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On the Kernel of the Reciprocity Map of Simple Normal Crossing Varieties over Finite Fields

Rin SUGIYAMA

Nagoya University

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Abstract. In this paper, we study the kernel of the reciprocity map of certain simple normal crossing varieties over a finite field and give an example of a simple normal crossing surface whose reciprocity map is not injective for any finite scalar extension.

1. Introduction

The reciprocity map of the unramified class field theory for a proper variety X over a finite field k is a homomorphism of the following form:

$$\rho_X : \mathrm{CH}_0(X) \longrightarrow \pi_1^{\mathrm{ab}}(X).$$

Here $\mathrm{CH}_0(X)$ is the Chow group of 0-cycles on X modulo rational equivalence, and $\pi_1^{\mathrm{ab}}(X)$ is the abelian étale fundamental group of X. The map ρ_X is defined by sending the class of a closed point x to the Frobenius substitution at x. If X is normal, ρ_X has dense image [5]. If X is smooth, ρ_X is injective [4]. We also know that there is a projective normal surface X for which ρ_X is not injective [6], and that there is a simple normal crossing surface X over k for which ρ_X/n is not injective but $\rho_{X\otimes E}/n$ is injective for any sufficiently large finite extension E/k and some $n > 1$ [7]. Here a normal crossing variety X over k is a equidimensional separated scheme of finite type over k which is everywhere étale locally isomorphic to

$$\mathrm{Spec}\big(k[T_0, \dots, T_d]/(T_0T_1\cdots T_r)\big) \quad (0 \le r \le d = \dim X)\,.$$

A normal crossing variety X over k is called *simple* if any irreducible component of X is smooth over k. For any simple normal crossing variety X, we have an exact sequence (cf. [3])

$$H_2(\Gamma_X, \mathbf{Z}/n) \xrightarrow{\varepsilon_{X,n}} \mathrm{CH}_0(X)/n \xrightarrow{\rho_X/n} \pi_1^{\mathrm{ab}}(X)/n\,, \tag{1.1}$$

where Γ_X is the dual graph of X which is a finite simplicial complex. By studying on the map $\varepsilon_{X,n}$, one can see as to whether ρ_X/n is injective or not. However $\varepsilon_{X,n}$ is abstract and difficult to compute directly. In this paper, we study $\varepsilon_{X,n}$ and $\mathrm{Ker}(\rho_X)$ for a certain simple normal

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crossing variety, and construct a simple normal crossing variety X over k for which $\rho_{X\otimes F}/n$ is not injective for any finite extension F/k and some $n>0$.

This paper is organized as follows: In Section 2, we investigate the kernel of the reciprocity map ρ_Y for a certain simple normal crossing variety Y by using a method of Matsumi-Sato-Asakura [6]. In Section 3, we construct a simple normal crossing surface over a finite field whose reciprocity map ρ_Y is potentially not injective. In Appendix, we prove some lemmas on simple normal crossing varieties over finite fields which are used in Section 2.

Notation

(1) For an abelian group A and a positive integer n, A/n denotes the cokernel of the map $A \xrightarrow{\times n} A$. A_{tors} denotes the torsion subgroup of A. $A^{\oplus n}$ denotes the direct sum of n copies of A.

(2) For a field k, $k^\times$ denotes the multiplicative group, k^{sep} denotes a fixed separable closure, G_k denotes the absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, G_k^{ab} denotes the maximal abelian quotient group of G_k. For a connected scheme X, $\pi_1^{\mathrm{ab}}(X)$ denotes the abelian étale fundamental group. Further, for a non-connected scheme V, $\pi_1^{\mathrm{ab}}(V)$ denotes $\bigoplus_i \pi_1^{\mathrm{ab}}(V_i)$ where V_i are connected components of V. For k-scheme X, $\pi_1^{\mathrm{geo}}(X)$ denotes $\mathrm{Ker}\big(\pi_1^{\mathrm{ab}}(X) \longrightarrow G_k^{\mathrm{ab}}\big)$.

(3) Let k be a field and X be a k-scheme. For a field extension F/k, $X\otimes_k F$ denotes $X\times_{\mathrm{Spec}(k)} \mathrm{Spec}(F)$. Especially, for a fixed separable closure k^{sep}/k, $\overline{X}$ denotes $X\times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^{\mathrm{sep}})$.

(4) For a scheme X of finite type over a field and an integer $q\geq 0$, X_q denotes the set of points on X which $\dim(\overline{\{x\}})=q$. For a point $x\in X$, $\kappa(x)$ denotes the residue field. For a scheme X of finite type over a field k, we define the following group:

$$\mathrm{CH}_0(X) := \mathrm{Coker}\Bigg(\partial_1 : \bigoplus_{x\in X_1} \kappa(x)^\times \longrightarrow \bigoplus_{x\in X_0} \mathbf{Z}\Bigg),$$

where ∂_1 is defined by the discrete valuation.

If X is proper over k, there is the degree map

$$\deg_{X/k} : \mathrm{CH}_0(X) \to \mathbf{Z},$$

$A_0(X)$ denotes its kernel.

(5) $H^r(-,-)$ denotes an étale cohomology group. Especially, $H^r(F,-)$ denotes $H^r(\mathrm{Spec}(F),-)$ for a field F.

For a separated scheme X of finite type over k and a natural number n, we define the étale homology with coefficient $\mathbf{Z}/n$ to

$$H_i(X,\mathbf{Z}/n) := \mathrm{Hom}\big(H_c^i(X,\mathbf{Z}/n), \mathbf{Q}/\mathbf{Z}\big).$$

Here $H_c^i(-,-)$ denotes an étale cohomology group with compact support. This functor $H_*(-,\mathbf{Z}/n)$ forms a homology theory on the category of separated schemes of finite type over k and proper k-morphisms ([2, (1.2), (2.1)]).

