex.

Note that $\mathrm{Mod}^*(S)$ naturally acts on $\mathcal{T}(S)$ by simplicial automorphisms. In this paper, we study not only automorphisms of $\mathcal{T}(S_{2,1})$ but also simplicial maps from $\mathcal{T}(S_{2,1})$ into itself satisfying strong injectivity, called superinjectivity. We mean by a *superinjective map* from

$\mathcal{T}(S)$ into itself a simplicial map $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ satisfying $i(\phi(\alpha), \phi(\beta)) \neq 0$ for any two vertices α, β of $\mathcal{T}(S)$ with $i(\alpha, \beta) \neq 0$. Any superinjective map from $\mathcal{T}(S)$ into itself is shown to be injective (see [12, Section 2.2]). Superinjectivity was first introduced by Irmak [8] for simplicial maps between the complexes of curves to study injective homomorphisms from a finite index subgroup of $\mathrm{Mod}^*(S)$ into $\mathrm{Mod}^*(S)$. Our main result is the following:

THEOREM 1.2. *We set $S = S_{2,1}$. Then the following assertions hold*:

(i) *Any superinjective map from $\mathcal{T}(S)$ into itself is induced by an element of* $\mathrm{Mod}^*(S)$. *Namely, for any superinjective map $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$, there exists an element γ_0 of* $\mathrm{Mod}^*(S)$ *such that $\phi(\alpha) = \gamma_0\alpha$ for any vertex α of $\mathcal{T}(S)$.*

(ii) *Let Γ be a finite index subgroup of $\mathcal{I}(S)$. Let $f\colon \Gamma \to \mathcal{I}(S)$ be an injective homomorphism. Then there exists a unique element γ_0 of* $\mathrm{Mod}^*(S)$ *such that $f(\gamma) = \gamma_0\gamma\gamma_0^{-1}$ for any $\gamma \in \Gamma$. In particular, any automorphism of $\mathcal{I}(S)$ is the conjugation by an element of* $\mathrm{Mod}^*(S)$.

We stress that even the description of any automorphism of $\mathcal{I}(S_{2,1})$ is new. A group Γ is called *co-Hopfian* if any injective homomorphism from Γ into itself is surjective. By assertion (ii), any finite index subgroup of $\mathcal{I}(S_{2,1})$ is co-Hopfian.

The process to derive assertion (ii) from assertion (i) is already discussed in [12, Section 6.3]. We thus omit the proof of assertion (ii).

REMARK 1.3. Farb-Ivanov [5] announced the computation of automorphisms of the Torelli geometry for a closed surface, which is the Torelli complex with a certain marking. As its consequence, they also announce the result that if $S = S_{g,0}$ is a surface with $g \geq 5$, then any isomorphism between finite index subgroups of $\mathcal{I}(S)$ is induced by an element of $\mathrm{Mod}^*(S)$. McCarthy-Vautaw [16] computed automorphisms of $\mathcal{I}(S)$ for $S = S_{g,0}$ with $g \geq 3$. Brendle-Margalit [3], [4] showed that any automorphism of $\mathcal{T}(S)$ and any isomorphism between finite index subgroups of $\mathcal{I}(S)$ are induced by an element of $\mathrm{Mod}^*(S)$ when $S = S_{g,0}$ with $g \geq 3$. The same conclusion for $S = S_{g,p}$ with either $g = 1$ and $p \geq 3$; $g = 2$ and $p \geq 2$; or $g \geq 3$ and $p \geq 0$ as Theorem 1.2 was obtained by the first author [13], based on [12], where the Torelli group $\mathcal{I}(S)$ is defined as the subgroup of $\mathrm{Mod}^*(S)$ generated by all elements of the forms t_α with $\alpha \in V_s(S)$ and $t_\beta t_\gamma^{-1}$ with $\{\beta, \gamma\} \in V_{bp}(S)$.

If $S = S_{2,0}$, then $\mathcal{T}(S)$ is zero-dimensional and consists of countably infinitely many vertices. The group $\mathcal{I}(S)$ is known to be isomorphic to the free group of infinite rank, due to Mess [17] (see [1] for another proof). It thus turns out that automorphisms of $\mathcal{T}(S)$ and $\mathcal{I}(S)$ are not necessarily induced by an element of $\mathrm{Mod}^*(S)$.

If $S = S_{2,1}$, then $\mathcal{T}(S)$ is one-dimensional and connected. The latter is proved by using the technique in [19, Lemma 2.1] to obtain connectivity of a simplicial complex on which $\mathrm{Mod}^*(S)$ acts. It also follows from Lemma 6.3.

REMARK 1.4. In [12], when either $g = 1$ and $p \geq 3$; $g = 2$ and $p \geq 2$; or $g \geq 3$ and $p \geq 0$, the first author observed simplices of $\mathcal{T}(S)$ of maximal dimension and the links of simplices in $\mathcal{T}(S)$ to prove that any superinjective map from $\mathcal{T}(S)$ into itself preserves $V_s(S)$

and $V_{bp}(S)$, respectively. When $g = 2$ and $p = 1$, this assertion does not immediately follow from only observations on simplices and their links because $\mathcal{T}(S)$ is one-dimensional. The link of a vertex is then zero-dimensional and has little information for our purpose. This is a substantial reason why our case is more delicate and difficult than the other cases.

We define $\mathcal{C}_s(S)$ as the full subcomplex of $\mathcal{T}(S)$ spanned by $V_s(S)$, and call it the *complex of separating curves* for S. This complex brings another difference between our case and the other cases. In [3], [4] and [12], automorphisms of $\mathcal{T}(S)$ are described by showing that any automorphism of $\mathcal{C}_s(S)$ is induced by an element of $\mathrm{Mod}^*(S)$. On the other hand, $\mathcal{C}_s(S_{2,1})$ consists of countably infinitely many $\aleph_0$-regular trees, and thus has continuously many automorphisms. This follows from [11, Theorem 7.1] (see also Theorem 3.2). In this respect, our computation of automorphisms of $\mathcal{T}(S_{2,1})$ takes a route completely different from those in the other cases.

The paper is organized as follows. In Section 2, we collect terminology employed throughout the paper. We recall the complex of curves for S, ideal triangulations of punctured surfaces considered by Mosher [18] and basic results on them. Setting $S = S_{2,1}$, through Sections 3–6, we observe hexagons in $\mathcal{T}(S)$, or equivalently, simple cycles in $\mathcal{T}(S)$ of length 6. In Section 7, applying results in those sections, we show that any superinjective map ϕ from $\mathcal{T}(S)$ into itself preserves $V_s(S)$ and $V_{bp}(S)$, respectively, and is surjective. We construct an automorphism Φ of the complex of curves for S inducing ϕ. It is known that Φ is induced by an element of $\mathrm{Mod}^*(S)$, due to Ivanov [9] (see Theorem 2.1). Theorem 1.2 (i) then follows. In Appendix A, we prove that there exists no simple cycle in $\mathcal{T}(S)$ of length at most 5. Hexagons in $\mathcal{T}(S)$ are thus simple cycles in $\mathcal{T}(S)$ of minimal length. This is a notable property of $\mathcal{T}(S)$ although we do not use it to prove Theorem 1.2 (i).

2. Preliminaries

2.1. Terminology. Let S be a connected, compact and orientable surface. Unless otherwise stated, we assume that a surface satisfies these conditions. Let us denote by $\mathrm{Mod}(S)$ the *mapping class group* of S, i.e., the subgroup of $\mathrm{Mod}^*(S)$ consisting of isotopy classes of orientation-preserving homeomorphisms from S onto itself. We define $\mathrm{PMod}(S)$ as the *pure mapping class group* of S, i.e., the subgroup of $\mathrm{Mod}^*(S)$ consisting of isotopy classes of homeomorphisms from S onto itself preserving an orientation of S and preserving each boundary component of S as a set.

We mean by a curve in S either an essential simple closed curve in S or its isotopy class if there is no confusion. A surface homeomorphic to $S_{1,1}$ is called a *handle*. A surface homeomorphic to $S_{0,3}$ is called a *pair of pants*. Let a be a separating curve in S. If a cuts off a handle from S, then a is called an *h-curve* in S. If a cuts off a pair of pants from S, then a is called a *p-curve* in S. We call an element of $V_s(S)$ corresponding to an h-curve and a p-curve in S an *h-vertex* and a *p-vertex*, respectively, and call an element of $V_{bp}(S)$ a *BP-vertex*.

Suppose that ∂S, the boundary of S, is non-empty. Let I be the closed unit interval. We mean by an *essential simple arc* in S the image of an injective continuous map $f\colon I \to S$

such that

- we have $f(\partial I) \subset \partial S$ and $f(I \setminus \partial I) \subset S \setminus \partial S$; and
- there exists no closed disk D embedded in S and whose boundary is the union of $f(I)$ and an arc in ∂S.

The boundary of an essential simple arc l is denoted by ∂l. Let $V_a(S)$ denote the set of isotopy classes of essential simple arcs in S, where isotopy may move the end points of arcs, keeping them staying in ∂S. We often identify an element of $V_a(S)$ with its representative if there is no confusion.

An essential simple arc l in S is called *separating* in S if the surface obtained by cutting S along l is not connected. Otherwise, l is called *non-separating* in S. These properties depend only on the isotopy class of l.

For $\sigma \in \Sigma(S)$, we mean by a *representative* of σ the union of mutually disjoint representatives of elements in σ. Given two elements $\alpha, \beta \in V(S) \sqcup \Sigma(S)$ and their representatives A, B, respectively, we say that A and B *intersect minimally* if we have $|A \cap B| = i(\alpha, \beta)$. For $\alpha, \beta \in V(S) \sqcup \Sigma(S)$, we say that α and β are *disjoint* if $i(\alpha, \beta) = 0$. Otherwise, we say that α and β *intersect*. For an element α of $V(S)$ (or its representative), we denote by S_α the surface obtained by cutting S along α. Similarly, for an element σ of $\Sigma(S)$ (or its representative), we denote by S_σ the surface obtained by cutting S along all curves in σ. Each component of S_σ is often identified with a complementary component in S of a tubular neighborhood of a one-dimensional submanifold representing σ if there is no confusion. For any component Q of S_σ, the set $V(Q)$ is naturally identified with a subset of $V(S)$.

2.2. The complex of curves. In the proof of Theorem 1.2 (i), we use a result on automorphisms of the complex of curves. The *complex of curves* for a surface S, denoted by $\mathcal{C}(S)$, is defined as the abstract simplicial complex so that the sets of vertices and simplices of $\mathcal{C}(S)$ are $V(S)$ and $\Sigma(S)$, respectively.

THEOREM 2.1 ([9, Theorem 1]). *If $S = S_{g,p}$ is a surface with $g \geq 2$ and $p \geq 0$, then any automorphism of $\mathcal{C}(S)$ is induced by an element of* $\mathrm{Mod}^*(S)$.

We refer to [14] and [15] for similar results for other surfaces. Theorem 1.2 (i) is obtained by showing that when $S = S_{2,1}$, for any superinjective map $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$, there exists an automorphism Φ of $\mathcal{C}(S)$ inducing ϕ, that is, satisfying the equalities

$$\Phi(\alpha) = \phi(\alpha) \quad \text{and} \quad \{\Phi(\beta), \Phi(\gamma)\} = \phi(\{\beta, \gamma\})$$

for any $\alpha \in V_s(S)$ and any $\{\beta, \gamma\} \in V_{bp}(S)$.

We note that the complex of separating curves for S, defined in Remark 1.4 and denoted by $\mathcal{C}_s(S)$, is the full subcomplex of $\mathcal{C}(S)$ spanned by $V_s(S)$.

2.3. Ideal triangulations of a punctured surface. We recall basic properties of ideal triangulations of a punctured surface discussed by Mosher [18], which will be used only in the proof of Lemma 7.6. Let S be a closed surface of positive genus g, and let P be a non-empty finite subset of S. The pair (S, P) is then called a *punctured surface*. Let I be the closed unit interval. We mean by an *ideal arc* in (S, P) the image of a continuous map $f : I \to S$ such that

- we have $f(\partial I) \subset P$ and $f(I \setminus \partial I) \subset S \setminus P$;
- f is injective on $I \setminus \partial I$; and
- there exists no closed disk D embedded in S such that $\partial D = f(I)$ and $(D \setminus \partial D) \cap P = \emptyset$.

Two ideal arcs l_1, l_2 in (S, P) are called *isotopic* if we have $l_1 \cap P = l_2 \cap P$; and l_1 and l_2 are isotopic relative to $l_1 \cap P$ as arcs in $(S \setminus P) \cup (l_1 \cap P)$. We mean by an *ideal triangulation* of (S, P) a cell division δ of S such that

(a) the set of 0-cells of δ is P;
(b) any 1-cell of δ is an ideal arc in (S, P); and
(c) any 2-cell of δ is a *triangle*, that is, it is obtained by attaching a Euclidean triangle τ to the 1-skeleton of δ, mapping each vertex of τ to a 0-cell of δ, and each edge of τ to a 1-cell of δ.

The following properties are noticed in [18, p. 14].

LEMMA 2.2. *The following assertions hold*:

(i) *Any cell division of S satisfying conditions* (a) *and* (c) *in the definition of an ideal triangulation necessarily satisfies condition* (b).
(ii) *Let δ be an ideal triangulation of* (S, P). *Then any two distinct* 1*-cells of δ are not isotopic.*

Let R be a surface of genus g with $|P|$ boundary components. Suppose that S is obtained from R by shrinking each component of ∂R into a point, and that P is the set of points into which components of ∂R are shrunken. The natural map from R onto S induces the bijection from $V_a(R)$ onto the set of isotopy classes of ideal arcs in (S, P).

3. Non-existence of some hexagons

Let $\mathcal{G}$ be a simplicial complex. We mean by a *hexagon* in $\mathcal{G}$ the full subcomplex of $\mathcal{G}$ spanned by six vertices $v_1, \ldots, v_6$ such that for any j mod 6, v_j and v_{j+1} are adjacent; v_j and v_{j+2} are not adjacent; and v_j and v_{j+3} are not adjacent. In this case, we say that the hexagon is defined by the 6-tuple $(v_1, \ldots, v_6)$.

Throughout this section, we set $S = S_{2,1}$. Examples of hexagons in $\mathcal{T}(S)$ are described in Sections 4–6. In this section, we show that there exists no hexagon in $\mathcal{T}(S)$ containing at most one BP-vertex. Note that any separating curve in S is an h-curve in S, and that any edge of $\mathcal{T}(S)$ consists of either two h-vertices or an h-vertex and a BP-vertex (see Figure 2). It follows that the number of BP-vertices of a hexagon in $\mathcal{T}(S)$ is at most 3.

LEMMA 3.1. *There exists no hexagon in $\mathcal{T}(S)$ consisting of only h-vertices.*

To prove this lemma, we use the following:

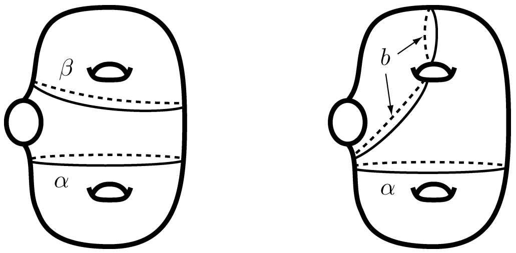


FIGURE 2. Each of $\{\alpha, \beta\}$ and $\{\alpha, b\}$ is an edge of $\mathcal{T}(S)$.

