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Existence and Non-existence of a Finite Invariant Measure

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Abstract. About fifty years ago, questions on the existence and non-existence of finite invariant measures were studied by various authors and from different directions. In this article, we examine these classical results and prove directly that all the conditions introduced by these authors are equivalent to each other. We begin at the fundamental level of a recurrent transformation whose properties can be strengthened to obtain the aforementioned classical results for the existence of a finite invariant measure. We conclude with the introduction of a new property, Strongly Weakly Wandering (sww) sequences, the existence of which is equivalent to the non-existence of a finite invariant measure. It is shown that every sww sequence is also an Exhaustive Weakly Wandering (eww) sequence for ergodic transformations. Although all ergodic transformations with no finite invariant measure are known to have eww sequences, there are exceedingly few actual examples for which explicit eww sequences can be exhibited. The significance of sww sequences is that it provides a condition which is easier to verify than the condition for eww sequences (Proposition 4.5). In a second paper, we will continue these studies and also connect them to some of the more recent derived conditions for finite invariant measures. The impetus for this work, began with the late Professor Shizuo Kakutani, with whom the authors worked and had many fruitful discussions on these topics.

1. Introduction

The results in this article reflect in a significant way properties of infinite ergodic transformations (ergodic measure preserving transformations defined on an infinite measure space). This is an area where the authors worked extensively with Professor Kakutani in the past, and we are now preparing a monograph [4] where we present our work in this area.

We consider only transformations T that are 1-1 and onto maps defined on a σ-finite non-atomic Lebesgue measure space $(X, \mathcal{B}, m)$. All the transformations T that we consider are measurable ($A \in \mathcal{B}$ if and only if $TA \in \mathcal{B}$) and nonsingular ($m(A) = 0$ if and only if $m(TA) = 0$). We say that m is an invariant measure for a transformation T if $m(TA) = m(A)$ for all $A \in \mathcal{B}$. Two measures m and μ are said to be *equivalent* (denoted, $m \sim \mu$), when $m(A) = 0$ if and only if $\mu(A) = 0$. If an invariant measure μ, equivalent to m, exists for T, then we say that T preserves the measure μ, or T is a measure preserving transformation. Throughout this article all the sets we mention are measurable, and often statements are made,

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ignoring sets of m-measure 0. In the sequel we assume that T is a measurable and nonsingular transformation defined on the finite measure space $(X, \mathcal{B}, m)$.

2. Recurrent transformations

DEFINITION 2.1. Let T be a measurable and nonsingular transformation defined on the measure space $(X, \mathcal{B}, m)$.

- T is a *recurrent* transformation if whenever A is a set of positive measure, then for a.a. $x \in A$ there is an integer $n > 0$ such that $T^n x \in A$.
- Two sets A and B are *finitely equivalent*, $A \approx B$, if for some integer $p > 0$

$$A = \bigcup_{i=1}^{p} A_i(disj), \quad B = \bigcup_{i=1}^{p} B_i(disj),$$

  and

$$T^{n_i} A_i = B_i \quad \text{for } 1 \leq i \leq p.$$

- A is a *wandering* set for T if $m(A) > 0$ and $T^i A \cap T^j A = \emptyset$ for $i \neq j, i, j \in \mathbb{Z}$.

Before stating our first theorem about the recurrence of a transformation T, we state the following lemma about wandering sets for a transformation; we use it in the proof of the Recurrence Theorem.

LEMMA 2.2 (Wandering Sets). *The following two conditions are equivalent for a transformation T.*

(1) *T does not admit any wandering sets.*

(2) *If $f(x)$ is a measurable function, and $f(Tx) \leq f(x)$ a.e. then $f(Tx) = f(x)$ a.e.*

PROOF.

(1) ⇒ (2): Assume condition (2) is not true. Then there is a measurable function f such that $f(Tx) \leq f(x)$ a.e. and $m\{x : f(Tx) < f(x)\} > 0$. It follows that there is a constant c such that if $W = \{x : f(Tx) \leq c < f(x)\}$ then $m(W) > 0$.

For $x \in W$, it follows that $f(T^n x) \leq f(T^{n-1}x) \leq \cdots \leq f(Tx) \leq c$.

For $x \in T^{-n}W$, since $T^n x \in W$, it follows that $c < f(T^n x)$. Thus, $T^{-n}W \cap W = \emptyset$ for all $n > 0$. Thus, for $i > j$, we have $T^i W \cap T^j W = T^i(W \cap T^{-(i-j)}W) = \emptyset$; this is a contradiction to (1).

(2) ⇒ (1): Assume condition (1) is not true. Let W be a wandering set for T, and let $W^* = \bigcup_{n=0}^{\infty} T^{-n}W$.