(6) For a simple normal crossing variety X over k, we use the following notation: Let $\{X_i\}_{i\in I}$ be the set of irreducible components of X. For a positive integer r, we define

$$X^{(r)} := \coprod_{\{i_1,i_2,\dots,i_r\}\subset I} X_{i_1} \times_X X_{i_2} \times_X \cdots \times_X X_{i_r} .$$

We define a simplicial complex Γ_X called the *dual graph* of X as follows:

Fix an ordering on I. The set of r-simplexes $\mathfrak{S}_r$ of Γ_X is the set of irreducible components of $X^{(r)}$. We determine the orientation on r-simplexes inductively on r by the fixed ordering on I (cf. [3, §3]).

Let F/k be an algebraic extension. We put $Y := X \otimes_k F$. Let $\{Y_j\}_{j\in J}$ be the set of irreducible components of Y. Then we define the semi-order on J as follows: for $j_1, j_2 \in J$,

$$j_1 < j_2 \Longleftrightarrow \phi(j_1) < \phi(j_2) ,$$

where $\phi : J \longrightarrow I$ is the map which sends j to $\phi(j)$ when Y_j lies above $X_{\phi(j)}$. By using this order on J, we define the homomorphism of the complexes

$$\sigma_{F/k} : \Gamma_Y \longrightarrow \Gamma_X .$$

Then the homomorphism $H_a(\Gamma_Y, \mathbf{Z}) \longrightarrow H_a(\Gamma_X, \mathbf{Z})$ induced by $\sigma_{F/k}$ is called *norm map*.

2. The kernel of the reciprocity map

In this section, we study the kernel of the reciprocity map ρ_Y for a variety Y of the following form by using a method of Matsumi-Sato-Asakura [6].

Let Y_0 is a projective smooth and geometrically irreducible variety over a finite field k and D be a simple normal crossing divisor on Y_0. We put $O := (0 : 1), \infty := (1 : 0) \in \mathbf{P}^1_k$. We then consider the following simple normal crossing variety:

$$Y := \big(Y_0 \times_k O\big) \cup \big(Y_0 \times_k \infty\big) \cup \big(D \times_k \mathbf{P}^1\big) \quad \subset Y_0 \times_k \mathbf{P}^1 .$$

We will construct the following map δ_Y whose image coincides with $\mathrm{Ker}(\rho_Y)$:

$$\delta_Y : H_1(\Gamma_D, \mathbf{Z}) \longrightarrow \mathrm{CH}_0(Y) .$$

We then consider the group

$$G(Y) := \mathrm{Im}(\delta_Y \circ \sigma) \subset \mathrm{Ker}(\rho_Y),$$

where $\sigma : H_1(\Gamma_{\overline{D}}, \mathbf{Z}) \to H_1(\Gamma_D, \mathbf{Z})$ is the norm map. The group $G(Y)$ is related to the following group and map

$$\Theta_\ell := \mathrm{Coker}\left(\bigoplus_j \pi_1^{\mathrm{ab}}(\overline{D}_j)^{\text{pro-}\ell} \longrightarrow \pi_1^{\mathrm{ab}}(\overline{Y_0})^{\text{pro-}\ell}\right),$$

$$\alpha^{(\ell)} : H_1(\Gamma_{\overline{D}}, \mathbf{Z}_\ell) \longrightarrow \Theta_\ell .$$

Here $\overline{Y_0} := Y_0 \otimes_k k^{\mathrm{sep}}$ and $\overline{D}_j$ denotes irreducible component of $\overline{D} := D \otimes_k k^{\mathrm{sep}}$, and $\pi_1^{\mathrm{ab}}(-)^{\text{pro-}\ell}$ denotes the maximal pro-ℓ-quotient of $\pi_1^{\mathrm{ab}}(-)$.

We will prove the following theorem which is an analogy of a result of [6] for certain simple normal crossing varieties over finite fields.

THEOREM 2.1. *Let ℓ be an arbitrary prime number.*

(1) *The ℓ-primary part $G(Y)\{\ell\}$ of $G(Y)$ is a subquotient of $\left(\Theta_\ell\right)_{\mathrm{tors}}$.*

(2) *Assume that*

  (i) *each connected components of $Y^{(2)}$ has a k-rational point,*

  (ii) *G_k acts on $\left(\Theta_\ell\right)_{\mathrm{tors}}$ trivially.*

Then $G(Y)\{\ell\}$ is isomorphic to the image of the map $\alpha^{(\ell)}$.

The remarkable points in Theorem 2.1 are that the map $\alpha^{(\ell)}$ does not vary for finite scalar extensions, and that the group $\mathrm{Im}(\alpha^{(\ell)})$ is related to $G(Y)$. Therefore by studying on $G(Y)$ and using the map $\alpha^{(\ell)}$, one can see as to whether the reciprocity map ρ_Y is potentially injective or not.

2.1. Construction of δ_Y

PROPOSITION 2.2. *There exists a homomorphism*

$$\delta_Y : H_1(\Gamma_D, \mathbf{Z}) \longrightarrow \mathrm{CH}_0(Y)$$

whose image coincides with $\mathrm{Ker}(\rho_Y)$.