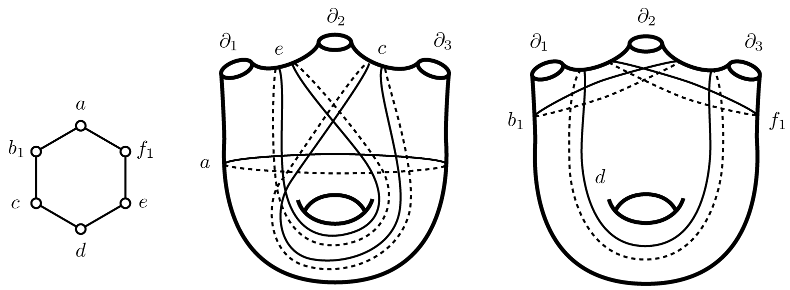


FIGURE 3. The 6-tuple (a, b_1, c, d, e, f_1) of the above curves defines a hexagon in $\mathcal{C}_s(S_{1,3})$. Let $S = S_{2,1}$ be a surface, and let α be a non-separating curve in S. If S_α is drawn as above so that ∂_1 and ∂_3 correspond to α and ∂_2 corresponds to ∂S, then the 6-tuple (a, b, c, d, e, f) with $b = \{\alpha, b_1\}$ and $f = \{\alpha, f_1\}$ defines a hexagon in $\mathcal{T}(S)$ of type 1.

THEOREM 3.2 ([11, Theorem 7.1]). *Let $S = S_{2,1}$ be a surface, and let $\bar{S}$ denote the closed surface obtained from S by attaching a disk to the boundary of S. We define*

$$\pi : \mathcal{C}(S) \to \mathcal{C}(\bar{S})$$

as the simplicial map associated to the inclusion of S into $\bar{S}$. Then for any vertex α of $\mathcal{C}(\bar{S})$, the full subcomplex of $\mathcal{C}(S)$ spanned by $\pi^{-1}(\alpha)$ is a tree.

PROOF OF LEMMA 3.1. We note that π sends two adjacent h-vertices of $\mathcal{C}(S)$ to the same vertex. If there were a hexagon Π in $\mathcal{T}(S)$ consisting of only h-vertices, then π would send Π to a single vertex. This contradicts Theorem 3.2. □

LEMMA 3.3. *There exists no hexagon in $\mathcal{T}(S)$ containing exactly one BP-vertex.*

PROOF. Suppose that there exists such a hexagon Π in $\mathcal{T}(S)$. Let (a, b, c, d, e, f) be a 6-tuple defining Π with a a BP-vertex. We then have the equality $\pi(b) = \pi(c) = \pi(d) = \pi(e) = \pi(f)$. The curves b and f are in the component of S_a that does not contain ∂S. The equality $\pi(b) = \pi(f)$ thus implies the equality $b = f$. This is a contradiction. □

4. Hexagons of type 1

Throughout this section, we set $S = S_{2,1}$. We say that a hexagon in $\mathcal{T}(S)$ is of *type 1* if it is defined by a 6-tuple $(v_1, \ldots, v_6)$ such that v_1, v_3, v_4 and v_5 are h-vertices and v_2 and v_6 are BP-vertices. To construct such a hexagon in $\mathcal{T}(S)$, we recall a hexagon in $\mathcal{C}_s(S_{1,3})$ (see Figure 3). A fundamental property of hexagons in $\mathcal{C}_s(S_{1,3})$ is the following:

THEOREM 4.1 ([12, Theorem 5.2]). *We set $X = S_{1,3}$. Then any two hexagons in $\mathcal{C}_s(X)$ are sent to each other by an element of* $\mathrm{PMod}(X)$.

We now present a hexagon in $\mathcal{T}(S)$ of type 1. Let α be a non-separating curve in S. Note that S_α is homeomorphic to $S_{1,3}$. We define a simplicial map

$$\lambda_\alpha \colon \mathcal{C}_s(S_\alpha) \to \mathcal{T}(S)$$

as follows. Pick $\beta \in V_s(S_\alpha)$. If the two boundary components of S_α corresponding to α are contained in distinct components of $S_{\{\alpha,\beta\}}$, then we have $\{\alpha, \beta\} \in V_{bp}(S)$ and set $\lambda_\alpha(\beta) = \{\alpha, \beta\}$. Otherwise, we have $\beta \in V_s(S)$ and set $\lambda_\alpha(\beta) = \beta$. The map λ_α is superinjective, that is, for any $\gamma, \delta \in V_s(S_\alpha)$, we have $i(\lambda_\alpha(\gamma), \lambda_\alpha(\delta)) = 0$ if and only if $i(\gamma, \delta) = 0$. Sending a hexagon in $\mathcal{C}_s(S_\alpha)$ through λ_α, we obtain a hexagon in $\mathcal{T}(S)$ of type 1 as precisely described in Figure 3.

The following theorem says that any hexagon in $\mathcal{T}(S)$ of type 1 can be obtained through the above procedure.

THEOREM 4.2. *The following assertions hold*:

(i) *For any hexagon Π in $\mathcal{T}(S)$ of type* 1, *there exist a non-separating curve α in S and a hexagon Π_0 in $\mathcal{C}_s(S_\alpha)$ with $\lambda_\alpha(\Pi_0) = \Pi$.*

(ii) *Any two hexagons in $\mathcal{T}(S)$ of type* 1 *are sent to each other by an element of* $\mathrm{Mod}(S)$.

PROOF. Assertion (ii) follows from assertion (i) and Theorem 4.1. To prove assertion (i), we pick a hexagon Π in $\mathcal{T}(S)$ of type 1. Let (a, b, c, d, e, f) be a 6-tuple defining Π with b and f BP-vertices. We choose representatives $A, \ldots, F$ of $a, \ldots, f$, respectively, such that any two of them intersect minimally.

Let R denote the component of S_C that is not a handle. Since B is a BP in R and is disjoint from A, the intersection $A \cap R$ consists of mutually isotopic, essential simple arcs in R which are non-separating in R (see Figure 4 (a)). Since D is an h-curve in R and is disjoint from E, the intersection $E \cap R$ consists of mutually isotopic, essential simple arcs in R which are separating in R (see Figure 4 (b)). Let l_1 be a component of $A \cap R$, and let l_2 be a component of $E \cap R$. If l_1 and l_2 could not be disjoint after isotopy which may move the end points of arcs, keeping them staying in ∂R, then the union of a subarc of l_1 and a subarc of l_2 would be a simple closed curve isotopic to ∂S. This is a contradiction because no simple closed curve in the component of S_F that is not a pair of pants is isotopic to ∂S as a curve in S. It thus turns out that l_1 and l_2 can be disjoint after isotopy. Note that B consists of two boundary components of a regular neighborhood of $l_1 \cup C$ in R, and that D is a boundary

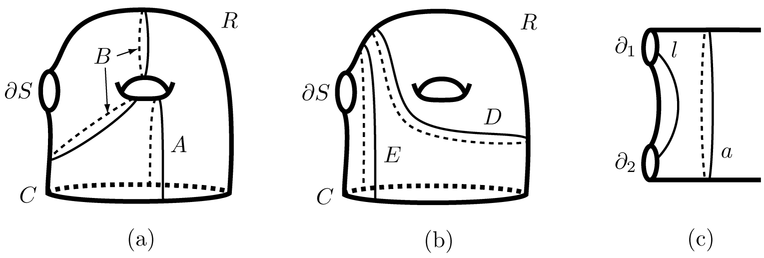


FIGURE 4

component of a regular neighborhood of $l_2 \cup C$ in R. It follows that exactly one component of B is contained in the handle Q cut off by D from S. We denote by α the isotopy class of that component of B.

Similarly, considering the component of S_E that is not a handle, instead of that of S_C, we can show that exactly one component of F is contained in Q. Let β denote the isotopy class of that component of F. Since B and F are disjoint from $A \cap Q$, that consists of essential simple arcs in the handle Q, we have $\alpha = \beta$. We define two curves b_1, f_1 so that $b = \{\alpha, b_1\}$ and $f = \{\alpha, f_1\}$. Any of a, c and e is disjoint from α because any of them is disjoint from b or f. The h-curve d is also disjoint from α because α is the isotopy class of a curve in Q. The map λ_α sends the hexagon in $\mathcal{C}_s(S_\alpha)$ defined by the 6-tuple (a, b_1, c, d, e, f_1) to Π. Assertion (i) is proved. $\square$

Let $\mathcal{G}$ be a simplicial graph and n a positive integer. We mean by an *n-path* in $\mathcal{G}$ a subgraph of $\mathcal{G}$ obtained as the image of a simple path in $\mathcal{G}$ of length n starting and terminating at vertices of $\mathcal{G}$. In the rest of this section, we observe two hexagons in $\mathcal{T}(S)$ of type 1 sharing a 3-path.

LEMMA 4.3. *If Π and Ω are hexagons in $\mathcal{T}(S)$ of type* 1 *such that $\Pi \cap \Omega$ contains a* 3*-path, then we have $\Pi = \Omega$.*

To prove this lemma, we make the following observation on hexagons in $\mathcal{C}_s(S_{1,3})$.

LEMMA 4.4. *We set $X = S_{1,3}$. Let H be a hexagon in $\mathcal{C}_s(X)$. Then for any* 3*-path L in H, H is the only hexagon in $\mathcal{C}_s(X)$ containing L.*

Before proving this lemma, we introduce terminology. Let $Y = S_{g,p}$ be a surface with $p \geq 2$. For an essential simple arc l in Y and two distinct components ∂_1, ∂_2 of ∂Y, we say that l *connects* ∂_1 *and* ∂_2 (or *connects* ∂_1 *with* ∂_2) if one of the end points of l lies in ∂_1 and another in ∂_2.

Suppose either $g \geq 1$ and $p \geq 2$ or $g = 0$ and $p \geq 5$. There is a one-to-one correspondence between elements of $V_s(Y)$ whose representatives are p-curves in Y and elements of $V_a(Y)$ whose representatives connect two distinct components of ∂Y. More specifically, for any p-curve a in Y, we have an essential simple arc in Y contained in the pair of pants cut off by a from Y and connecting two distinct components of ∂Y, which uniquely exists up to isotopy. Conversely, for any essential simple arc l in Y connecting two distinct components ∂_1, ∂_2 of ∂Y, we have the p-curve in Y that is a boundary component of a regular neighborhood of $l \cup \partial_1 \cup \partial_2$ in Y (see Figure 4 (c)).

PROOF OF LEMMA 4.4. Let (a, b, c, d, e, f) be a 6-tuple defining H such that a, c and e are h-curves in X and b, d and f are p-curves in X. To prove the lemma, it is enough to show that H is the only hexagon in $\mathcal{C}_s(X)$ containing a, b, c and d.

Choose representatives $A, \ldots, F$ of $a, \ldots, f$, respectively, such that any two of them intersect minimally. We can then find essential simple arcs l_B, l_D and l_F in X such that

- for any $G \in \{B, D, F\}$, the arc l_G lies in the pair of pants cut off by G from X, and connects two distinct components of ∂X;
- the arcs l_B, l_D and l_F are pairwise disjoint; and
- any of $A \cap l_D$, $C \cap l_F$ and $E \cap l_B$ consists of exactly two points

(see Figure 5 (a)). Label components of ∂X as ∂_1, ∂_2 and ∂_3 so that l_B connects ∂_1 and ∂_2 and l_D connects ∂_1 and ∂_3. Let R denote the component of X_A homeomorphic to $S_{0,4}$, and let ∂_4 denote the component of ∂R corresponding to A (see Figure 5 (b)). The intersection $l_D \cap R$ then consists of an arc connecting ∂_1 with ∂_4 and an arc connecting ∂_3 with ∂_4. If we cut R along l_B and $l_D \cap R$, then we obtain a disk K such that each of ∂_2 and ∂_3 corresponds to a single arc in ∂K. It follows that up to isotopy, there exists at most one simple arc in X connecting ∂_2 with ∂_3, meeting ∂X only at its end points, and disjoint from A, l_B and l_D.

We proved that any hexagon in $\mathcal{C}_s(X)$ containing a, b, c and d contains f. The lemma follows because e is the only separating curve in X disjoint from d and f. $\square$

PROOF OF LEMMA 4.3. Let Π and Ω be hexagons in $\mathcal{T}(S)$ of type 1 such that $\Pi \cap \Omega$ contains a 3-path. Let (a, b, c, d, e, f) be a 6-tuple defining Π with b and f BP-vertices. We define α as the curve contained in b and f. The number of BP-vertices in $\Pi \cap \Omega$ is either 1

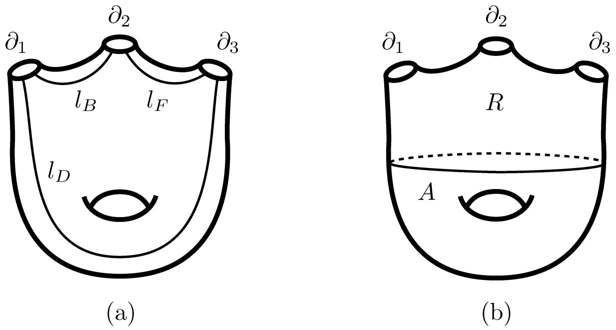


FIGURE 5

or 2. If $\Pi \cap \Omega$ has two BP-vertices, then both Π and Ω are hexagons in $\lambda_\alpha(\mathcal{C}_s(S_\alpha))$, where $\lambda_\alpha \colon \mathcal{C}_s(S_\alpha) \to \mathcal{T}(S)$ is the simplicial map defined right after Theorem 4.1. The equality $\Pi = \Omega$ holds by Lemma 4.4.

Assuming that $\Pi \cap \Omega$ contains only one BP-vertex, we deduce a contradiction. Without loss of generality, we may assume that b is contained in $\Pi \cap \Omega$. It then follows that c and d are also contained in $\Pi \cap \Omega$. Since α is determined as the curve in b disjoint from d, the two BPs in Ω share α. Both Π and Ω are hexagons in $\lambda_\alpha(\mathcal{C}_s(S_\alpha))$, and the equality $\Pi = \Omega$ holds by Lemma 4.4. This contradicts our assumption. □

5. Hexagons of type 2

Throughout this section, we set $S = S_{2,1}$. We say that a hexagon in $\mathcal{T}(S)$ is of *type* 2 if it is defined by a 6-tuple $(v_1, \ldots, v_6)$ such that v_2, v_3, v_5 and v_6 are h-vertices and v_1 and v_4 are BP-vertices. We construct a hexagon of type 2 by gluing two pentagons in the Torelli complex of $S_{1,3}$.

Let $\mathcal{G}$ be a simplicial complex. We mean by a *pentagon* in $\mathcal{G}$ the full subcomplex of $\mathcal{G}$ spanned by five vertices $v_1, \ldots, v_5$ such that for any j mod 5, v_j and v_{j+1} are adjacent, and v_j and v_{j+2} are not adjacent. In this case, we say that the pentagon is defined by the 5-tuple $(v_1, \ldots, v_5)$.

Fix a non-separating curve δ in S, and let X be the surface obtained by cutting S along δ, which is homeomorphic to $S_{1,3}$. Let ∂_δ^1 and ∂_δ^2 denote the two boundary components of X corresponding to δ. We have the pentagon in $\mathcal{T}(X)$ defined by the 5-tuple (a, b, c, d, z) in Figure 6 (a), where we put $a = \{a_0, a_1\}$ and $z = \{a_0, \zeta\}$.