Let $f(x) = I_{W^*}(x)$, the characteristic function of the set W^*. Then, $f(Tx) = I_{W^*}(Tx) = I_{T^{-1}W^*}(x) \leq f(x)$, and for $x \in W$, we have $0 = f(Tx) < f(x) = 1$. This is a contradiction to (2) and completes the proof of Lemma 2.2. □

The Recurrence Theorem 2.3 shows various conditions that are equivalent for a transformation T.

THEOREM 2.3 (Recurrence). *Let T be a measurable and nonsingular transformation on $(X, \mathcal{B}, m)$. Then, the following conditions are equivalent*:

(1R) *If A is a measurable set of positive measure, then for a.a. $x \in A$, there is an integer $n > 0$ such that $T^n x \in A$ (i.e., T is recurrent).*

(2R) *If A is a measurable set of positive measure, then for a.a. $x \in A$, there are infinitely many integers $n > 0$ such that $T^n x \in A$.*

(3R) *If f is measurable and $f(x) > 0$ a.e., then $\sum_{n=1}^{\infty} f(T^n x) = \infty$ a.e.*

(4R) *If A is a measurable set of positive measure, then there is an integer $n > 0$ such that $m(T^n A \cap A) > 0$.*

(5R) *T does not admit any wandering sets (i.e., T is* conservative).

(6R) *If A is a measurable set of positive measure and $A \approx B$, with $A \supset B$ then $m(A - B) = 0$ (i.e., T is* finitely bounded).

PROOF.
(1R) $\Rightarrow$ (2R): Repeated applications of condition (1R) implies (2R) .
(2R) $\Rightarrow$ (3R): Assume condition (3R) is not true. Then there is a function f, with $f(x) > 0$ and a set A, of positive measure, such that $\sum_{n=1}^{\infty} f(T^n x) < \infty$ for $x \in A$. For some $\varepsilon > 0$, by possibly removing a small subset of A, we may assume that $m(A) > 0$ and $f(x) \geq \varepsilon > 0$ for $x \in A$.

For $p \geq 1$ let $A_p = \{x \in A : \sum_{n=1}^{\infty} f(T^n x) \leq p\}$. Then, $A = \bigcup_{p=1}^{\infty} A_p$, and $m(A) > 0$ implies that for some $p > 0$ we have $m(A_p) > 0$; for $x \in A_p$, $f(x) \geq \varepsilon > 0$ and $\sum_{n=1}^{\infty} f(T^n x) \leq p$.

In other words, for $x \in A_p$, the cardinality of $\{n : T^n x \in A_p\}$ is not greater than p/ε a finite number. This contradicts (2R).
(3R) $\Rightarrow$ (4R): Assume condition (4R) is not true. Then there is a set A, of positive measure, such that $T^n A \cap A = \emptyset$ for all $n > 0$. We let

$$f(x) = \begin{cases} 1 & \text{if } x \in X - \bigcup_{i=0}^{\infty} T^i A \\ 1/2^n & \text{if } x \in T^n A \quad \text{for } n > 0 \end{cases}$$

Then, $f(x) > 0$, and for $x \in A$,

$$\sum_{n=0}^{\infty} f(T^n x) = \sum_{n=0}^{\infty} 1/2^n < \infty .$$

This is a contradiction to (3R).
(4R) $\Rightarrow$ (5R): The existence of a wandering set contradicts property (4R).
(5R) $\Rightarrow$ (6R): Suppose there are sets A, B with $A \supset B$ and for some $k > 0$

$$A = \bigcup_{i=1}^{k} A_i(disj), \ B = \bigcup_{i=1}^{k} B_i(disj), \quad \text{and} \quad A_i = T^{n_i} B_i \quad \text{for } 0 \leq i \leq k .$$

Let $I_1 = \{i : 1 \le i \le k, n_i > 0\}$ and $I_2 = \{i : 1 \le i \le k, n_i < 0\}$. Define

$$f(x) = \sum_{i \in I_1} \sum_{j=0}^{n_i - 1} I_{A_i}(T^j x) - \sum_{i \in I_2} \sum_{j=n_i}^{-1} I_{A_i}(T^j x) .$$

Then, since $I_{A_i}(T^{n_i} x) = I_{B_i}(x)$ for each i, we obtain

$$\begin{aligned} f(Tx) &= f(x) - \sum_{i \in I_1} \big(I_{A_i}(x) - I_{B_i}(x)\big) + \sum_{i \in I_2} \big(I_{B_i}(x) - I_{A_i}(x)\big) \\ &= f(x) - (I_A(x) - I_B(x)) \ . \end{aligned}$$

Therefore, $f(x) - f(Tx)$ equals the characteristic function I_{A-B}, which is nonnegative a.e. and equals 1 on $A - B$. Lemma 2.2 then implies $m(A - B) = 0$, and this proves (6R).
(6R) $\Rightarrow$ (1R): Assume condition (1R) is not true. Then there is a set C of positive measure such that for $x \in C$, we have $T^n x \notin C$ for any $n > 0$. Let

$$A = \bigcup_{n=0}^{\infty} T^n C , \quad \text{and} \quad B = \bigcup_{n=1}^{\infty} T^n C .$$

Then, $A \supset B$, $m(A - B) = m(C) > 0$ and $TA = B$. This contradicts condition (6R) and completes the proof of the Recurrence Theorem 2.3. $\square$

3. Existence of a finite invariant measure $\mu \sim m$

Let T be a measurable and nonsingular transformation defined on a finite measure space $(X, \mathcal{B}, m)$. In this section we study necessary and sufficient conditions for the existence of a finite invariant measure $\mu \sim m$ for T.