PROOF. We consider the following variety and two closed subschemes:

$$\begin{aligned} S &:= \left(Y_0 \times_k O\right) \sqcup \left(Y_0 \times_k \infty\right) \sqcup \left(D \times_k \mathbf{P}^1\right), \\ Z &:= \left(Y_0 \times_k O\right) \cup \left(Y_0 \times_k \infty\right) \subset Y, \\ Z' &:= \left(Y_0 \times_k O\right) \sqcup \left(Y_0 \times_k \infty\right) \sqcup \left(D \times_k \{O, \infty\}\right) \subset S. \end{aligned}$$

Then we have

$$Y \backslash Z \cong S \backslash Z' \simeq D \times \mathbf{G}_m\,. \tag{2.1}$$

From this isomorphism, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd} \mathrm{CH}_0(Z') \arrow[r, "\beta"] \arrow[d] & \mathrm{CH}_0(S) \arrow[r] \arrow[d] & \mathrm{CH}_0(D \times \mathbf{G}_m) \arrow[d, equal] \\ \mathrm{CH}_0(Z) \arrow[r] & \mathrm{CH}_0(Y) \arrow[r] & \mathrm{CH}_0(D \times \mathbf{G}_m)\,. \end{tikzcd} \tag{2.2}$$

Now we have $\mathrm{CH}_0(D \times \mathbf{G}_m) = 0$. We compute the kernel of the map β. Since there are the following isomorphisms

$$\mathrm{CH}_0(Z') \simeq \mathrm{CH}_0(Y_0)^{\oplus 2} \oplus \mathrm{CH}_0(D)^{\oplus 2}\,,$$

$$\mathrm{CH}_0(S) \simeq \mathrm{CH}_0(Y_0)^{\oplus 2} \oplus \mathrm{CH}_0(D \times \mathbf{P}^1)\,,$$
$$\mathrm{CH}_0(D \times \mathbf{P}^1) \simeq \mathrm{CH}_0(D)\,,$$

we have

$$\mathrm{Ker}(\beta) = \bigl\{(0, 0, c, -c) \bigm| c \in \mathrm{CH}_0(D)\bigr\} \simeq \mathrm{CH}_0(D)\,.$$

Hence, by the diagram (2.2) and $\mathrm{CH}_0(Z) \simeq \mathrm{CH}_0(Y_0)^{\oplus 2}$, we have an exact sequence

$$\mathrm{CH}_0(D) \longrightarrow \mathrm{CH}_0(Y_0)^{\oplus 2} \longrightarrow \mathrm{CH}_0(Y) \longrightarrow 0\,. \tag{2.3}$$

On the other hand, considering the localization sequence of étale homology groups, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd} \pi_1^{\mathrm{ab}}(Z') \arrow[r, "\beta'"] \arrow[d] & \pi_1^{\mathrm{ab}}(S) \arrow[r] \arrow[d] & H_1(D \times \mathbf{G}_m, \mathbf{Q}/\mathbf{Z}) \arrow[d, equal] \\ \pi_1^{\mathrm{ab}}(Z) \arrow[r] & \pi_1^{\mathrm{ab}}(Y) \arrow[r] & H_1(D \times \mathbf{G}_m, \mathbf{Q}/\mathbf{Z})\,. \end{tikzcd} \tag{2.4}$$

Similarly to the above, we have $\mathrm{Ker}\bigl(\beta'\bigr) \simeq \pi_1^{\mathrm{ab}}(D)$. Hence, by the diagram (2.4) and $\pi_1^{\mathrm{ab}}(Z) \simeq \pi_1^{\mathrm{ab}}(Y_0)^{\oplus 2}$, we have an exact sequence

$$\pi_1^{\mathrm{ab}}(D) \longrightarrow \pi_1^{\mathrm{ab}}(Y_0)^{\oplus 2} \longrightarrow \pi_1^{\mathrm{ab}}(Y)\,. \tag{2.5}$$

From (2.3) and (2.5), we have the following diagram with exact rows (cf. Proposition A.2):

$$\begin{tikzcd} \mathrm{CH}_0(D) \arrow[r] \arrow[d] & \mathrm{CH}_0(Y_0)^{\oplus 2} \arrow[r] \arrow[d, "\simeq"] & \mathrm{CH}_0(Y) \arrow[d, "\rho_Y"] \\ \pi_1^{\mathrm{ab}}(D) \arrow[r] \arrow[d, two heads] & \pi_1^{\mathrm{ab}}(Y_0)^{\oplus 2} \arrow[r] & \pi_1^{\mathrm{ab}}(Y) \\ H_1(\Gamma_D, \hat{\mathbf{Z}})\,. & & \end{tikzcd}$$

Here $H_1(\Gamma_X, \hat{\mathbf{Z}}) := \varprojlim_n H_1(\Gamma_X, \mathbf{Z}/n)$. Let $\hat{\delta} : H_1(\Gamma_D, \hat{\mathbf{Z}}) \to \mathrm{Ker}(\rho_Y)$ be the surjective map induced by the above diagram. We then define δ_Y by the composite

$$H_1(\Gamma_D, \mathbf{Z}) \longrightarrow H_1(\Gamma_D, \hat{\mathbf{Z}}) \xrightarrow{\hat{\delta}} \mathrm{Ker}(\rho_Y) \hookrightarrow \mathrm{CH}_0(Y)\,.$$

Then we have $\mathrm{Im}(\delta_Y) = \mathrm{Ker}(\rho_Y)$, since $\mathrm{Ker}(\rho_Y)$ is finite and the map $H_1(\Gamma_D, \mathbf{Z}) \to H_1(\Gamma_D, \hat{\mathbf{Z}})$ has dense image with respect to the pro-finite topology. $\square$

REMARK 2.3. From the structure of Y, we obtain a suspension isomorphism

$$H_1(\Gamma_D, \mathbf{Z}) \simeq H_2(\Gamma_Y, \mathbf{Z}).$$

Therefore we have the map

$$H_1(\Gamma_D, \mathbf{Z}) \simeq H_2(\Gamma_Y, \mathbf{Z}) \xrightarrow{\varepsilon_Y} \mathrm{CH}_0(Y)$$

whose image coincides with $\mathrm{Ker}(\rho_Y)$. Here ε_Y is the map in (1.1). This map coincides with the map δ_Y.