Fix a non-zero integer m, and put $b' = t_\zeta^m(b)$ and $c' = t_\zeta^m(c)$. We then have the hexagon Π in $\mathcal{T}(S)$ defined by the 6-tuple $(a, b, c, \{\delta, d\}, c', b')$, where a and $\{\delta, d\}$ are BPs in S, and b, c, c' and b' are h-curves in S (see Figure 6 (b)). Note that z is not a vertex of $\mathcal{T}(S)$. Let n be a non-zero integer distinct from m, and put $b'' = t_\zeta^n(b)$ and $c'' = t_\zeta^n(c)$. The hexagon

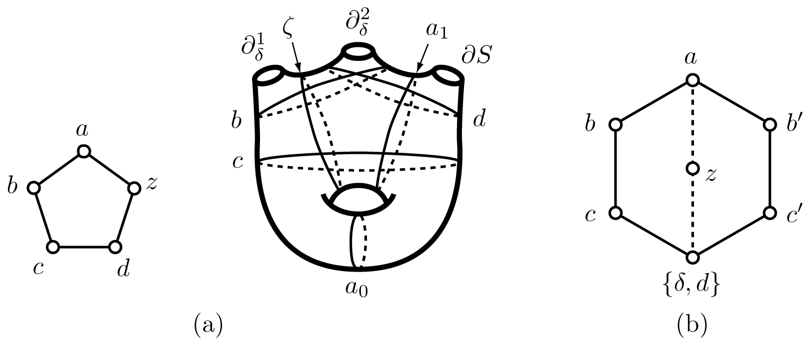


FIGURE 6

in $\mathcal{T}(S)$ defined by the 6-tuple $(a, b, c, \{\delta, d\}, c'', b'')$ is distinct from Π and shares a 3-path with Π. This property is in contrast with Lemma 4.3 on hexagons of type 1.

The aim of this section is to show that any hexagon in $\mathcal{T}(S)$ of type 2 can be obtained through this construction, and to describe the number of hexagons sharing a 3-path with a given hexagon of type 1 or type 2. Uniqueness of the pentagon in $\mathcal{T}(S_{1,3})$ in Figure 6 (a) is proved in the following:

LEMMA 5.1. *We set $X = S_{1,3}$. Then the following assertions hold*:

(i) *Any pentagon in $\mathcal{T}(X)$ having exactly two BP-vertices is defined by a 5-tuple $(v_1, \ldots, v_5)$ with v_1 and v_5 BP-vertices, v_2 and v_4 p-vertices, and v_3 an h-vertex.*

(ii) *Any two pentagons in $\mathcal{T}(X)$ having exactly two BP-vertices are sent to each other by an element of* $\mathrm{Mod}(X)$.

To prove this lemma, we need uniqueness of pentagons in the one-dimensional complex $\mathcal{C}(S_{0,5})$.

LEMMA 5.2. *We set $T = S_{0,5}$. Then for any two 5-tuples $(u_1, \ldots, u_5)$, $(v_1, \ldots, v_5)$ defining pentagons in $\mathcal{C}(T)$, there exists an element g of* $\mathrm{Mod}(T)$ *with $g(u_j) = v_j$ for any $j = 1, \ldots, 5$.*

PROOF. As noticed right before the proof of Lemma 4.4, there is a one-to-one correspondence between isotopy classes of curves in T and isotopy classes of essential simple arcs in T connecting two distinct components of ∂T. Let $(u_1, \ldots, u_5)$ be a 5-tuple defining a pentagon in $\mathcal{C}(T)$. For each $j = 1, \ldots, 5$, let l_j be an essential simple arc in T corresponding to u_j.

We claim that for any j mod 5, l_j and l_{j+2} can be isotoped so that they are disjoint, and exactly one component of ∂T, denoted by ∂_j, contains a point of ∂l_j and a point of ∂l_{j+2} (see Figure 7 (a)). Although this follows from [14, Theorem 3.2] or [15, Lemma 4.2], we give a proof for the reader's convenience. Fix $j = 1, \ldots, 5$. The indices are regarded as numbers modulo 5. Let Q be the component of $T_{u_{j+1}}$ homeomorphic to $S_{0,4}$. The curves u_j and u_{j+2} lie in Q. Since u_{j-1} is disjoint from u_j, the intersection $u_{j-1} \cap Q$ consists of

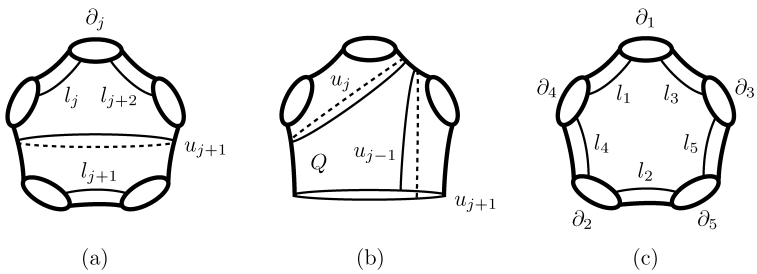


FIGURE 7

mutually isotopic, essential simple arcs in Q (see Figure 7 (b)). Since u_{j+3} is disjoint from u_{j+2}, the intersection $u_{j+3} \cap Q$ also consists of mutually isotopic, essential simple arcs in Q. Any component of $u_{j-1} \cap Q$ and any component of $u_{j+3} \cap Q$ are not isotopic because otherwise we would have $u_j = u_{j+2}$. The curves u_{j-1} and u_{j+3} are disjoint, and $u_{j-1} \cap Q$ and $u_{j+3} \cap Q$ are therefore disjoint. It follows that l_j and l_{j+2} can be isotoped so that they are disjoint, as drawn in Figure 7 (a). The claim follows.

We may therefore assume that $l_1, \ldots, l_5$ are mutually disjoint. We next claim that $\partial_1, \ldots, \partial_5$ are mutually distinct. For any j mod 5, ∂_j and ∂_{j+1} are distinct because they are contained in the pairs of pants cut off by the curves u_j and u_{j+1}, respectively, that are disjoint and distinct. For any j mod 5, ∂_j and ∂_{j+2} are distinct because they are contained in the pairs of pants cut off by the curves u_j and u_{j+4}, respectively, that are disjoint and distinct. The claim follows.

Let $(v_1, \ldots, v_5)$ be a 5-tuple defining a pentagon in $\mathcal{C}(T)$. For each $j = 1, \ldots, 5$, we choose an essential simple arc r_j in T corresponding to v_j so that $r_1, \ldots, r_5$ are mutually disjoint. Applying an element of $\mathrm{Mod}(T)$ to $(v_1, \ldots, v_5)$, we may assume that for any j mod 5, ∂_j contains a point of ∂r_j and a point of ∂r_{j+2}. Cutting T along $\bigcup_{j=1}^{5} l_j$, we obtain two disks. The boundary of each of those disks consists of arcs contained in

$$\partial_1\,,\ \ l_1\,,\ \ \partial_4\,,\ \ l_4\,,\ \ \partial_2\,,\ \ l_2\,,\ \ \partial_5\,,\ \ l_5\,,\ \ \partial_3\,,\ \ l_3\,,$$

along the boundary (see Figure 7 (c)). The same property holds for the arcs $r_1, \ldots, r_5$. We can thus find a homeomorphism of T onto itself sending ∂_j to itself and sending l_j to r_j for any $j = 1, \ldots, 5$. The lemma is proved. □

PROOF OF LEMMA 5.1. To prove assertion (i), we use the following properties:

(1) The link of any BP-vertex in $\mathcal{T}(X)$ consists of BP-vertices and p-vertices.

(2) The link of any p-vertex in $\mathcal{T}(X)$ consists of BP-vertices and h-vertices.

Let P be a pentagon in $\mathcal{T}(X)$ with exactly two BP-vertices. If the two BP-vertices of P were not adjacent, then property (1) would imply that the other three vertices of P are p-vertices. We then have two adjacent p-vertices of P, and this contradicts property (2). It follows that the two BP-vertices of P are adjacent. Properties (1) and (2) imply assertion (i).

To prove assertion (ii), we pick two pentagons P, P' in $\mathcal{T}(X)$ having exactly two BP-vertices. Let (a, b, c, d, e) be a 5-tuple defining P with a and e BP-vertices. Let (a', b', c', d', e') be a 5-tuple defining P' with a' and e' BP-vertices. Any two distinct and disjoint BPs in X have a common curve in X. Let α be the curve in $a \cap e$. We define curves a_1 and e_1 in X so that $a = \{\alpha, a_1\}$ and $e = \{\alpha, e_1\}$. We may assume that α is also the curve in $a' \cap e'$ after applying an element of $\mathrm{PMod}(X)$ to P'. We define curves a'_1 and e'_1 in X so that $a' = \{\alpha, a'_1\}$ and $e' = \{\alpha, e'_1\}$. Let R denote the component of S_c homeomorphic to $S_{0,4}$. The two p-curves b and d *fill* R, i.e., there is no essential simple closed curve in R disjoint from both b and d. Since α is disjoint from b and d, the curve α is disjoint from c. Similarly, α is disjoint from b', c' and d'.

Let X_α be the surface obtained by cutting X along α, which is homeomorphic to $S_{0,5}$.

Each of the 5-tuples (a_1, b, c, d, e_1) and (a_1', b', c', d', e_1') defines a pentagon in $\mathcal{C}(X_\alpha)$. By Lemma 5.2, we obtain an element g of $\mathrm{Mod}(X_\alpha)$ sending (a_1, b, c, d, e_1) to (a_1', b', c', d', e_1'). The two boundary components of X_α corresponding to α lie in the pair of pants cut off by c from X_α and in that cut off by c' from X_α. The equality $g(c) = c'$ implies that g preserves those two boundary components of X_α. Assertion (ii) follows. $\square$

We now present several properties of hexagons in $\mathcal{T}(S)$ of type 2. Let $\bar{S}$ denote the closed surface obtained by attaching a disk to ∂S. Let $\pi : \mathcal{C}(S) \to \mathcal{C}(\bar{S})$ be the simplicial map associated with the inclusion of S into $\bar{S}$. The map π sends any BP in S to a non-separating curve in $\bar{S}$.

LEMMA 5.3. *Let (a, b, c, d, e, f) be a 6-tuple defining a hexagon in $\mathcal{T}(S)$ of type 2 with a and d BP-vertices. Then the equalities $\pi(b) = \pi(c)$ and $\pi(e) = \pi(f)$ hold, and $\pi(a)$ and $\pi(d)$ are disjoint and distinct.*

PROOF. The first two equalities hold because any of b, c, e and f are h-vertices, b and c are adjacent, and e and f are adjacent. Let A, B, C and D be representatives of a, b, c and d, respectively, such that any two of them intersect minimally. We identify a curve in S with a curve in $\bar{S}$ through the inclusion of S into $\bar{S}$. Let H denote the handle cut off by C from $\bar{S}$ and containing ∂S. Let I denote another handle cut off by C from $\bar{S}$. The BP A lies in the handle cut off by B from $\bar{S}$ and containing ∂S. This handle contains I, and the two curves B and C are isotopic in $\bar{S}$. It follows that in $\bar{S}$, the two curves in A can be isotoped into curves in I. On the other hand, the BP D lies in H. It turns out that $\pi(a)$ and $\pi(d)$ lie in distinct components of $\bar{S}_{\pi(c)}$. In particular, $\pi(a)$ and $\pi(d)$ are disjoint and distinct. $\square$

LEMMA 5.4. *Let (a, b, c, d, e, f) be a 6-tuple defining a hexagon in $\mathcal{T}(S)$ of type 2 with a and d BP-vertices. Then there exist a curve a_0 in a and a curve d_0 in d such that each of a_0 and d_0 is disjoint from any of $a, \dots, f$, and the surface obtained by cutting S along $a_0 \cup d_0$ is homeomorphic to $S_{0,5}$.*

PROOF. Choose representatives $A, \dots, F$ of $a, \dots, f$, respectively, such that any two of them intersect minimally. Let R denote the component of S_B homeomorphic to $S_{1,2}$. Since A is a BP in R and is disjoint from F, the intersection $F \cap R$ consists of mutually isotopic, essential simple arcs in R which are non-separating in R. Let l_F be a component of $F \cap R$. Since C is an h-curve in R and is disjoint from D, the intersection $D \cap R$ consists of mutually isotopic, essential simple arcs in R which are separating in R. Let l_D be a component of $D \cap R$. We find a desired curve a_0 in the following two cases individually: (1) There exists a component of $E \cap R$ which is separating in R. (2) There exists no component of $E \cap R$ that is separating in R.

In case (1), since $E \cap R$ is disjoint from $D \cap R$, any component of $E \cap R$ that is separating in R is isotopic to l_D. Let l_E^0 be a component of $E \cap R$ which is separating in R. Since $E \cap R$ is disjoint from $F \cap R$, the arc l_E^0 is disjoint from l_F. As drawn in Figure 8 (a), we can find

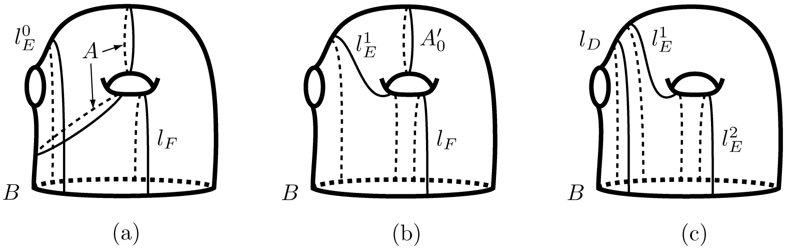


FIGURE 8

the unique component of A disjoint from l_E^0. Let a_0 be the isotopy class of that component of A. The curve a_0 is disjoint from any of b, d and f. Since C is a boundary component of a regular neighborhood of $B \cup D$, the curve a_0 is disjoint from c. Similarly, since E is a boundary component of a regular neighborhood of $D \cup F$, the curve a_0 is disjoint from e.

In case (2), $E \cap R$ consists of essential simple arcs in R which are non-separating in R. Let $\bar{S}$ be the closed surface obtained from S by attaching a disk to ∂S. We identify a curve in S with a curve in $\bar{S}$ through the inclusion of S into $\bar{S}$. Let $\bar{R}$ denote the component of $\bar{S}_B$ containing ∂S. Since any component of $E \cap R$ is non-separating in R, the two curves B and E intersect minimally even as curves in $\bar{S}$, by the criterion on minimal intersection in [6, Exposé 3, Proposition 10]. Similarly, B and F also intersect minimally even as curves in $\bar{S}$. The two curves E and F are isotopic in $\bar{S}$ because they are disjoint h-curves in S. By [6, Exposé 3, Proposition 12], there exists a homeomorphism of $\bar{S}$ onto itself isotopic to the identity, fixing B as a set and sending $E \cap R$ to $F \cap R$. Any component of $E \cap R$ is thus isotopic to l_F in $\bar{R}$.

If any component of $E \cap R$ were isotopic to l_F in R, then e and a would be disjoint. This is a contradiction. There thus exists a component of $E \cap R$ which is not isotopic to l_F in R. Let l_E^1 be such a component of $E \cap R$.

Assuming that there exists no component of $E \cap R$ isotopic to l_F in R, we deduce a contradiction. Any component of $E \cap R$ is then isotopic to l_E^1. Note that if r_1 and r_2 are non-separating arcs in R which are disjoint and non-isotopic in R, but are isotopic in $\bar{R}$, then there exists a homeomorphism of R onto itself sending r_1 and r_2 to l_E^1 and l_F, respectively. It follows that as drawn in Figure 8 (b), there exists a non-separating curve A'_0 in R which is disjoint from l_F and $E \cap R$ and is a boundary component of a regular neighborhood of $l_F \cup B$ in R. This curve A'_0 is isotopic to a component of A. There exists a path in R connecting a point of A'_0 with a point of l_F without touching $E \cap R$ because any component of $E \cap R$ is isotopic to l_E^1. This contradicts the following:

CLAIM 5.5. *Let α be a BP in S. Let β and γ be h-curves in S such that each of $\{\alpha, \beta\}$ and $\{\beta, \gamma\}$ is an edge of $\mathcal{T}(S)$. If a curve α_0 in the BP α is disjoint from γ, then α_0 lies in the handle cut off by γ from S. In particular, there exists no path in S connecting a point of α_0*

with a point of β without touching γ.