We write $H_1(\Gamma_X, \hat{\mathbf{Z}})_{\overline{X}}$ for the image of the norm map $H_1(\Gamma_{\overline{X}}, \hat{\mathbf{Z}}) \longrightarrow H_1(\Gamma_X, \hat{\mathbf{Z}})$. We then define the map

$$\delta_Y^{\mathrm{geo}} : H_1(\Gamma_D, \hat{\mathbf{Z}})_{\overline{D}} \longrightarrow \mathrm{CH}_0(Y)$$

to be that induced by the following commutative diagram with exact rows (cf. Proposition A.4):

$$\begin{tikzcd}
A_0(D) \arrow[r] \arrow[d] & A_0(Y_0)^{\oplus 2} \arrow[r] \arrow[d, "\simeq"] & A_0(Y) \arrow[d] \\
\pi_1^{\mathrm{geo}}(D) \arrow[r] \arrow[d, two heads] & \pi_1^{\mathrm{geo}}(Y_0)^{\oplus 2} \arrow[r] & \pi_1^{\mathrm{geo}}(Y) \\
H_1(\Gamma_D, \hat{\mathbf{Z}})_{\overline{D}}. & &
\end{tikzcd}$$

Here the diagram follows from (2.3) and (2.5). The bijectivity of the middle vertical map is due to Kato-Saito [4].

From the constructions of δ_Y and δ_Y^{geo}, the following diagram commutes:

$$\begin{tikzcd}
H_1(\Gamma_{\overline{D}}, \mathbf{Z}) \arrow[r] \arrow[d, "\sigma"] & H_1(\Gamma_D, \hat{\mathbf{Z}})_{\overline{D}} \arrow[d, "\delta_Y^{\mathrm{geo}}"] \\
H_1(\Gamma_D, \mathbf{Z}) \arrow[r, "\delta_Y"] & \mathrm{CH}_0(Y).
\end{tikzcd} \tag{2.6}$$

2.2. Proof of Theorem 2.1. Let ℓ be a prime number. We write Θ_ℓ for the G_k-module

$$\mathrm{Coker}\big(\pi_1^{\mathrm{ab}}(\overline{D}^{(1)})^{\text{pro-}\ell} \longrightarrow \pi_1^{\mathrm{ab}}(\overline{Y_0})^{\text{pro-}\ell}\big).$$

We consider the following G_k-equivariant homomorphism

$$\alpha^{(\ell)} : H_1(\Gamma_{\overline{D}}, \mathbf{Z}_\ell) \longrightarrow \Theta_\ell$$

induced by the following commutative diagram with exact rows

$$\begin{array}{ccccccc} & & \pi_1^{\mathrm{ab}}(\overline{D}^{(1)})^{\text{pro-}\ell} & \longrightarrow & \pi_1^{\mathrm{ab}}(\overline{D})^{\text{pro-}\ell} & \longrightarrow & H_1(\Gamma_{\overline{D}}, \mathbf{Z}_\ell) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_1^{\mathrm{ab}}(\overline{Y_0})^{\text{pro-}\ell} & \xrightarrow{\mathrm{id}} & \pi_1^{\mathrm{ab}}(\overline{Y_0})^{\text{pro-}\ell} & \longrightarrow & 0\,. & & \end{array}$$

By the weight argument, Matsumi, Sato and Asakura [6, Thm. 3.3] proved the following:

LEMMA 2.4 (Matsumi-Sato-Asakura). *Let ℓ be an arbitrary prime number.*
(1) *The image of $\alpha^{(\ell)}$ is contained in $(\Theta_\ell)_{\mathrm{tors}}$.*
(2) *Assume that G_k acts on $(\Theta_\ell)_{\mathrm{tors}}$ trivially. Then the composite of canonical maps*

$$(\Theta_\ell)_{\mathrm{tors}} \xrightarrow{f_1} ((\Theta_\ell)_{\mathrm{tors}})_{G_k} \xrightarrow{f_2} (\Theta_\ell)_{G_k}$$

is injective.

PROOF OF THEOREM 2.1. (1) For a finite abelian group M, we write $M^{(\ell)}$ for the maximal ℓ-quotient. Since $A_0(Y)$ is finite (cf. Lemma A.1(1)), the ℓ-primary part $G(Y)\{\ell\}$ is identified with $G(Y)^{(\ell)}$, and hence identified with the image of the composite map

$$(\delta_Y \circ \sigma)^{(\ell)} : H_1(\Gamma_{\overline{D}}, \mathbf{Z}) \longrightarrow A_0(Y) \longrightarrow A_0(Y)^{(\ell)}\,.$$