PROOF. Let α_0 be a curve in the BP α disjoint from γ. Let Q be the component of S_β that is not a handle. Any BP in S disjoint from β lies in Q. The BP α thus lies in Q. The curve γ is an h-curve in Q. Since α_0 is disjoint from γ, the curve α_0 lies in the handle cut off by γ from Q. □

We have therefore proved that there exists a component of $E \cap R$ isotopic to l_F in R. Let l_E^2 be a component of $E \cap R$ isotopic to l_F in R. Cutting R along $l_E^1 \cup l_E^2$, we obtain two annuli, one of which contains ∂S. The arc l_D lies in the annulus containing ∂S because l_D is disjoint from $E \cap R$ (see Figure 8 (c)). We have the unique component of A isotopic to a curve lying in another annulus. Let a_0 be the isotopy class of that component of A. The curve a_0 is disjoint from any of b, d and f, and is thus disjoint from any of $a, \dots, f$.

We obtained a curve a_0 in a disjoint from any of $a, \dots, f$ in both cases (1) and (2). By symmetry, we can also find a curve d_0 in d disjoint from any of $a, \dots, f$. By Lemma 5.3, $\pi(a_0)$ and $\pi(d_0)$ lie in distinct components of $\bar{S}_{\pi(b)}$. It turns out that a_0 and d_0 are distinct, and the surface obtained by cutting S along $a_0 \cup d_0$ is homeomorphic to $S_{0,5}$. □

Let X be a surface. For a BP b in X and a boundary component ∂ of X, we say that *b cuts off ∂* if b cuts off a pair of pants containing ∂ from X. For two distinct boundary components ∂_1, ∂_2 of X and a p-curve α in X, we say that *α cuts off ∂_1 and ∂_2* if α cuts off a pair of pants containing ∂_1 and ∂_2 from X.

LEMMA 5.6. *Let (a, b, c, d, e, f) be a 6-tuple defining a hexagon in $\mathcal{T}(S)$ of type 2 with a and d BP-vertices. Let a_0 and d_0 be the curves obtained in Lemma 5.4. Then there exists a non-separating curve ζ in S satisfying the following three conditions:*

(a) *The curve ζ is disjoint from a and d, and belongs to neither a nor d.*

(b) *Let S_{d_0} denote the surface obtained by cutting S along d_0, which is homeomorphic to $S_{1,3}$. The pair $\{a_0, \zeta\}$ is then a BP in S_{d_0}, and cuts off one of the two boundary components of S_{d_0} corresponding to d_0.*

(c) *The condition obtained by exchanging a_0 and d_0 in condition* (b) *holds.*

Moreover, such a curve ζ uniquely exists up to isotopy.

Let Π be a hexagon in $\mathcal{T}(S)$ of type 2, and let (a, b, c, d, e, f) be a 6-tuple defining Π with a and d BP-vertices. We denote by $\zeta(\Pi)$ the curve ζ obtained by applying Lemma 5.6 to Π. Let d_1 be the curve in d distinct from d_0. In the surface S_{d_0}, a and $\{a_0, \zeta\}$ are BPs, b and d_1 are p-curves, and c is an h-curve. The 5-tuple $(a, b, c, d_1, \{a_0, \zeta\})$ defines a pentagon in $\mathcal{T}(S_{d_0})$.

PROOF OF LEMMA 5.6. Choose representatives $A, \dots, F$ of $a, \dots, f$, respectively, such that any two of them intersect minimally. Let A_0 and A_1 denote the two components of A so that the isotopy class of A_0 is a_0. Let D_0 and D_1 denote the two components of D so that the isotopy class of D_0 is d_0. We define T as the surface obtained by cutting S along $A_0 \cup D_0$, which is homeomorphic to $S_{0,5}$. We label boundary components of T as ∂, ∂_a^1, ∂_a^2,

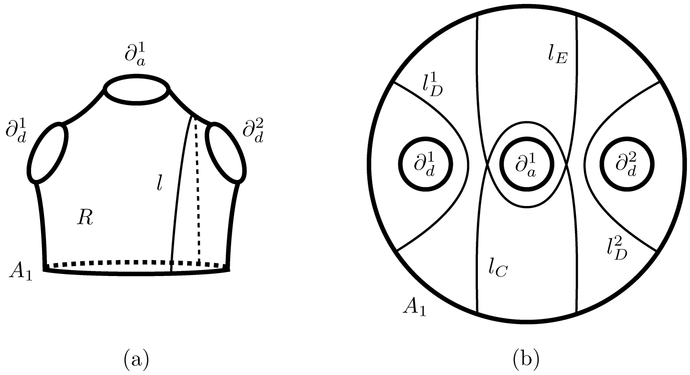


FIGURE 9. The arc l in (a) cuts off ∂_d^2.

∂_d^1 and ∂_d^2 so that ∂ corresponds to ∂S, ∂_a^1 and ∂_a^2 correspond to A_0, and ∂_d^1 and ∂_d^2 correspond to D_0. Without loss of generality, we may assume that A_1 is a p-curve in T cutting off ∂ and ∂_a^2. Each of B and F is a p-curve in T cutting off ∂_d^1 and ∂_d^2. Similarly, each of C and E is a p-curve in T cutting off ∂_a^1 and ∂_a^2.

Let R be the component of T_{A_1} homeomorphic to $S_{0,4}$. The surface R contains ∂_a^1, ∂_d^1 and ∂_d^2. For each essential simple arc l in R whose boundary lies in A_1 and for each $\partial_j^k \in \{\partial_a^1, \partial_d^1, \partial_d^2\}$, we say that l *cuts off* ∂_j^k if ∂_j^k lies in the annulus cut off by l from R (see Figure 9 (a)). Since B is a curve in R and is disjoint from C, the intersection $C \cap R$ consists of mutually isotopic, essential simple arcs in R cutting off ∂_a^1. Similarly, since F is a curve in R and is disjoint from E, the intersection $E \cap R$ consists of mutually isotopic, essential simple arcs in R cutting off ∂_a^1. Pick a component l_C of $C \cap R$ and a component l_E of $E \cap R$.

CLAIM 5.7. *The two arcs l_C and l_E are non-isotopic, and cannot be isotoped so that they are disjoint.*

PROOF. The former assertion holds because otherwise B and F would be isotopic. The latter assertion holds because l_C and l_E are non-isotopic and because both l_C and l_E cut off ∂_a^1. □

CLAIM 5.8. *The intersection $D_1 \cap R$ consists of mutually isotopic, essential simple arcs in R.*

PROOF. Assuming the contrary, we deduce a contradiction. Any family of essential simple arcs in R which are mutually disjoint and non-isotopic and whose boundaries lie in A_1 has at most three elements. If $D_1 \cap R$ had three components which are mutually non-isotopic, then l_C and l_E would be isotopic because l_C and l_E are disjoint from $D_1 \cap R$. This contradicts

Claim 5.7. We thus assume that $D_1 \cap R$ contains exactly two essential simple arcs in R up to isotopy. Let l_D^1 and l_D^2 be components of $D_1 \cap R$ which are non-isotopic.

If either l_D^1 or l_D^2 cut off ∂_a^1, then l_C and l_E would be isotopic to that arc. This also contradicts Claim 5.7. It follows that one of l_D^1 and l_D^2 cuts off ∂_d^1 and another cuts off ∂_d^2. Without loss of generality, we may assume that l_D^1 cuts off ∂_d^1, and l_D^2 cuts off ∂_d^2. Claim 5.7 implies that l_C and l_E are drawn as in Figure 9 (b). For each $k = 1, 2$, there exists a path in R connecting a point of ∂_d^k with a point of l_D^k without touching neither $C \cap R$ nor $E \cap R$.

The curve D_1 is a p-curve in T cutting off ∂ and one of ∂_d^1 and ∂_d^2. Suppose that D_1 cuts off ∂ and ∂_d^2. We define U as the surface obtained from T by attaching a disk to ∂_d^1. The two curves C and D_1 are isotopic in U because C and D_1 are disjoint and the pair of pants cut off from T by each of them does not contain ∂_d^1. Similarly, D_1 and E are also isotopic in U. It turns out that C and E are isotopic in U. On the other hand, C and E intersect minimally as curves in T, and $C \cap E$ is non-empty. By [6, Exposé 3, Proposition 10], there exist a subarc in C and a subarc in E whose union is a simple closed curve in T isotopic to ∂_d^1. The curve D_1 is disjoint from C and E. Any path in T connecting a point of ∂_d^1 with a point of D_1 therefore intersects either C or E. This contradicts the property obtained in the end of the last paragraph. Exchanging ∂_d^1 and ∂_d^2, we can deduce a contradiction if we suppose that D_1 cuts off ∂ and ∂_d^1. □

By Claim 5.8, there exists an essential simple closed curve in R disjoint from $D_1 \cap R$, which is unique up to isotopy. Let ζ denote the isotopy class of that curve. This is a desired one. In fact, condition (a) holds by definition. Claim 5.7 implies that any component of $D_1 \cap R$ cuts off either ∂_d^1 or ∂_d^2. The curve ζ is therefore a p-curve in R cutting off ∂_a^1 and one of ∂_d^1 and ∂_d^2. Conditions (b) and (c) follow. The uniqueness of ζ holds because there exists at most one curve in T disjoint from the two curves a_1 and d_1 that intersect. □

In the proof of the subsequent two theorems, we use the following:

Graph $\mathcal{F}$. Let R be a surface homeomorphic to $S_{0,4}$. We define a simplicial graph $\mathcal{F} = \mathcal{F}(R)$ so that the set of vertices of $\mathcal{F}$ is $V(R)$, and two vertices α, β of $\mathcal{F}$ are connected by an edge of $\mathcal{F}$ if and only if $i(\alpha, \beta) = 2$.

It is well known that this graph is isomorphic to the Farey graph realized as an ideal triangulation of the Poincaré disk (see [15, Section 3.2] or Figure 11 (a)). We mean by a *triangle* in $\mathcal{F}$ a subgraph of $\mathcal{F}$ consisting of exactly three vertices and exactly three edges. Note that for any two ordered triples of vertices in $\mathcal{F}$ defining triangles in $\mathcal{F}$, there exists a unique simplicial automorphism of $\mathcal{F}$ sending the first triple to the second one.

The following theorem characterizes hexagons in $\mathcal{T}(S)$ of type 2.

THEOREM 5.9. *Let Π be a hexagon in $\mathcal{T}(S)$ of type 2, and let (a, b, c, d, e, f) be a 6-tuple defining Π with a and d BP-vertices. Put $\zeta = \zeta(\Pi)$. Then there exists a unique non-zero integer m with $f = t_\zeta^m(b)$ and $e = t_\zeta^m(c)$.*

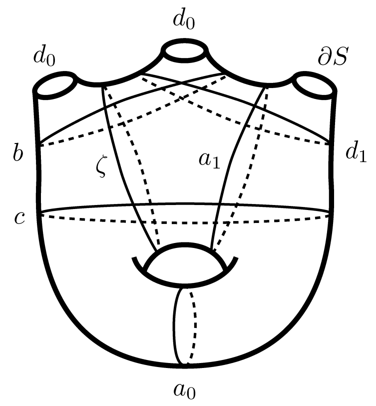


FIGURE 10

PROOF. Let a_0 and d_0 be the curves in the BPs a and d, respectively, obtained in Lemma 5.4. The surface S_{d_0} is homeomorphic to $S_{1,3}$. In S_{d_0}, the curve in d distinct from d_0, denoted by d_1, is a p-curve, b is a p-curve, c is an h-curve, and a and $\{a_0, \zeta\}$ are BPs. The 5-tuple $(a, b, c, d_1, \{a_0, \zeta\})$ defines a pentagon in $\mathcal{T}(S_{d_0})$ (see Figure 10). Similarly, the 5-tuple $(a, f, e, d_1, \{a_0, \zeta\})$ also defines a pentagon in $\mathcal{T}(S_{d_0})$ such that in S_{d_0}, e is an h-curve and f is a p-curve. Cut S_{d_0} along a. The obtained surface consists of a pair of pants and a surface homeomorphic to $S_{0,4}$. Let R denote the latter component.

Let ∂_d^1 and ∂_d^2 denote the two boundary components of R corresponding to d_0. The curves b and f lie in R and cut off ∂_d^1 and ∂_d^2. The curve ζ cuts off a pair of pants from R containing exactly one of ∂_d^1 and ∂_d^2. By Lemma 5.1 (ii), we have $i(\zeta, b) = i(\zeta, f) = 2$. Looking at the action of the Dehn twist t_ζ on the graph $\mathcal{F}(R)$, we see that t_ζ acts on the link of ζ in $\mathcal{F}(R)$ freely. Moreover, t_ζ transitively acts on the set of all vertices in the link of ζ that correspond to curves in R cutting off ∂_d^1 and ∂_d^2. It follows that there exists a unique integer m with $t_\zeta^m(b) = f$. Since b and f are distinct, the integer m is non-zero.

The 6-tuple $(a, b, c, d, t_\zeta^m(c), t_\zeta^m(b))$ defines a hexagon in $\mathcal{T}(S)$, as shown in the beginning of this section. There exists at most one h-curve in S disjoint from the BP d and the h-curve $t_\zeta^m(b) = f$. We therefore have $t_\zeta^m(c) = e$. □

THEOREM 5.10. *Let Π be a hexagon in $\mathcal{T}(S)$ of type* 2, *and let (a, b, c, d, e, f) be a* 6-*tuple defining Π with a and d BP-vertices. Put $\zeta = \zeta(\Pi)$. Then the following assertions hold*:

(i) *If neither $f = t_\zeta(b)$ nor $f = t_\zeta^{-1}(b)$, then Π is the only hexagon in $\mathcal{T}(S)$ of type* 2 *containing f, a, b and c.*

(ii) *If either $f = t_\zeta(b)$ or $f = t_\zeta^{-1}(b)$, then there exists exactly one hexagon in $\mathcal{T}(S)$ of* *type* 2 *that is distinct from* Π *and contains* f, a, b *and* c.

Before proving Theorem 5.10, we prepare two lemmas.

LEMMA 5.11. *Let* Π *be a hexagon in* $\mathcal{T}(S)$ *of type* 2, *and let* (a, b, c, d, e, f) *be a* 6-*tuple defining* Π *with* a *and* d *BP-vertices. Let* Ω *be a hexagon in* $\mathcal{T}(S)$ *of type* 2 *containing* f, a, b *and* c. *If* $\zeta(\Pi) = \zeta(\Omega)$, *then* $\Pi = \Omega$.

PROOF. Put $\zeta = \zeta(\Pi) = \zeta(\Omega)$. By Theorem 5.9, there exists a unique integer m with $t_{\zeta}^{m}(b) = f$ and $t_{\zeta}^{m}(c) = e$. Let (a, b, c, d', e', f) be the 6-tuple defining Ω. Applying the same theorem to Ω, we obtain a unique integer n with $t_{\zeta}^{n}(b) = f$ and $t_{\zeta}^{n}(c) = e'$. The equality $t_{\zeta}^{m}(b) = t_{\zeta}^{n}(b)$ then holds. We thus have $m = n$ and $e = e'$. Since at most one BP in S disjoint from c and e exists, we have $d = d'$. □

We set $R = S_{0,4}$. For any edge τ of the graph $\mathcal{F} = \mathcal{F}(R)$, the complement of $\tau \cup \partial\tau$ in the geometric realization of $\mathcal{F}$ has exactly two connected components. We call those components *sides* of τ.

LEMMA 5.12. *We set* $R = S_{0,4}$ *and* $\mathcal{F} = \mathcal{F}(R)$. *Let* α *and* β *be curves in* R *with* $i(\alpha, \beta) = 2$. *We denote by* γ *the only curve in* R *such that each of* $\{\alpha, \beta, \gamma\}$ *and* $\{\alpha, \beta, t_{\alpha}(\gamma)\}$ *defines a triangle in* $\mathcal{F}$. *Let* δ *be a curve in* R *with* $\delta \neq \alpha$ *and* $i(\beta, \delta) = 2$. *Let* m *and* n *be non-zero integers. If the equality* $t_{\alpha}^{m}(\beta) = t_{\delta}^{n}(\beta)$ *holds, then either* $\delta = \gamma$ *and* $(m, n) = (-1, 1)$ *or* $\delta = t_{\alpha}(\gamma)$ *and* $(m, n) = (1, -1)$.

PROOF. Realize the graph $\mathcal{F}$ geometrically as an ideal triangulation of the Poincaré disk D. The set $\partial D \setminus \{\alpha, \beta, \gamma, t_{\alpha}(\gamma)\}$ consists of the four connected components L_1, L_2, L_3 and L_4 as in Figure 11 (b). For any positive integer j, $t_{\alpha}^{j}(\beta)$ lies in L_4. For any negative integer k, $t_{\alpha}^{k}(\beta)$ lies in L_3.

The vertex δ is in the link of β in $\mathcal{F}$ and distinct from α. Assuming that δ is equal to neither γ nor $t_{\alpha}(\gamma)$, we deduce a contradiction. The vertex δ then lies in either L_1 or L_2. We have the two triangles in $\mathcal{F}$ containing the edge $\{\beta, \delta\}$. Each of those triangles has the edge containing δ and distinct from $\{\beta, \delta\}$. Let τ and σ denote those edges. If δ lies in L_1, then the interior of τ and that of σ lie in the side of the edge $\{\beta, t_{\alpha}(\gamma)\}$ containing δ. The argument in the previous paragraph shows that for any non-zero integer j, $t_{\delta}^{j}(\beta)$ lies in L_1. This contradicts the equality $t_{\alpha}^{m}(\beta) = t_{\delta}^{n}(\beta)$. We can deduce a contradiction similarly if we assume that δ lies in L_2. It turns out that δ is equal to either γ or $t_{\alpha}(\gamma)$.

We first suppose the equality $\delta = \gamma$. Let ε denote the vertex $t_{\alpha}^{-1}(\beta) = t_{\gamma}(\beta)$, which lies in L_3 and forms a triangle in $\mathcal{F}$ together with α and γ. Let L_{31} and L_{32} be the two components of $L_3 \setminus \{\varepsilon\}$ so that the closure of L_{31} contains γ and that of L_{32} contains α (see Figure 11 (c)). For any positive integer j, $t_{\alpha}^{j}(\beta)$ lies in L_4. For any integer k with $k < -1$, $t_{\alpha}^{k}(\beta)$ lies in L_{32}. For any integer j with $j > 1$, $t_{\gamma}^{j}(\beta)$ lies in L_{31}. For any negative integer k, $t_{\gamma}^{k}(\beta)$ lies in L_2. The equality $t_{\alpha}^{m}(\beta) = t_{\gamma}^{n}(\beta)$ therefore implies $(m, n) = (-1, 1)$. If we suppose the equality

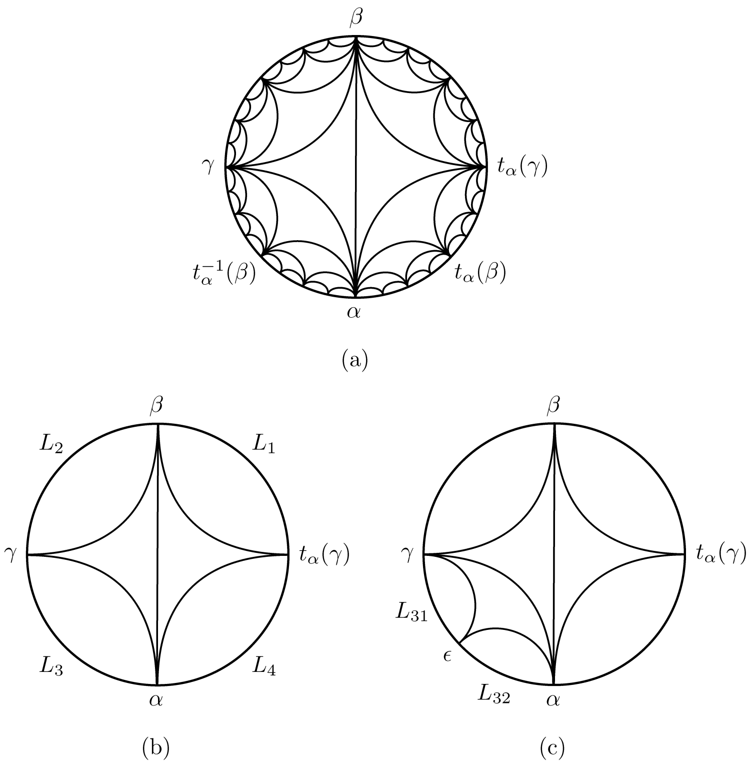


FIGURE 11

$\delta = t_\alpha(\gamma)$ in place of the equality $\delta = \gamma$, then we obtain $(m, n) = (1, -1)$ along a similar argument. $\square$

We are now ready to prove Theorem 5.10.

PROOF OF THEOREM 5.10 (i). Let Π be a hexagon in $\mathcal{T}(S)$ of type 2. Let (a, b, c, d, e, f) be a 6-tuple defining Π with a and d BP-vertices. Pick a hexagon Ω in $\mathcal{T}(S)$ of type 2 containing f, a, b and c. Let (a, b, c, d', e', f) be the 6-tuple defining Ω. We put $\zeta = \zeta(\Pi)$ and $\eta = \zeta(\Omega)$. By Theorem 5.9, we have the non-zero integers m, n with

$$f = t_\zeta^m(b) = t_\eta^n(b), \quad e = t_\zeta^m(c) \quad \text{and} \quad e' = t_\eta^n(c).$$

Applying Lemma 5.4 to Π, we obtain the curve a_0 in a and the curve d_0 in d that are disjoint from any of $a, \ldots, f$. In the component of S_a homeomorphic to $S_{1,2}$, the curve d_0 is the only curve disjoint from b and f. Applying Lemma 5.4 to the hexagon Ω, which contains a, b and f, we see that d_0 is also contained in the BP d'. Let R denote the subsurface of S filled by b and f, which is homeomorphic to $S_{0,4}$. Any of b, f, ζ and η is a curve in R. By Lemmas 5.1

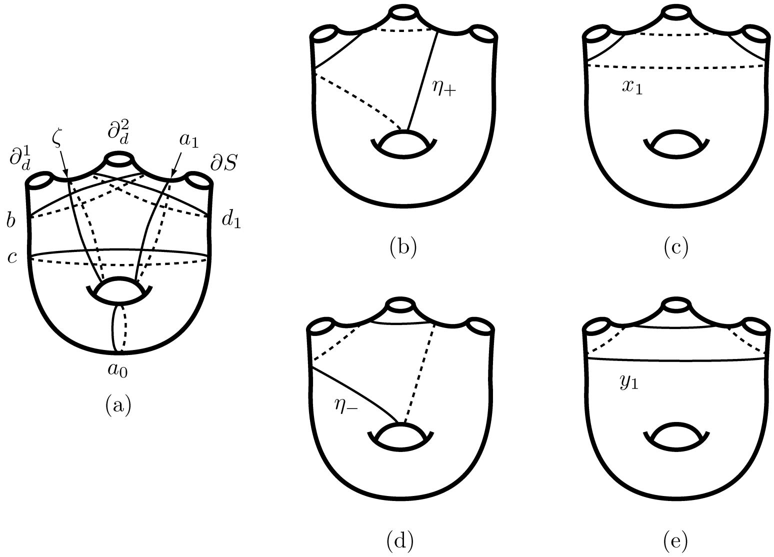


FIGURE 12

and 5.6, we have $i(b, \zeta) = i(b, \eta) = 2$.

If ζ and η are distinct, then by Lemma 5.12, the equality $t_\zeta^m(b) = t_\eta^n(b)$ implies that either $(m, n) = (1, -1)$ or $(m, n) = (-1, 1)$. It follows that either $f = t_\zeta(b)$ or $f = t_\zeta^{-1}(b)$. Under the assumption that neither $f = t_\zeta(b)$ nor $f = t_\zeta^{-1}(b)$ holds, we therefore have the equality $\zeta = \eta$. By Lemma 5.11, we then have $\Pi = \Omega$. Theorem 5.10 (i) is proved. □

PROOF OF THEOREM 5.10 (ii). Let Π be a hexagon in $\mathcal{T}(S)$ of type 2. Let (a, b, c, d, e, f) be a 6-tuple defining Π with a and d BP-vertices. Put $\zeta = \zeta(\Pi)$. Let a_0 and d_0 be the curves in the BPs a and d, respectively, obtained in Lemma 5.4. We define curves a_1 and d_1 so that $a = \{a_0, a_1\}$ and $d = \{d_0, d_1\}$. Let R denote the subsurface of S filled by b and f, which is homeomorphic to $S_{0,4}$ because b and f are disjoint from a and d_0. We set $\mathcal{F} = \mathcal{F}(R)$. Any of b, f and ζ is a curve in R. By Lemmas 5.1 and 5.6, the curves a_0, a_1, b, c, d_1 and ζ in S_{d_0} are drawn as in Figure 12 (a), where ∂_d^1 and ∂_d^2 denote the two boundary components of S_{d_0} corresponding to d_0.

We first suppose the equality $f = t_\zeta(b)$. We construct a hexagon in $\mathcal{T}(S)$ of type 2 containing f, a, b and c and distinct from Π. The assumption $f = t_\zeta(b)$ implies that there exists a unique curve η_+ in R such that each of the triples $\{b, \zeta, \eta_+\}$ and $\{f, \zeta, \eta_+\}$ forms a triangle in $\mathcal{F}$, as in Figure 13 (a). We have the equality $f = t_\zeta(b) = t_{\eta_+}^{-1}(b)$. The curve η_+ is then determined as in Figure 12 (b). We define x_1 as the curve drawn in Figure 12 (c), and set $x = \{d_0, x_1\}$. The 5-tuple $(a, b, c, x_1, \{a_0, \eta_+\})$ defines a pentagon in $\mathcal{T}(S_{d_0})$. The 6-tuple

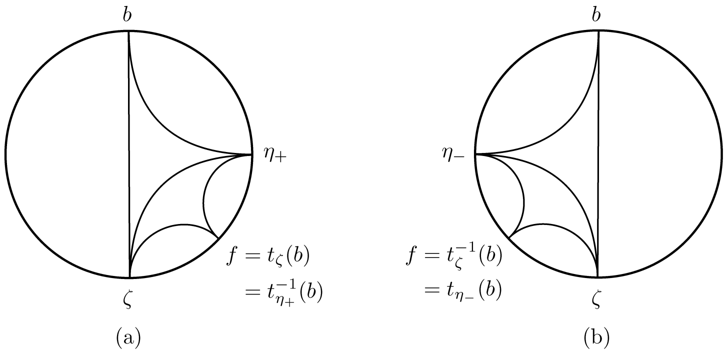


FIGURE 13

$(a, b, c, x, t_{\eta_+}^{-1}(c), f)$ therefore defines a hexagon in $\mathcal{T}(S)$ of type 2, denoted by Ω_+.

Let Ω be a hexagon in $\mathcal{T}(S)$ of type 2 containing f, a, b and c. Put $\eta = \zeta(\Omega)$. Applying Theorem 5.9 to Ω, we have a non-zero integer n with $f = t_\eta^n(b)$. In the first paragraph in the proof of Theorem 5.10 (i), we showed that η is also a curve in R, and we have $i(b, \zeta) = i(b, \eta) = 2$. The equality $f = t_\zeta(b) = t_\eta^n(b)$ and Lemma 5.12 imply that either $\eta = \zeta$ or $\eta = \eta_+$ and $n = -1$. By Lemma 5.11, we have either $\Omega = \Pi$ or $\Omega = \Omega_+$. Theorem 5.10 (ii) is therefore proved if $f = t_\zeta(b)$.

We next suppose the equality $f = t_\zeta^{-1}(b)$. There exists a unique curve η_- in R such that each of the triples $\{b, \zeta, \eta_-\}$ and $\{f, \zeta, \eta_-\}$ forms a triangle in $\mathcal{F}$, as in Figure 13 (b). We have the equality $f = t_\zeta^{-1}(b) = t_{\eta_-}(b)$. The curve η_- is then determined as in Figure 12 (d). We define a curve y_1 as in Figure 12 (e), and set $y = \{d_0, y_1\}$. The 5-tuple $(a, b, c, y_1, \{a_0, \eta_-\})$ defines a pentagon in $\mathcal{T}(S_{d_0})$. The 6-tuple $(a, b, c, y, t_{\eta_-}(c), f)$ defines a hexagon in $\mathcal{T}(S)$ of type 2, denoted by Ω_-. As in the previous paragraph, we can show that if Ω is a hexagon in $\mathcal{T}(S)$ of type 2 containing f, a, b and c, then either $\Omega = \Pi$ or $\Omega = \Omega_-$. $\square$

In the rest of this section, we observe hexagons in $\mathcal{T}(S)$ sharing a 3-path with a given hexagon of type 1 or type 2. Note that a hexagon in $\mathcal{T}(S)$ has exactly two BP-vertices if and only if it is of either type 1 or type 2.

LEMMA 5.13. *Let Π be a hexagon in $\mathcal{T}(S)$ of type* 1, *and let (a, b, c, d, e, f) be a* 6-*tuple defining Π with b and f BP-vertices. Let Ω be a hexagon in $\mathcal{T}(S)$ such that $\Pi \cap \Omega$ contains a* 3-*path. Then the following assertions hold*:

(i) *The hexagon Ω contains at least one of b and f.*
(ii) *If Ω contains exactly one of b and f, then Ω has exactly two BP-vertices.*
(iii) *If Ω contains both b and f, then the equality $\Omega = \Pi$ holds.*

PROOF. Assertion (i) holds because any 3-path in Π contains at least one of b and f. If Ω contains exactly one of b and f, then Ω contains two adjacent h-vertices, and thus has

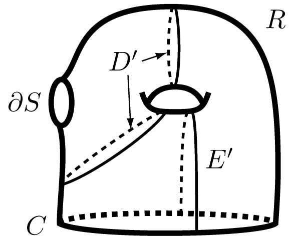


FIGURE 14

exactly two BP-vertices by Lemmas 3.1 and 3.3. Assertion (ii) follows.

Assuming that Ω contains b and f, we prove assertion (iii). Without loss of generality, we may assume that Π and Ω contain f, a, b and c. Let (a, b, c, d', e', f) be the 6-tuple defining Ω. The vertex e' is an h-vertex because f is a BP-vertex. Assuming that d' is a BP-vertex, we deduce a contradiction. Let α be the curve contained in b and f.

Choose representatives $A, \ldots, F$, D' and E' of $a, \ldots, f$, d' and e', respectively, such that any two of them intersect minimally. Let $\mathfrak{a}$ denote the component of F whose isotopy class is α. Let R denote the component of S_C that is not a handle. Note that $\mathfrak{a}$ is a curve in R. Since D' is a BP in R, the intersection $E' \cap R$ consists of mutually isotopic, essential simple arcs in R which are non-separating in R (see Figure 14). It follows from $E' \cap F = \emptyset$ that $E' \cap R$ is disjoint from $\mathfrak{a}$. Since the two components of D' are boundary components of a regular neighborhood of $(E' \cap R) \cup C$ in R, one of components of D' is isotopic to $\mathfrak{a}$. It turns out that d' contains α.