From the commutativity of the diagram (2.6) and the constructions of δ_Y^{geo} and $\alpha^{(\ell)}$, the map $(\delta_Y \circ \sigma)^{(\ell)}$ is decomposed as follows:

$$H_1(\Gamma_{\overline{D}}, \mathbf{Z}) \xrightarrow{\alpha^{(\ell)}} \Theta_\ell \longrightarrow (\Theta_\ell)_{G_k} \xrightarrow{\eta^{(\ell)}} A_0(Y)^{(\ell)}\,,$$

where $\eta^{(\ell)}$ denotes the following composite map:

$$\begin{aligned}(\Theta_\ell)_{G_k} &\simeq \mathrm{Coker}\big(\pi_1^{\mathrm{geo}}(D^{(1)})^{\text{pro-}\ell} \to \pi_1^{\mathrm{geo}}(Y_0)^{\text{pro-}\ell}\big) \\ &\simeq \mathrm{Coker}\big(A_0(D^{(1)})^{(\ell)} \to A_0(Y_0)^{(\ell)}\big) \\ &\to \mathrm{Coker}\big(A_0(D)^{(\ell)} \to A_0(Y_0)^{(\ell)}\big) \xrightarrow{\psi^{(\ell)}} A_0(Y)^{(\ell)}\,.\end{aligned}$$

From Lemma 2.4(1), the image of $\alpha^{(\ell)}$ is contained in $(\Theta_\ell)_{\mathrm{tors}}$. Thus, $G(Y)\{\ell\}$ is a subquotient of $(\Theta_\ell)_{\mathrm{tors}}$.

(2) It suffices to show that the composite of canonical maps

$$\mathrm{Im}(\alpha^{(\ell)}) \hookrightarrow (\Theta_\ell)_{\mathrm{tors}} \xrightarrow{f_1} ((\Theta_\ell)_{\mathrm{tors}})_{G_k} \xrightarrow{f_2} (\Theta_\ell)_{G_k} \xrightarrow{\eta^{(\ell)}} A_0(Y)^{(\ell)}$$

is injective under the assumptions. Here the first map is injective by (1). From Lemma 2.4(2), the composite map $f_2 \circ f_1$ is injective. Under the assumption (i), $\eta^{(\ell)}$ coincides with the map

$\psi^{(\ell)}$ from Lemma A.1(2). The injectivity of $\psi^{(\ell)}$ follows from the following commutative diagram with exact rows

$$\begin{tikzcd} A_0(D) \arrow[r] \arrow[d, equal] & A_0(Y_0) \arrow[r] \arrow[d, "\xi"] & \mathrm{Coker} \arrow[d, "\psi"] \\ A_0(D) \arrow[r] & A_0(Y_0)^{\oplus 2} \arrow[r] & A_0(Y)\,, \end{tikzcd}$$

where ξ maps an element a of $A_0(Y_0)$ to an element $(a, -a)$ of $A_0(Y_0)^{\oplus 2}$. □

3. Example of potentially non-injectivity

We here construct a simple normal crossing surface over a finite field for which the reciprocity map is potentially not injective by using a surface considered in [6].

Let k be a finite field. Let $n > 1$ be an integer such that $(n, 6 \cdot ch(k)) = 1$. We assume that k contains a primitive n-th root ζ of unity. Let V be a Fermat surface in $\mathbf{P}^3_k$ defined by the following equation:

$$T_0^n + T_1^n + T_2^n + T_3^n = 0\,.$$

We define an action τ on V as follows:

$$\tau : \big(T_0 : T_1 : T_2 : T_3\big) \longmapsto \big(T_0 : \zeta T_1 : \zeta^2 T_2 : \zeta^3 T_3\big)\,,$$

which does not have fixed points. We then have a projective smooth surface $Y_0 := V/\langle\tau\rangle$.

Now we consider $2n$ lines on $V : j = 1, \ldots, n-1$

$$L_1 : T_0 + T_1 = T_2 + T_3 = 0\,, \quad {L_1}^{\tau^j} : T_0 + \zeta^j T_1 = T_2 + \zeta^j T_3 = 0$$

$$L_2 : T_0 + T_1 = T_2 + \zeta T_3 = 0\,, \quad {L_2}^{\tau^j} : T_0 + \zeta^j T_1 = T_2 + \zeta^{j+1} T_3 = 0\,.$$

Let L be the following divisor on V:

$$L = L_1 \cup L_2 \cup {L_1}^{\tau} \cup \cdots \cup {L_1}^{\tau^{n-1}} \cup {L_2}^{\tau^{n-1}}\,.$$

Then the divisor L is a connected simple normal crossing divisor and stable under the action of $\langle\tau\rangle$.

Let $\varphi : V \longrightarrow Y_0$ and $C_i = \varphi_*(L_i)$ $(i = 1, 2)$. Since C_i is isomorphic to L_i, C_i is a nonsingular rational curve on Y_0 and $D = C_1 \cup C_2$ is a simple normal crossing divisor on Y_0. Moreover every singular points of D are k-rational. We then put

$$Y := \big(Y_0 \times_k O\big) \cup \big(Y_0 \times_k \infty\big) \cup \big(D \times_k \mathbf{P}^1\big).$$

Now we have to show that for the above surface Y, the map ρ_Y/n is not injective. Since $\overline{V}$ is a hypersurface in $\mathbf{P}^3_{k^{\mathrm{sep}}}$, $\pi_1^{\mathrm{ab}}(\overline{V}) = 0$ (cf. [7, Lemma 3.5.]). Hence we have

$$\pi_1^{\mathrm{ab}}(\overline{Y_0}) \simeq \langle\tau\rangle \simeq \mathbf{Z}/n\,.$$

Since C_i is rational curves, we have $\pi_1^{\mathrm{ab}}(\overline{C_i}) = 0$. Therefore, we have

$$\Theta_{\mathrm{tors}} = \mathrm{Coker}\left(\bigoplus_j \pi_1^{\mathrm{ab}}(\overline{D}_j) \longrightarrow \pi_1^{\mathrm{ab}}(\overline{Y_0})\right)_{\mathrm{tors}} = \pi_1^{\mathrm{ab}}(\overline{Y_0}),$$

and G_k acts on the above group trivially.