We define curves b_1, d'_1 and f_1 so that $b = \{\alpha, b_1\}$, $d' = \{\alpha, d'_1\}$ and $f = \{\alpha, f_1\}$. The 6-tuple $(a, b_1, c, d'_1, e, f_1)$ then defines a hexagon in $\mathcal{C}_s(S_\alpha)$ such that each of the curves b_1, d'_1 and f_1 in S_α cuts off a pair of pants containing ∂S from S_α. This is a contradiction because by Theorem 4.1, for any hexagon H in $\mathcal{C}_s(S_\alpha)$, there is a p-curve in H cutting off a pair of pants containing the two boundary components of S_α that correspond to α, from S_α.

We proved that d' is an h-vertex. It follows that Ω is of type 1. By Lemma 4.3, we have the equality $\Omega = \Pi$. □

Finally, we obtain the following:

THEOREM 5.14. *Let Π be a hexagon in $\mathcal{T}(S)$. Then the following assertions hold*:

(i) *If Π is of type* 1, *then for any* 3*-path K in Π containing the two BP-vertices of Π, there exists no hexagon in $\mathcal{T}(S)$ distinct from Π and containing K.*

(ii) *If Π is of type* 2, *then for any* 3*-path L in Π containing exactly one BP-vertex of Π, there exist at most two hexagons in $\mathcal{T}(S)$ distinct from Π and containing L.*

PROOF. Assertion (i) follows from Lemma 5.13 (iii). Suppose that Π is of type 2, and pick a 3-path L in Π containing exactly one BP-vertex of Π. By Lemmas 3.1 and 3.3, any

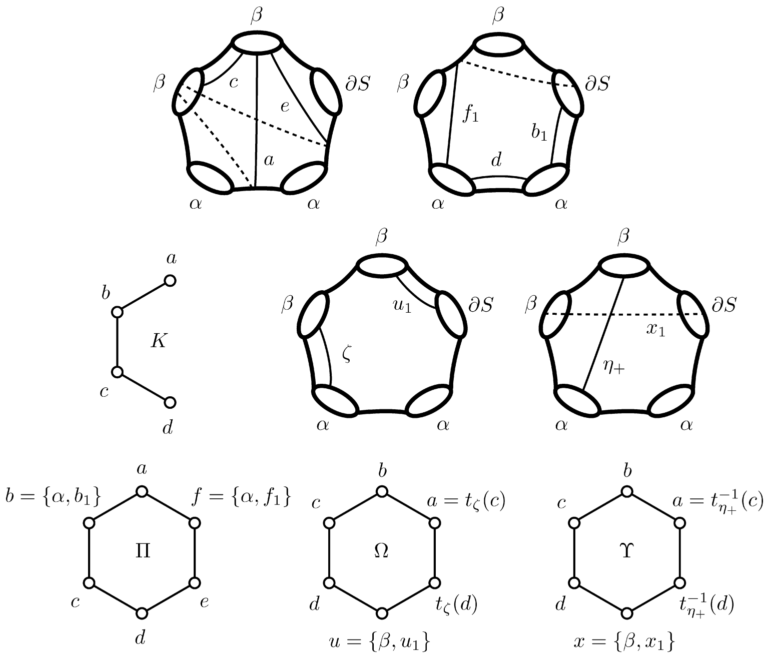


FIGURE 15

hexagon in $\mathcal{T}(S)$ has at least two BP-vertices. Any hexagon in $\mathcal{T}(S)$ containing L is thus of either type 1 or type 2 because L contains two adjacent h-vertices. By Lemma 4.3, the number of hexagons in $\mathcal{T}(S)$ of type 1 containing L is at most 1. By Theorem 5.10, the number of hexagons in $\mathcal{T}(S)$ of type 2 distinct from Π and containing L is at most 1. Assertion (ii) is therefore proved. □

REMARK 5.15. In addition to Theorem 5.14, we have the following description of the number of hexagons sharing a 3-path with a given hexagon of type 1 or type 2, whose proof is not presented here because it is not used in the rest of the paper.

Let Π be a hexagon in $\mathcal{T}(S)$ defined by a 6-tuple (a, b, c, d, e, f). Assume that Π is of type 1 with b and f BP-vertices. Let K be a 3-path in Π containing exactly one of b and f. If K does not contain a, then Π is the only hexagon in $\mathcal{T}(S)$ containing K by Lemma 4.3. If K contains a, then any hexagon in $\mathcal{T}(S)$ distinct from Π and containing K is of type 2 by Lemma 4.3, and there exist exactly two hexagons in $\mathcal{T}(S)$ of type 2 containing K.

Those two hexagons are drawn in Figure 15. Let α and β be disjoint and non-isotopic curves in S such that the surface obtained by cutting S along $\alpha \cup \beta$, denoted by T, is connected. Any essential simple arc l in T connecting two distinct boundary components ∂_1, ∂_2 associates a curve $c(l)$ in T. Namely, $c(l)$ is defined as a boundary component of a regular neighborhood of $l \cup \partial_1 \cup \partial_2$ in T. In Figure 15, the surface T is drawn, and in place of curves, essential simple arcs associating them are drawn. Given a hexagon Π in $\mathcal{T}(S)$ of type 1 and a 3-path K in Π as drawn in Figure 15, we have the two hexagons Ω, Υ in $\mathcal{T}(S)$ of type 2 containing K. It follows from Theorem 5.14 (ii) that there is no other hexagon in $\mathcal{T}(S)$ containing K.

We next assume that Π is of type 2 with a and d BP-vertices. Let ζ be the curve $\zeta(\Pi)$ obtained in Lemma 5.6. Let L be a 3-path in Π. Any hexagon in $\mathcal{T}(S)$ containing L is either of type 1 or type 2 because L contains two adjacent h-vertices. We first suppose that L contains exactly one of a and d. If either $f = t_\zeta(b)$ or $f = t_\zeta^{-1}(b)$, then there exist exactly two hexagons in $\mathcal{T}(S)$ distinct from Π and containing L, one of which is of type 1 and another of which is of type 2. We omit to describe those two hexagons because they are obtained by using Figure 15 after exchanging symbols appropriately. If neither $f = t_\zeta(b)$ nor $f = t_\zeta^{-1}(b)$, then Π is the only hexagon in $\mathcal{T}(S)$ containing L. Finally, we suppose that L contains a and d. In the fourth paragraph of this section, we have observed that there are infinitely many hexagons in $\mathcal{T}(S)$ of type 2 containing L.

6. Hexagons of type 3

Throughout this section, we set $S = S_{2,1}$. We say that a hexagon in $\mathcal{T}(S)$ is of *type 3* if it contains exactly three BP-vertices. In this section, we focus on the hexagons of type 3 drawn in Figure 16, and present their property that no hexagon of type 1 or type 2 satisfies.

We note that there is a one-to-one correspondence between elements of $V_{bp}(S)$ and elements of $V_a(S)$ whose representatives are non-separating in S. In fact, each BP b in S associates an essential simple arc in S contained in the pair of pants cut off by b from S, which is non-separating in S and is uniquely determined up to isotopy. Conversely, given an essential simple arc l in S which is non-separating in S, one obtains the BP in S whose curves are boundary components of a regular neighborhood of $l \cup \partial S$ in S.

In Figure 16, in place of BPs, essential simple arcs corresponding to them are drawn. This replacement makes the drawing much plainer. We define a, c and e as the h-curves in S in Figure 16, and define b, d and f as the BPs in S corresponding to the arcs in Figure 16. Let Θ denote the hexagon in $\mathcal{T}(S)$ of type 3 defined by the 6-tuple (a, b, c, d, e, f). Let α, β and γ be the non-separating curves in Figure 17. The curve α is disjoint from a and d, the curve β is disjoint from b and e, and the curve γ is disjoint from c and f. The following lemma is in contrast with Theorem 5.14 on hexagons of type 1 and type 2.

PROPOSITION 6.1. *For any* 3*-path L in Θ, there exist infinitely many hexagons in $\mathcal{T}(S)$ containing L.*

Before proving this proposition, we show the following:

LEMMA 6.2. *Let v_1, v_3 and v_5 be h-vertices of $\mathcal{T}(S)$, and v_2, v_4 and v_6 BP-vertices of $\mathcal{T}(S)$ such that*

- *for any j mod* 6*, v_j and v_{j+1} are adjacent, for any $k = 1, 2$, v_k and v_{k+2} are distinct and not adjacent, and v_1 and v_4 are not adjacent; and*

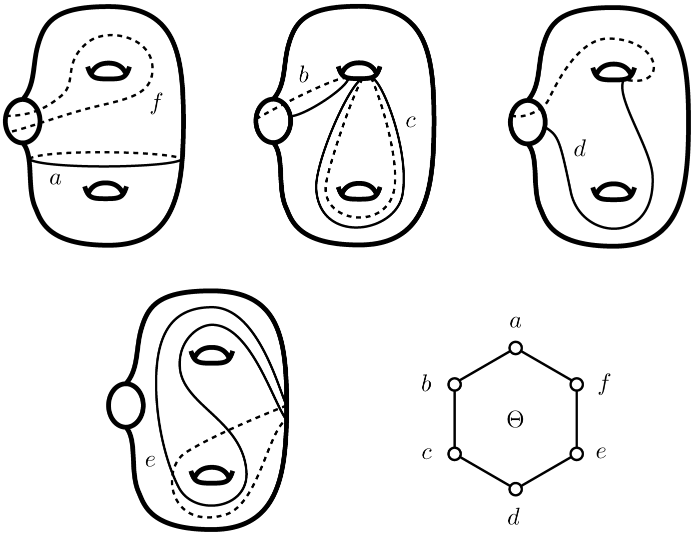


FIGURE 16

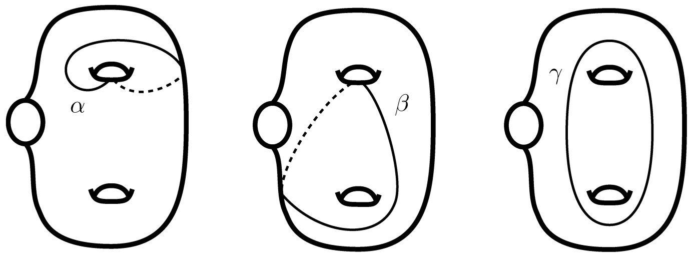


FIGURE 17

- *we have* $v_6 \neq v_2$.

Then the 6*-tuple* $(v_1, \dots, v_6)$ *defines a hexagon in* $\mathcal{T}(S)$.

PROOF. The assumption $v_6 \neq v_2$ implies that v_6 and v_2 intersect. Since v_1 is disjoint from v_6, but intersects v_4, the BPs v_6 and v_4 are distinct, and thus intersect. In general, for any two distinct h-curves x, y in S, there exists at most one BP in S disjoint from x and y if it exists. If v_6 and v_3 were disjoint, then the two BPs v_6 and v_2 would be disjoint from the two distinct h-curves v_1 and v_3. This contradicts $v_6 \neq v_2$. It follows that v_6 and v_3 intersect.

Since v_6 is disjoint from v_5, but intersects v_3, the h-curves v_5 and v_3 are distinct. The curves v_5 and v_3 intersect because they are disjoint from the BP v_4. Since v_4 is disjoint from

v_5, but intersects v_1, the h-curves v_5 and v_1 are distinct. The curves v_5 and v_1 intersect because they are disjoint from the BP v_6. If v_5 and v_2 were disjoint, then the two BPs v_6 and v_2 would be disjoint from the two distinct h-curves v_1 and v_5. As noted in the previous paragraph, this contradicts $v_6 \neq v_2$. It follows that v_5 and v_2 intersect. □

PROOF OF PROPOSITION 6.1. We prove the proposition in the case where L consists of a, b, c and d. The proof of the other cases are obtained along a verbatim argument after exchanging symbols appropriately. We show that for all but one of integers n, the 6-tuple $(a, b, c, d, t_\alpha^n(e), t_\alpha^n(f))$ defines a hexagon in $\mathcal{T}(S)$. For any integer n, the pair $\{t_\alpha^n(e), t_\alpha^n(f)\}$ is an edge of $\mathcal{T}(S)$. Since α is disjoint from a and d, we have $t_\alpha^n(a) = a$ and $t_\alpha^n(d) = d$. Each of $\{d, t_\alpha^n(e)\}$ and $\{t_\alpha^n(f), a\}$ is thus an edge of $\mathcal{T}(S)$. To prove the proposition, it suffices to show the following three assertions:

(1) At most one integer n satisfies the equality $t_\alpha^n(f) = b$.

(2) For any integer n with $t_\alpha^n(f) \neq b$, the 6-tuple $(a, b, c, d, t_\alpha^n(e), t_\alpha^n(f))$ defines a hexagon in $\mathcal{T}(S)$.

(3) For any integers n_1, n_2 with $n_1 \neq n_2$, we have $t_\alpha^{n_1}(f) \neq t_\alpha^{n_2}(f)$.

Assertions (1) and (3) hold because α and f intersect. Applying Lemma 6.2 when $(v_1, \ldots, v_6) = (a, b, c, d, t_\alpha^n(e), t_\alpha^n(f))$, we obtain assertion (2). □

The following lemma will be used in Section 7.

LEMMA 6.3. *Let Θ be the hexagon in $\mathcal{T}(S)$ drawn in Figure* 16. *For any two vertices u, v of $\mathcal{T}(S)$, there exists a sequence of hexagons in $\mathcal{T}(S)$, $\Pi_1, \Pi_2, \ldots, \Pi_n$, satisfying the following three conditions*:

- *For any $k = 1, \ldots, n$, there exists $\gamma \in \mathrm{Mod}(S)$ with $\Pi_k = \gamma(\Theta)$.*
- *We have $u \in \Pi_1$ and $v \in \Pi_n$.*
- *For any $k = 1, \ldots, n-1$, the intersection $\Pi_k \cap \Pi_{k+1}$ contains a 2-path.*

PROOF. Pick two vertices u, v of $\mathcal{T}(S)$. For any $\gamma \in \mathrm{Mod}(S)$, the lemma holds for u and v if and only if it holds for γu and γv. To prove the lemma, we may therefore assume

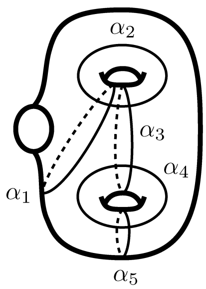


FIGURE 18

that u is a vertex of Θ. For $j = 1, \ldots, 5$, let t_j denote the Dehn twist about the curve α_j drawn in Figure 18. We set $T = \{t_1^{\pm 1}, \ldots, t_5^{\pm 1}\}$. The group Mod($S$) is known to be generated by elements of T (see [7]). Since Mod(S) transitively acts on $V_s(S)$ and on $V_{bp}(S)$, we can find an element h of Mod(S) with $v \in \{h(a), h(b)\}$, where a and b are the h-vertex and the BP-vertex of Θ, respectively, drawn in Figure 16. Write h as a product $h = h_1 \cdots h_n$ so that $h_j \in T$ for any j. For any $r \in T$, the intersection $r(\Theta) \cap \Theta$ contains a 2-path. The sequence of hexagons in $\mathcal{T}(S)$,

$$\Theta,\ h_1(\Theta)\,, \quad h_1h_2(\Theta), \ldots, h_1 \cdots h_n(\Theta) = h(\Theta)\,,$$

is thus a desired one. □

7. Construction of an automorphism of the complex of curves

Throughout this section, we set $S = S_{2,1}$. For any superinjective map ϕ from $\mathcal{T}(S)$ into itself, we construct an automorphism of $\mathcal{C}(S)$ inducing ϕ.

7.1. Surjectivity of a superinjective map. In this subsection, we show that any superinjective map from $\mathcal{T}(S)$ into itself preserves h-vertices and BP-vertices, respectively, and is surjective.

LEMMA 7.1. *Let Θ be the hexagon in $\mathcal{T}(S)$ drawn in Figure* 16. *Then for any superinjective map $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ and any $\gamma \in$* Mod(S), *the hexagon $\phi(\gamma(\Theta))$ in $\mathcal{T}(S)$ is of type* 3.

PROOF. Pick $\gamma \in$ Mod(S). The same property as that in Proposition 6.1 is satisfied by the hexagon $\gamma(\Theta)$, and hence by the image $\phi(\gamma(\Theta))$ because ϕ is superinjective. By Theorem 5.14, the hexagon $\phi(\gamma(\Theta))$ is of neither type 1 nor type 2, and is thus of type 3. □

LEMMA 7.2. *Any superinjective map from $\mathcal{T}(S)$ into itself preserves h-vertices and BP-vertices of $\mathcal{T}(S)$, respectively.*

PROOF. Let $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ be a superinjective map. Assuming that there exists an h-vertex u of $\mathcal{T}(S)$ with $\phi(u)$ a BP-vertex, we deduce a contradiction. Pick an h-vertex v of $\mathcal{T}(S)$. By Lemma 6.3, there exists a sequence of hexagons in $\mathcal{T}(S)$, $\Pi_1, \Pi_2, \ldots, \Pi_n$, such that

- any Π_k is of the form $\gamma(\Theta)$ for some $\gamma \in$ Mod(S);
- we have $u \in \Pi_1$ and $v \in \Pi_n$; and
- the intersection $\Pi_k \cap \Pi_{k+1}$ contains a 2-path for any $k = 1, \ldots, n-1$.

We note that for any $k = 1, \ldots, n$, the hexagon $\phi(\Pi_k)$ is of type 3 by Lemma 7.1, and that any edge of a hexagon in $\mathcal{T}(S)$ of type 3 consists of an h-vertex and a BP-vertex. Since u is an h-vertex and $\phi(u)$ is a BP-vertex, the map ϕ sends h-vertices of Π_1 to BP-vertices of $\phi(\Pi_1)$, and sends BP-vertices of Π_1 to h-vertices of $\phi(\Pi_1)$. Using the property that $\Pi_k \cap \Pi_{k+1}$ contains a 2-path for any $k = 1, \ldots, n-1$, we inductively see that for any $k = 1, \ldots, n$, the map ϕ sends h-vertices of Π_k to BP-vertices of $\phi(\Pi_k)$, and sends BP-vertices of Π_k to h-vertices of $\phi(\Pi_k)$. It turns out that $\phi(v)$ is a BP-vertex.

We have shown that ϕ sends any h-vertex to a BP-vertex. It therefore follows that ϕ sends an edge consisting of two h-vertices to an edge consisting of two BP-vertices. This is a contradiction because $\mathcal{T}(S)$ contains no edge consisting of two BP-vertices.

We can also deduce a contradiction along a verbatim argument if we assume that there exists a BP-vertex u of $\mathcal{T}(S)$ with $\phi(u)$ an h-vertex. $\square$

We set $Y = S_{1,2}$. To prove surjectivity of a superinjective map from $\mathcal{T}(S)$ into itself, we recall the following simplicial complexes associated to Y.

Complex $\mathcal{A}(Y)$. We define $\mathcal{A}(Y)$ to be the abstract simplicial complex such that the set of vertices of $\mathcal{A}(Y)$ is $V_a(Y)$, and a non-empty finite subset σ of $V_a(Y)$ is a simplex of $\mathcal{A}(Y)$ if and only if there exist mutually disjoint representatives of elements of σ.

Complex $\mathcal{D}(Y)$. We define $\mathcal{D}(Y)$ to be the full subcomplex of $\mathcal{A}(Y)$ spanned by all vertices that correspond to essential simple arcs in Y connecting the two boundary components of Y.

REMARK 7.3. Let us describe simplices of $\mathcal{D}(Y)$ of maximal dimension. We denote by Y_0 the surface obtained from Y by shrinking each component of ∂Y to a point. Let $P = \{x_1, x_2\}$ denote the set of the two points of Y_0 into which components of ∂Y are shrunken. The natural map from Y onto Y_0 induces the bijection from $V_a(Y)$ onto the set of isotopy classes of ideal arcs in the punctured surface (Y_0, P). It turns out that a simplex of $\mathcal{A}(Y)$ of maximal dimension corresponds to an ideal triangulation of (Y_0, P), and that a simplex of $\mathcal{D}(Y)$ of maximal dimension corresponds to an ideal squaring of (Y_0, P) defined as follows. We mean by an *ideal squaring* of (Y_0, P) a cell division δ of Y_0 such that

- the set of 0-cells of δ is P;
- any 1-cell of δ is an ideal arc in (Y_0, P) connecting x_1 and x_2; and
- any 2-cell of δ is a *square*, that is, it is obtained by attaching a Euclidean square τ to the 1-skeleton of δ, mapping each vertex of τ to a 0-cell of δ, and each edge of τ to a 1-cell of δ.

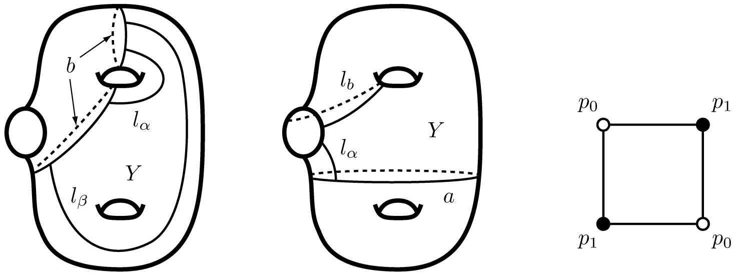
By argument on the Euler characteristic of Y_0, for any ideal squaring δ of (Y_0, P), the numbers of 1-cells and 2-cells of δ are equal to 4 and 2, respectively.

We will use the following:

PROPOSITION 7.4. *We set $Y = S_{1,2}$. Then any injective simplicial map from $\mathcal{D}(Y)$ into itself is surjective.*

This proposition follows from [13, Proposition 3.1 and Lemma 3.2]. For a vertex v of $\mathcal{T}(S)$, we denote by $\mathrm{Lk}(v)$ the link of v in $\mathcal{T}(S)$.

LEMMA 7.5. *Let b be a BP-vertex of $\mathcal{T}(S)$ and $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ a superinjective map. Then the equality $\phi(\mathrm{Lk}(b)) = \mathrm{Lk}(\phi(b))$ holds.*



(a) (b) (c)

FIGURE 19

PROOF. We may assume $\phi(b) = b$. Let Y denote the component of S_b homeomorphic to $S_{1,2}$. As noted right after Lemma 4.4, there is a one-to-one correspondence between elements of $V_s(Y)$ and elements of $V_a(Y)$ whose representatives connect the two components of ∂Y. For $\alpha \in V_s(Y)$, we denote by l_α the element of $V_a(Y)$ corresponding to α. By Theorem 4.2 (ii), for any two distinct vertices $\alpha, \beta \in V_s(Y)$, there is a hexagon in $\mathcal{T}(S)$ of type 1 containing b, α and β if and only if l_α and l_β are disjoint (see Figure 19 (a) for such two disjoint arcs). The map ϕ induces an injective simplicial map from $\mathcal{D}(Y)$ into itself because ϕ preserves hexagons in $\mathcal{T}(S)$ of type 1 by Lemma 7.2. Proposition 7.4 implies the equality in the lemma. □

LEMMA 7.6. *Let a be an h-vertex of $\mathcal{T}(S)$ and $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ a superinjective map. Then the equality $\phi(\mathrm{Lk}(a)) = \mathrm{Lk}(\phi(a))$ holds.*

PROOF. We may assume $\phi(a) = a$. Let Y denote the component of S_a homeomorphic to $S_{1,2}$. Note that $V_s(Y)$ and $V_{bp}(Y)$ are naturally identified with sets of vertices of $\mathrm{Lk}(a)$. An argument similar to the proof of Lemma 7.5 shows that ϕ induces an injective simplicial map $\tilde{\phi}\colon \mathcal{D}(Y) \to \mathcal{D}(Y)$, which is surjective by Proposition 7.4. It follows that ϕ sends $V_s(Y)$ onto itself.

We prove that ϕ sends $V_{bp}(Y)$ onto itself. As noted in the second paragraph of Section 6, there is a one-to-one correspondence between elements of $V_{bp}(Y)$ and elements of $V_a(Y)$ whose representatives are non-separating in Y and connect two points in the component of ∂Y corresponding to ∂S. For $b \in V_{bp}(Y)$, we denote by l_b the element of $V_a(Y)$ corresponding to b. We use the same symbol as in the proof of Lemma 7.5. Namely, for $\alpha \in V_s(Y)$, we denote by l_α the element of $V_a(Y)$ corresponding to α. By Theorem 4.2 (ii), for any $b \in V_{bp}(Y)$ and $\alpha \in V_s(Y)$, there exists a hexagon in $\mathcal{T}(S)$ of type 1 containing a, b and α if and only if l_b and l_α are disjoint (see Figure 19 (b) for such two disjoint arcs).

Pick a simplex σ of $\mathcal{D}(Y)$ of maximal dimension. Let Y_0 denote the surface obtained from Y by shrinking each component of ∂Y to a point. Let P denote the set of points of Y_0

into which components of ∂Y are shrunken. We then obtain the punctured surface (Y_0, P). Let p_0 denote the point of P into which ∂S is shrunken. Let p_1 denote the other point of P. As discussed in Remark 7.3, we have the ideal squaring of (Y_0, P) corresponding to σ, and the set of whose 2-cells consists of two squares. In each of those two squares, as in Figure 19 (c), two opposite vertices correspond to p_0, the other two vertices correspond to p_1, and we have an arc connecting the two vertices corresponding to p_0 and dividing the square into two triangles. It follows from Lemma 2.2 that up to isotopy, there exist exactly two ideal arcs in (Y_0, P) disjoint from any ideal arc corresponding to an element of σ and both of whose end points are p_0. We define $L(\sigma)$ as the subset of $V_a(Y)$ consisting of the two elements that correspond to those ideal arcs. Any arc in $L(\sigma)$ is non-separating in Y because for any essential simple arc l in Y that is separating in Y, a vertex of $\mathcal{D}(Y)$ whose representative is disjoint from l uniquely exists and because we have $|\sigma| = 4$.

The claim in the end of the second paragraph of the proof and injectivity of ϕ imply that for any simplex σ of $\mathcal{D}(Y)$ of maximal dimension, the map ϕ induces a bijection from $L(\sigma)$ onto $L(\tilde{\phi}(\sigma))$. For any $b \in V_{bp}(Y)$, there exists a simplex of $\mathcal{D}(Y)$ of maximal dimension any of whose arcs is disjoint from l_b. Surjectivity of the map $\tilde{\phi}\colon \mathcal{D}(Y) \to \mathcal{D}(Y)$ therefore implies that ϕ sends $V_{bp}(Y)$ onto itself. □

The last two lemmas and connectivity of $\mathcal{T}(S)$ imply the following:

THEOREM 7.7. *Any superinjective map from $\mathcal{T}(S)$ into itself is surjective and is thus an automorphism of $\mathcal{T}(S)$.*

7.2. Construction of a map from $V(S)$ into itself. Let ϕ be an automorphism of $\mathcal{T}(S)$. We define a map $\Phi\colon V(S) \to V(S)$ as follows. Pick an element α of $V(S)$. If α is separating in S, then we set $\Phi(\alpha) = \phi(\alpha)$. If α is non-separating in S, then pick a hexagon Π in $\mathcal{T}(S)$ of type 1 such that α is contained in the two BP-vertices of Π, and define $\Phi(\alpha)$ to be the non-separating curve in S contained in the two BP-vertices of the hexagon $\phi(\Pi)$ of type 1.

We will prove that Φ is well-defined as a consequence of Lemma 7.9. To prove it, let us introduce the following:

Graph $\mathcal{E}$. We define $\mathcal{E}$ to be the simplicial graph so that the set of vertices of $\mathcal{E}$ is $V_{bp}(S)$, and two distinct vertices u, v of $\mathcal{E}$ are connected by an edge of $\mathcal{E}$ if and only if there exists a hexagon in $\mathcal{T}(S)$ of type 1 containing u and v.

We mean by a *square* in $\mathcal{E}$ the full subgraph of $\mathcal{E}$ spanned by exactly four vertices $v_1, \ldots, v_4$ such that for any k mod 4, v_k and v_{k+1} are adjacent, and v_k and v_{k+2} are not adjacent. In this case, let us say that the square is defined by the 4-tuple $(v_1, \ldots, v_4)$.

LEMMA 7.8. *Let $(v_1, \ldots, v_4)$ be a 4-tuple defining a square in $\mathcal{E}$. Then there exists a non-separating curve α in S with $\alpha \in v_k$ for any $k = 1, \ldots, 4$.*

PROOF. By the definition of $\mathcal{E}$, for any two adjacent vertices of $\mathcal{E}$, the two BPs in S corresponding to them share a non-separating curve in S. For each k mod 4, let β_k denote the non-separating curve in S contained in v_k and v_{k+1}. Without loss of generality, it suffices to deduce a contradiction under the assumption $\beta_1 \neq \beta_2$.

Let $\bar{S}$ denote the closed surface obtained from S by attaching a disk to ∂S, and let $\pi: \mathcal{C}(S) \to \mathcal{C}(\bar{S})$ be the simplicial map associated with the inclusion of S into $\bar{S}$. Since π sends the two curves in any BP in S to the same curve in $\bar{S}$, all curves in the BPs $v_1, \ldots, v_4$ are sent to the same curve in $\bar{S}$, denoted by α_0. In other words, all curves in the BPs $v_1, \ldots, v_4$ are in $\pi^{-1}(\alpha_0)$. Let T denote the full subcomplex of $\mathcal{C}(S)$ spanned by $\pi^{-1}(\alpha_0)$, which is a tree by Theorem 3.2. The sequence, $\beta_1, \beta_2, \beta_3, \beta_4, \beta_1$, forms a closed path in T.

We assume $\beta_1 \neq \beta_2$. The equality $v_2 = \{\beta_1, \beta_2\}$ then holds. We have $\beta_3 \neq \beta_1$ and $\beta_4 \neq \beta_2$ because $v_3 \neq v_2$ and $v_1 \neq v_2$. Let γ and δ denote the curves in S with $v_1 = \{\beta_1, \gamma\}$ and $v_3 = \{\beta_2, \delta\}$. Each of γ and δ is equal to neither β_1 nor β_2 because $v_1 \neq v_2$ and $v_3 \neq v_2$. We have either $\beta_4 = \gamma$ or $\beta_3 = \delta$ because otherwise we would have $v_2 = v_4$.

If $\beta_4 = \gamma$, then we have $\beta_3 \neq \beta_2$ and $\beta_3 \neq \gamma$ because otherwise the sequence, $\beta_1, \beta_2, \gamma, \beta_1$, would form a simple closed path in T. It turns out that $\beta_1, \beta_2, \beta_3$ and β_4 are mutually distinct. This is a contradiction.

If $\beta_3 = \delta$, then we have $\beta_4 \neq \beta_1$ and $\beta_4 \neq \delta$ because otherwise the sequence, $\beta_1, \beta_2, \delta, \beta_1$, would form a simple closed path in T. It turns out that $\beta_1, \beta_2, \beta_3$ and β_4 are mutually distinct. This is also a contradiction. □

LEMMA 7.9. *Let ϕ be an automorphism of $\mathcal{T}(S)$. Let α be a non-separating curve in S. Pick two hexagons Π, Ω in $\mathcal{T}(S)$ of type* 1 *such that any BP-vertex of Π and Ω contains α. Then the non-separating curve in S contained in the two BP-vertices of $\phi(\Pi)$ is equal to that of $\phi(\Omega)$.*

PROOF. Let a_1 and a_2 denote the two BP-vertices of Π. Let b_1 and b_2 denote the two BP-vertices of Ω.