On the other hand, the map

$$\alpha : H_1(\Gamma_{\overline{D}}, \mathbf{Z}) \longrightarrow \pi_1^{\mathrm{ab}}(\overline{Y_0})$$

is surjective, because φ induces the completely splitting covering $L \longrightarrow D$.

From Theorem 2.1, $\mathrm{Ker}(\rho_Y) \simeq \mathbf{Z}/n$. Thus ρ_Y is not injective. Moreover, we have $\mathrm{Ker}(\rho_{Y\otimes F}) \simeq \mathbf{Z}/n$ for any finite extension F/k, therefore the map $\rho_{Y\otimes F}$ is not injective. We also see that the map $\rho_{Y\otimes F}/n$ is not injective.

Considering a product $Y \times_k X$, we obtain a higher dimensional variety for which the reciprocity map is potentially not injective. Here Y is the above surface and X is a projective smooth and geometrically irreducible variety over k. Indeed, $\rho_{Y\times_k X}$ is not injective for any finite scalar extension. This follows from the following commutative diagram with exact rows:

$$\begin{tikzcd} H_1(\Gamma_{D\times X}, \mathbf{Z}) \arrow[r, "\delta_{Y\times X}"] \arrow[d, equal] & \mathrm{CH}_0(Y\times X) \arrow[r, "\rho_{Y\times X}"] \arrow[d] & \pi_1^{\mathrm{ab}}(Y\times X) \arrow[d] \\ H_1(\Gamma_D, \mathbf{Z}) \arrow[r, "\delta_Y"] & \mathrm{CH}_0(Y) \arrow[r, "\rho_Y"] & \pi_1^{\mathrm{ab}}(Y). \end{tikzcd}$$

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A. Appendix

We here prove some lemmas about simple normal crossing varieties over finite fields. In case of curves, lemmas similar to that in this section is proved in [6]. We extend to higher dimensional cases by an argument similar to that in [6].

Let X be a simple normal crossing variety over a finite field k which is proper over k. Let $\{X_i\}_{i\in I}$ be the set of irreducible components of X. Fix an ordering on I. For $i < j$ $(i, j \in I)$, $X_{i,j}$ denotes $X_i \times_X X_j$. Then $X^{(2)} = \coprod_{i<j} X_{i,j}$ (cf. notation (6)).

LEMMA A.1. (1) *There is an exact sequence of Chow groups*

$$\bigoplus_{i<j} \mathrm{CH}_0(X_{i,j}) \xrightarrow{\phi} \bigoplus_{i\in I} \mathrm{CH}_0(X_i) \xrightarrow{\psi} \mathrm{CH}_0(X) \to 0,$$

where ϕ is the alternate sum of the push-forward maps $\mathrm{CH}_0(X_{i,j}) \to \mathrm{CH}_0(X_i)$ and $\mathrm{CH}_0(X_{i,j}) \to \mathrm{CH}_0(X_j)$, and ψ is the push-forward map for the canonical map $\coprod_{i\in I} X_i \to X$.

(2) *The degree* 0*-part* $A_0(X)$ *of* $\mathrm{CH}_0(X)$ *is finite.*

(3) *Assume that each connected component of* $X^{(2)}$ *has a* k*-rational point. Then the canonical map* $\bigoplus_{i\in I} A_0(X_i) \longrightarrow A_0(X)$ *is surjective.*

PROOF. The proof of (1) is straight-forward and left to the reader.

(2) We consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & & & \bigoplus_{i<j}\mathrm{CH}_0(X_{i,j}) & \xrightarrow{g} & \bigoplus_{i<j}\mathbf{Z} & \longrightarrow & \bigoplus_{i<j}\mathbf{Z}/m_{ij} \to 0 \\
 & & & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\nu'} \\
0 & \longrightarrow & \bigoplus_{i\in I} A_0(X_i) & \longrightarrow & \bigoplus_{i\in I}\mathrm{CH}_0(X_i) & \xrightarrow{f} & \bigoplus_{i\in I}\mathbf{Z} & \longrightarrow & \bigoplus_{i\in I}\mathbf{Z}/m_i \to 0 \\
 & & \downarrow{\scriptstyle\psi'} & & \downarrow{\scriptstyle\psi} & & \downarrow & & \\
0 & \longrightarrow & A_0(X) & \longrightarrow & \mathrm{CH}_0(X) & \xrightarrow{\deg_{X/k}} & \mathbf{Z}\,. & &
\end{array}$$

Here $m_{ij} = [\Gamma(X_{i,j}, \mathcal{O}_{X_{i,j}}) : k]$, $m_i = [\Gamma(X_i, \mathcal{O}_{X_i}) : k]$ and the above maps are defined as follows:

$f := \bigoplus_{i\in I} \deg_{X_i/k}$,

$g := \bigoplus_{i<j} \deg_{X_{i,j}/k}$,

ν : the alternate sum of the identity maps,

ν': the map induced by ν,

ψ': the restriction of ψ.

By the above diagram, we have a surjective map from the kernel of ν' to the cokernel of ψ'. Since X_i is smooth for all $i \in I$, $A_0(X_i)$ is finite by a theorem of Kato-Saito [4]. Hence we see that the cokernel of ψ' is finite and that $A_0(X)$ is finite.