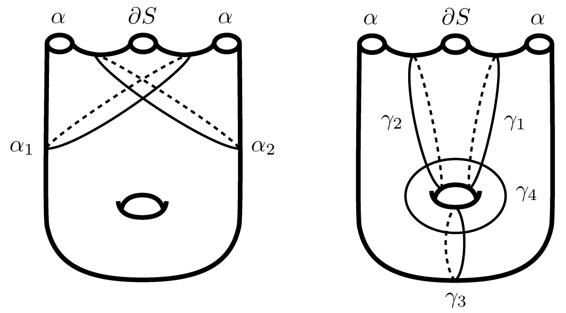
CLAIM 7.10. *There exists a sequence of squares in $\mathcal{E}$, $\Delta_1, \ldots, \Delta_n$, satisfying the following three conditions*:

- *The square Δ_1 contains a_1 and a_2, and the square Δ_n contains b_1 and b_2.*
- *For any $k = 1, \ldots, n$, any vertex of Δ_k contains α.*
- *For any $k = 1, \ldots, n-1$, the intersection $\Delta_k \cap \Delta_{k+1}$ contains an edge of $\mathcal{E}$.*

PROOF. Let $\alpha_1, \alpha_2, \beta_1$ and β_2 be the curves in S with $a_j = \{\alpha, \alpha_j\}$ and $b_j = \{\alpha, \beta_j\}$ for $j = 1, 2$. The curves α_1 and α_2 can be drawn as in Figure 20 (a), where the surface S_α obtained by cutting S along α is drawn. We define $\gamma_1, \ldots, \gamma_4$ as the curves in S drawn in Figure 20 (b). The equality $i(\alpha_1, \gamma_1) = i(\alpha_2, \gamma_2) = 0$ then holds. For $j = 1, \ldots, 4$, let $t_j \in \mathrm{Mod}(S)$ denote the Dehn twist about γ_j. Let $\mathrm{Mod}(S)_\alpha$ denote the stabilizer of α in $\mathrm{Mod}(S)$, and define

$$q: \mathrm{Mod}(S)_\alpha \to \mathrm{Mod}(S_\alpha)$$

as the natural homomorphism. The group $\mathrm{PMod}(S_\alpha)$ is known to be generated by $q(t_1), \ldots, q(t_4)$ (see [7]). By Theorem 4.1, there exists an element h of $\mathrm{PMod}(S_\alpha)$ with



(a) (b)

FIGURE 20

$\{h(\alpha_1), h(\alpha_2)\} = \{\beta_1, \beta_2\}$. Write h as a product $h = q(h_1)\cdots q(h_n)$ so that $h_j \in \{t_1^{\pm 1}, \ldots, t_4^{\pm 1}\}$ for any j.

The 4-tuple $(a_1, a_2, t_2(a_1), t_1(a_2))$ defines a square in $\mathcal{E}$. We denote it by Δ. For any $w \in \{t_1^{\pm 1}, \ldots, t_4^{\pm 1}\}$, the intersection $\Delta \cap w(\Delta)$ contains an edge of $\mathcal{E}$. The sequence of squares in $\mathcal{E}$,

$$\Delta, \quad h_1(\Delta), \quad h_1h_2(\Delta), \ldots, h_1h_2\cdots h_n(\Delta) = h(\Delta),$$

is therefore a desired one. □

By the definition of $\mathcal{E}$, the automorphism ϕ of $\mathcal{T}(S)$ induces an automorphism of $\mathcal{E}$. If $\Delta_1, \ldots, \Delta_n$ are the squares in $\mathcal{E}$ chosen in Claim 7.10, then $\phi(\Delta_1), \ldots, \phi(\Delta_n)$ are also squares in $\mathcal{E}$ such that

- $\phi(\Delta_1)$ contains $\phi(a_1)$ and $\phi(a_2)$, and $\phi(\Delta_n)$ contains $\phi(b_1)$ and $\phi(b_2)$; and
- for any $k = 1, \ldots, n-1$, the intersection $\phi(\Delta_k) \cap \phi(\Delta_{k+1})$ contains an edge of $\mathcal{E}$.

It follows from Lemma 7.8 that for any $k = 1, \ldots, n$, there exists a non-separating curve in S contained in any vertex of $\phi(\Delta_k)$. The above second condition implies that this curve does not depend on k. In particular, the curve shared by $\phi(a_1)$ and $\phi(a_2)$ is equal to the curve shared by $\phi(b_1)$ and $\phi(b_2)$. □

Lemma 7.9 implies that the map $\Phi\colon V(S) \to V(S)$ constructed in the beginning of this subsection is well-defined.

LEMMA 7.11. *Let ϕ be an automorphism of $\mathcal{T}(S)$, and let $\Phi\colon V(S) \to V(S)$ be the map defined in the beginning of this subsection. Then Φ defines a simplicial map from $\mathcal{C}(S)$ into itself.*

PROOF. Let α and β be distinct elements of $V(S)$ with $i(\alpha, \beta) = 0$. We have to show $i(\Phi(\alpha), \Phi(\beta)) = 0$. If both α and β are separating in S, then this equality follows from the definition of Φ and simpliciality of ϕ. If α and β are non-separating curves in S with $\{\alpha, \beta\}$ a BP in S, then $\Phi(\alpha)$ and $\Phi(\beta)$ are curves in the BP $\phi(\{\alpha, \beta\})$ by the definition of Φ. The equality $i(\Phi(\alpha), \Phi(\beta)) = 0$ thus holds.

For $\gamma \in V_s(S)$, we denote by H_γ the component of S_γ that is a handle. If α and β are non-separating curves in S such that $\{\alpha, \beta\}$ is not a BP in S, then there exist $\gamma, \delta \in V_s(S)$ with $\gamma \neq \delta$, $i(\gamma, \delta) = 0$, $\alpha \in V(H_\gamma)$ and $\beta \in V(H_\delta)$. We can then find two hexagons Π, Ω in $\mathcal{T}(S)$ of type 1 such that

- the two BP-vertices of Π contain α, and those of Ω contain β; and
- both Π and Ω contain the h-vertices γ and δ.

By the definition of Φ, we have $\Phi(\alpha) \in V(H_{\phi(\gamma)})$ and $\Phi(\beta) \in V(H_{\phi(\delta)})$. Since we have $\phi(\gamma) \neq \phi(\delta)$ and $i(\phi(\gamma), \phi(\delta)) = 0$, the equality $i(\Phi(\alpha), \Phi(\beta)) = 0$ holds.

If α is non-separating in S and β is separating in S, then one can find two distinct BPs a_1, a_2 in S containing α and a hexagon Π in $\mathcal{T}(S)$ of type 1 such that Π contains a_1, a_2 and β as its vertices. By the definition of Φ, the curve $\Phi(\alpha)$ is disjoint from any curve corresponding to an h-vertex of $\phi(\Pi)$. We therefore have $i(\Phi(\alpha), \Phi(\beta)) = 0$. $\square$

Let ϕ be an automorphism of $\mathcal{T}(S)$. We have constructed the simplicial map $\Phi\colon \mathcal{C}(S) \to \mathcal{C}(S)$ associated to ϕ. The simplicial map from $\mathcal{C}(S)$ into itself associated to ϕ^{-1} is then the inverse of Φ. The map Φ is therefore an automorphism of $\mathcal{C}(S)$ and is induced by an element of $\mathrm{Mod}^*(S)$ by Theorem 2.1. If $\{\alpha, \beta\}$ is a BP in S, then $\Phi(\alpha)$ and $\Phi(\beta)$ are curves in the BP $\phi(\{\alpha, \beta\})$ by the definition of Φ, and are distinct because Φ is an automorphism of $\mathcal{C}(S)$. We thus have the equality $\{\Phi(\alpha), \Phi(\beta)\} = \phi(\{\alpha, \beta\})$. It follows that ϕ is induced by an element of $\mathrm{Mod}^*(S)$. Combining this with Theorem 7.7, we obtain the following:

THEOREM 7.12. *Any superinjective map from $\mathcal{T}(S)$ into itself is induced by an element of* $\mathrm{Mod}^*(S)$.

A. The minimal length of simple cycles

Let $\mathcal{G}$ be a simplicial graph. We mean by a *simple cycle* in $\mathcal{G}$ a subgraph of $\mathcal{G}$ obtained as the image of a simple closed path in $\mathcal{G}$. A simple cycle in $\mathcal{G}$ is called *non-trivial* if its length is positive, where any edge of $\mathcal{G}$ is defined to be of length 1.

Throughout this appendix, we set $S = S_{2,1}$. We aim to show the following:

PROPOSITION A.1. *There exists no non-trivial simple cycle in $\mathcal{T}(S)$ of length at most* 5.

It turns out that hexagons in $\mathcal{T}(S)$ are simple cycles in $\mathcal{T}(S)$ of minimal positive length. We first prove the following:

LEMMA A.2. *There exists no non-trivial simple cycle in $\mathcal{T}(S)$ of length at most* 4.

PROOF. Since $\mathcal{T}(S)$ is one-dimensional, there exists no simple cycle in $\mathcal{T}(S)$ of length 3. Assume that there are four vertices $v_1, \ldots, v_4$ of $\mathcal{T}(S)$ with $i(v_j, v_{j+1}) = 0$ and $i(v_j, v_{j+2}) \neq 0$ for any j mod 4. We can find $\alpha_1, \ldots, \alpha_4 \in V(S)$ such that

- for any $j = 1, \ldots, 4$, we have either $\alpha_j = v_j \in V_s(S)$ or $v_j \in V_{bp}(S)$ and $\alpha_j \in v_j$; and
- for any $k = 1, 2$, we have $i(\alpha_k, \alpha_{k+2}) \neq 0$.

For a surface X, we denote by $\chi(X)$ the Euler characteristic of X. For $k = 1, 2$, we define Q_k as the subsurface of S filled by α_k and α_{k+2}. If $|\chi(Q_k)| \geq 2$, then set $R_k = Q_k$. If $|\chi(Q_k)| = 1$, then Q_k is a handle, and α_k and α_{k+2} are non-separating in S. It follows that v_k and v_{k+2} are BPs in S. The curve in v_k distinct from α_k, denoted by β_k, intersects the h-curve in S corresponding to the boundary of Q_k. In the case of $|\chi(Q_k)| = 1$, we define R_k as the subsurface of S filled by the three curves α_k, β_k and α_{k+2}. In both cases, R_1 and R_2 can be realized so that they are disjoint, and we have $|\chi(R_1)| \geq 2$ and $|\chi(R_2)| \geq 2$. This contradicts $|\chi(S)| = 3$. The lemma follows. $\square$

The proof of Proposition A.1 reduces to showing the following:

LEMMA A.3. *There exists no pentagon in $\mathcal{T}(S)$.*

PROOF. Let $\bar{S}$ denote the closed surface obtained from S by attaching a disk to ∂S. We have the simplicial maps

$$\pi\colon \mathcal{C}(S) \to \mathcal{C}(\bar{S}), \quad \theta\colon \mathcal{T}(S) \to \mathcal{C}(\bar{S})$$

associated with the inclusion of S into $\bar{S}$. Note that θ sends each BP-vertex of $\mathcal{T}(S)$ to a vertex of $\mathcal{C}(\bar{S})$ corresponding to a non-separating curve in $\bar{S}$, and that both π and θ send any two adjacent h-vertices to the same h-vertex of $\mathcal{C}(\bar{S})$. Since the fiber of π over each vertex of $\mathcal{C}(\bar{S})$ is a tree by Theorem 3.2, there exists no pentagon in $\mathcal{T}(S)$ consisting of only h-vertices. We thus have to show non-existence of pentagons in $\mathcal{T}(S)$ having one or two BP-vertices.

CLAIM A.4. *There exists no pentagon in $\mathcal{T}(S)$ defined by a 5-tuple (a, b, c, d, e) such that a, c and e are h-vertices and b and d are BP-vertices.*

PROOF. Assuming that such a 5-tuple (a, b, c, d, e) exists, we deduce a contradiction. Choose representatives $A, \ldots, E$ of $a, \ldots, e$, respectively, such that any two of them intersect minimally. Let R denote the component of S_C that is not a handle. Since B is a BP disjoint from A and C, the intersection $A \cap R$ consists of mutually isotopic, essential simple arcs in R which are non-separating in R. Similarly, $E \cap R$ also consists of mutually isotopic, essential simple arcs in R which are non-separating in R. Let l_A be a component of $A \cap R$, and let l_E be a component of $E \cap R$. The arcs l_A and l_E are not isotopic because otherwise the equality $b = d$ would hold. Since A and E are disjoint, l_A and l_E are also disjoint.

We first assume that along C, the end points of l_A first appear, and those of l_E then appear. Cut R along l_A. We then obtain a pair of pants. Up to a homeomorphism of that pair

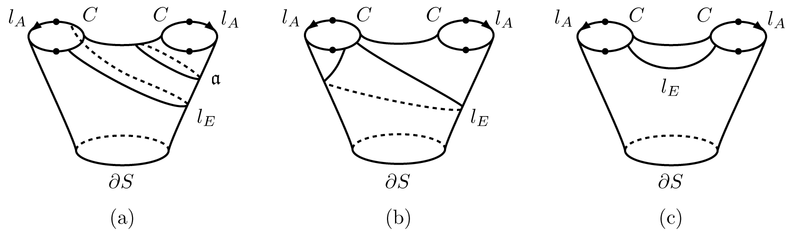


FIGURE 21. The pair of pants obtained by cutting R along l_A.

of pants fixing points of its boundary, the arc l_E is drawn as in Figure 21 (a) or (b). In Figure 21 (b), the union of l_E and a subarc of C cuts off an annulus containing ∂S from S, and this contradicts that l_E is non-separating in R. The arc l_E is thus drawn as in Figure 21 (a). The curve $\mathfrak{a}$ in Figure 21 (a) is a boundary component of a regular neighborhood of $l_A \cup C$ in S, and is also that of $l_E \cup C$. It follows that the isotopy class of $\mathfrak{a}$ is contained in b and d. The surface $S_{\mathfrak{a}}$ obtained by cutting S along $\mathfrak{a}$ is homeomorphic to $S_{1,3}$. The curve A is an h-curve in $S_{\mathfrak{a}}$ because A is disjoint from the BP B in S. Similarly, E is also an h-curve in $S_{\mathfrak{a}}$ because E is disjoint from the BP D in S. Since A and E are disjoint h-curves in $S_{\mathfrak{a}}$, they have to be isotopic. This is a contradiction.

We next assume that along C, the end points of l_A and l_E appear alternately. Cut R along l_A. We then obtain a pair of pants. Up to a homeomorphism of that pair of pants fixing points of its boundary, the arc l_E is drawn as in Figure 21 (c). We then have $i(\theta(b), \theta(d)) = 1$. The curves $\theta(b)$ and $\theta(d)$ fill a component of $\bar{S}_{\theta(c)}$. The equality $\theta(a) = \theta(e)$ holds because a and e are adjacent h-vertices. It follows that $\theta(a)$ is an h-curve in $\bar{S}$ disjoint from $\theta(b)$ and $\theta(d)$. We thus have the equality $\theta(a) = \theta(c)$. On the other hand, A and C are curves in the component of S_B that does not contain ∂S. The equality $\theta(a) = \theta(c)$ implies the equality $a = c$. This is a contradiction. □

CLAIM A.5. *There exists no pentagon in $\mathcal{T}(S)$ defined by a 5-tuple (a, b, c, d, e) such that a, c, d and e are h-vertices and b is a BP-vertex.*

PROOF. Assume that such a 5-tuple (a, b, c, d, e) exists. We have the equality $\theta(a) = \theta(e) = \theta(d) = \theta(c)$. We can deduce a contradiction along the argument in the end of the proof of Claim A.4. □

Claims A.4 and A.5 complete the proof of Lemma A.3. □

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