(3) Under the assumption, the map g is surjective. The assertion follows from the above diagram. □

We describe the cokernel of the reciprocity map ρ_X for X in terms of the dual graph of X (cf. [3], [4]).

PROPOSITION A.2. *For a positive integer* n, *there is an exact sequence*

$$\mathrm{CH}_0(X)/n \xrightarrow{\rho_X/n} \pi_1^{\mathrm{ab}}(X)/n \xrightarrow{(*1)} H_1(\Gamma_X, \mathbf{Z}/n) \longrightarrow 0\,.$$

PROOF. We consider the following exact sequence of étale sheaves on $X_{\text{ét}}$:

$$0 \longrightarrow \mathbf{Z}/n_X \longrightarrow \bigoplus_{i\in I}\mathbf{Z}/n_{X_i} \longrightarrow \cdots \longrightarrow \bigoplus_{t\in T}\mathbf{Z}/n_{X_t^{(d+1)}} \longrightarrow 0\,.$$

Here $\{X_t^{(d+1)}\}_{t\in T}$ is the set of irreducible components of $X^{(d+1)}$, and we have omitted the indication of direct image functors of sheaves. From this exact sequence, we obtain a spectral sequence

$$E_1^{p,q} = H^q(X^{(p)}, \mathbf{Z}/n) \Longrightarrow H^{p+q}(X, \mathbf{Z}/n)\,.$$

By computing E_2-terms, we have an exact sequence

$$0 \longrightarrow H^1(\Gamma_X, \mathbf{Z}/n) \longrightarrow H^1(X, \mathbf{Z}/n) \longrightarrow \bigoplus_{i\in I} H^1(X_i, \mathbf{Z}/n)\,,$$

where $H^1(\Gamma_X, \mathbf{Z}/n) := \mathrm{Hom}(H_1(\Gamma_X, \mathbf{Z}/n), \mathbf{Q}/\mathbf{Z})$. From the Pontryagin dual of this sequence and Lemma A.1 (1), we obtain the following commutative diagram:

$$\begin{tikzcd}
\bigoplus_{i\in I} \mathrm{CH}_0(X_i)/n \arrow[r] \arrow[d, "\bigoplus \rho_{X_i}/n"] & \mathrm{CH}_0(X)/n \arrow[r] \arrow[d, "\rho_X/n"] & 0 \arrow[d] & \\
\bigoplus_{i\in I} \pi_1^{\mathrm{ab}}(X_i)/n \arrow[r] & \pi_1^{\mathrm{ab}}(X)/n \arrow[r, "(*1)"] & H_1(\Gamma_X, \mathbf{Z}/n) \arrow[r] & 0\,.
\end{tikzcd}$$

Here ρ_X (resp. ρ_{X_i}) is the reciprocity map for X (resp. X_i) and the left vertical map is an isomorphism by Kato-Saito [4]. Hence the assertion follows from the above diagram. $\square$

We regard X as a trivial right G_k-scheme and define the right action of G_k on $\overline{X}$ by the natural right action of G_k on $\mathrm{Spec}(k^{\mathrm{sep}})$. Let $\{W_s\}_{s\in S}$ be the set of connected components of $\overline{X}$, and let $\{V_j\}_{j\in J}$ be the set of irreducible components of $\overline{X}$. We define the semi-order on J as in notation (6). Then we have the norm map $\sigma : H_1(\Gamma_{\overline{X}}, \mathbf{Z}/n) \to H_1(\Gamma_X, \mathbf{Z}/n)$.

LEMMA A.3. *There is an exact sequence of finite left G_k-modules*:

$$\bigoplus_{j\in J} \pi_1^{\mathrm{ab}}(V_j)/n \longrightarrow \bigoplus_{s\in S} \pi_1^{\mathrm{ab}}(W_s)/n \xrightarrow{(*2)} H_1(\Gamma_{\overline{X}}, \mathbf{Z}/n) \longrightarrow 0\,. \tag{A.1}$$

Further the following diagram commutes:

$$\begin{tikzcd}
\bigoplus_{s\in S} \pi_1^{\mathrm{ab}}(W_s)/n \arrow[r] \arrow[d, "(*2)"'] & \pi_1^{\mathrm{ab}}(X)/n \arrow[d, "(*1)"'] \\
H_1(\Gamma_{\overline{X}}, \mathbf{Z}/n) \arrow[r, "\sigma"] & H_1(\Gamma_X, \mathbf{Z}/n)\,.
\end{tikzcd} \tag{A.2}$$

PROOF. Since we have the canonical isomorphisms

$$\bigoplus_{s\in S} \pi_1^{\mathrm{ab}}(W_s)/n \simeq \mathrm{Hom}\big(H^1(\overline{X}, \mathbf{Z}/n), \mathbf{Q}/\mathbf{Z}\big)\,,$$

$$\bigoplus_{j\in J}\pi_1^{\mathrm{ab}}(V_j)/n \simeq \mathrm{Hom}\Bigl(\bigoplus_{j\in J} H^1(V_j,\mathbf{Z}/n),\mathbf{Q}/\mathbf{Z}\Bigr),$$

the finiteness of groups in (A.1) follows from the finiteness of étale cohomology groups [1, XVI, 5.2] and the definition of dual graph. Furthermore it is sufficient to prove that there is the following exact sequence:

$$0 \longrightarrow H^1(\Gamma_{\overline{X}},\mathbf{Z}/n) \longrightarrow H^1(\overline{X},\mathbf{Z}/n) \longrightarrow \bigoplus_{j\in J} H^1(V_j,\mathbf{Z}/n),$$

where $H^1(\Gamma_{\overline{X}},\mathbf{Z}/n) := \mathrm{Hom}(H_1(\Gamma_{\overline{X}},\mathbf{Z}/n),\mathbf{Q}/\mathbf{Z})$. The above exact sequence is obtained from the same argument as that in the proof of Proposition A.2 and the following exact sequence of étale sheaves on $\overline{X}_{\text{ét}}$:

$$0 \longrightarrow \mathbf{Z}/n_{\overline{X}} \longrightarrow \bigoplus_{j\in J}\mathbf{Z}/n_{V_j} \longrightarrow \cdots \longrightarrow \bigoplus_{u\in U}\mathbf{Z}/n_{\overline{X}_u^{(d+1)}} \longrightarrow 0.$$

Here $\{\overline{X}_u^{(d+1)}\}_{u\in U}$ is the set of irreducible components of $\overline{X}^{(d+1)}$, and we have omitted the indication of direct image functors of sheaves.

The commutativity of (A.2) follows from the following commutative diagram of étale sheaves on $X_{\text{ét}}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}/n_X \arrow[r] \arrow[d] & \bigoplus_{i\in I}\mathbf{Z}/n_{X_i} \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{t\in T}\mathbf{Z}/n_{X_t^{(d+1)}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{Z}/n_{\overline{X}} \arrow[r] & \bigoplus_{j\in J}\mathbf{Z}/n_{V_j} \arrow[r] & \cdots \arrow[r] & \bigoplus_{u\in U}\mathbf{Z}/n_{\overline{X}_u^{(d+1)}} \arrow[r] & 0,
\end{tikzcd}$$

and the fact that the map $(*1)$ comes from the upper row (cf. Proposition A.2). □

We put $H_1(\Gamma_X,\hat{\mathbf{Z}}) := \varprojlim_n H_1(\Gamma_X,\mathbf{Z}/n)$. We write $H_1(\Gamma_X,\hat{\mathbf{Z}})_{\overline{X}}$ for the image of the norm map $H_1(\Gamma_{\overline{X}},\hat{\mathbf{Z}}) \longrightarrow H_1(\Gamma_X,\hat{\mathbf{Z}})$. The following proposition is the 'geometric' part of the unramified class field theory for a simple normal crossing variety. If $\dim X = 1$, the map ρ_X^{geo} in the proposition is injective by Kato-Saito [4].

PROPOSITION A.4. *There is an exact sequence*

$$A_0(X) \xrightarrow{\rho_X^{\mathrm{geo}}} \pi_1^{\mathrm{geo}}(X) \xrightarrow{(*3)} H_1(\Gamma_X,\hat{\mathbf{Z}})_{\overline{X}} \longrightarrow 0.$$

Further the following diagram commutes:

$$\begin{tikzcd}
\pi_1^{\mathrm{geo}}(X) \arrow[r, hook] \arrow[d, "(*3)"'] & \pi_1^{\mathrm{ab}}(X) \arrow[d, "(*1)^\wedge"'] \\
H_1(\Gamma_X, \hat{\mathbf{Z}})_{\overline{X}} \arrow[r, hook] & H_1(\Gamma_X, \hat{\mathbf{Z}})\,,
\end{tikzcd}$$

where the map $(*1)^\wedge$ *is induced by the map* $(*1)$ *in Proposition* A.2.

PROOF. We write $\mathrm{CH}_0(X)^\wedge$ for $\varprojlim_n \mathrm{CH}_0(X)/n$, and $\hat{\mathbf{Z}}$ for $\varprojlim_n \mathbf{Z}/n$. We consider the following commutative diagram with exact rows (cf. Proposition A.2):

$$\begin{tikzcd}
0 \arrow[r] & A_0(X) \arrow[r] \arrow[d, "\rho_X^{\mathrm{geo}}"'] & \mathrm{CH}_0(X)^\wedge \arrow[r, "\deg_{X/k}^\wedge"] \arrow[d, "\rho_X^\wedge"'] & \hat{\mathbf{Z}} \arrow[d, "\rho_k^\wedge"'] & \\
0 \arrow[r] & \pi_1^{\mathrm{geo}}(X) \arrow[r] & \pi_1^{\mathrm{ab}}(X) \arrow[r] \arrow[d, "(*1)^\wedge"'] & G_k^{\mathrm{ab}} \arrow[r] & 0 \\
& & H_1(\Gamma_X, \hat{\mathbf{Z}})\,, & &
\end{tikzcd}$$

where the map $\rho_X^\wedge$ is induced by ρ_X. Here we have used the finiteness of $A_0(X)$ (cf. Lemma A.1(2)), and the fact that the pro-finite completion of an exact sequence of finitely generated abelian groups is exact. Since the map $\rho_k^\wedge$ is injective (in fact bijective), the cokernel of the map ρ_X^{geo} is isomorphic to the image of $\pi_1^{\mathrm{geo}}(X)$ in $H_1(\Gamma_X, \hat{\mathbf{Z}})$. Therefore $\mathrm{Coker}(\rho_X^{\mathrm{geo}})$ coincides with $H_1(\Gamma_X, \hat{\mathbf{Z}})_{\overline{X}}$ from Lemma A.3. □

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Present Address:
GRADUATE SCHOOL OF MATHEMATICS,
NAGOYA UNIVERSITY,
FURO-CHO, CHIKUSA-KU, NAGOYA, 464–8602 JAPAN.
e-mail: rin-sugiyama@math.nagoya-u.ac.jp