Suppose T admits a wandering set W with $m(W) > 0$. Then for any measure μ, equivalent to m, it is clear that $\mu(W) > 0$ also. If μ is an invariant measure for T then the infinite number of mutually disjoint images of W under T will have the same positive μ measure. It follows that the condition: T does not admit wandering sets, namely condition (4R) of Theorem 2.3, is a necessary condition for the existence of a finite invariant measure μ equivalent to m. In view of the Recurrence Theorem any one of the equivalent conditions (1R)–(6R) of Theorem 2.3 is also a necessary condition for the existence of an m-equivalent finite invariant measure μ for T. However, as we shall see later, the converse is not true; in other words, any one of the conditions of the Recurrence Theorem is not a sufficient condition for the existence of a finite T-invariant measure μ equivalent to m. To obtain sufficient conditions for the existence of such a measure μ, the conditions in the Recurrence Theorem need to be strengthened.

DEFINITION 3.1. Let T be a measurable transformation defined on the finite measure space $(X, \mathcal{B}, m)$. Let us define the following, the first of which we repeat from the previous section for emphasis.

- Two sets A and B are *finitely equivalent*, $A \approx B$, if

$$A = \bigcup_{i=1}^{p} A_i (disj) , \quad B = \bigcup_{i=1}^{p} B_i (disj) ,$$

and for some finite set of integers $\{n_i : 1 \leq i \leq p\}$,

$$T^{n_i} A_i = B_i \quad \text{for } 1 \leq i \leq p.$$

- Two sets A and B are *countably equivalent* or simply *equivalent*, $A \sim B$, if

$$A = \bigcup_{i=1}^{\infty} A_i (disj) , \quad B = \bigcup_{i=1}^{\infty} B_i (disj) ,$$

and

$$T^{n_i} A_i = B_i \quad \text{for a sequence } \{n_i : i \geq 1\} .$$

- A set A is *strongly recurrent* if $\{n : m(T^n A \cap A) > 0\}$ is relatively dense in $\mathbb{Z}$, or equivalently: There exists an integer $k > 0$ such that

$$\max_{0 \leq i \leq k} m(T^{n+i} A \cap A) > 0 , \qquad \text{for all } n \in \mathbb{Z} .$$

- An infinite subset of integers $\{n_i : i \geq 0\}$ is a *weakly wandering* (ww) sequence if there is a set W of positive measure such that

$$T^{n_i} W \cap T^{n_j} W = \emptyset \quad \text{for } i \neq j; i, j \geq 0 .$$

We call W a weakly wandering set for T.

In the past a number of authors have been interested in finding necessary and sufficient conditions for the existence of a finite invariant measure $\mu \sim m$ for a given transformation T. We list below conditions on a transformation T that various authors introduced and proved them to be necessary and sufficient conditions for the existence of a finite invariant measure $\mu \sim m$.

In 1932 E. Hopf [10], introduced the condition:

(H) *If* $m(A) > 0$, $A \supset B$ *and* $A \sim B$ *then* $m(A - B) = 0$.

In 1955, Y. Dowker [2] introduced the condition:

(D1) *If* $m(A) > 0$ *then* $\liminf_{n\to\infty} m(T^n A) > 0$.

In 1955, A. Calderón [1] introduced the condition:

(C) *If* $m(A) > 0$ *then* $\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} m(T^i A) > 0$.

In 1956, Y. Dowker [3] introduced the condition:

(D2) *If* $m(A) > 0$ *then* $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} m(T^i A) > 0$.

In 1964, A. Hajian and S. Kakutani [5] introduced the condition:

(W) *If* $m(A) > 0$ *then* A *is not a weakly wandering set.*

In 1964, A. Hajian [6] introduced the condition:

(S) *If* $m(A) > 0$ *then* A *is strongly recurrent.*

In 1964, A. Hajian and Y. Ito [7] introduced the condition:

(F) *If* $f(x) > 0$ *is measurable then* $\sum_{i=1}^{\infty} f(T^{n_i}x) = \infty$ *for any sequence* $\{n_i\}$.

In trying to find an invariant measure $\mu \sim m$, it is useful to study the asymptotic behavior of the measure of the iterates $m(T^n A)$ of a given measurable set A. Actually, the necessary and sufficient conditions (D1), (C), and (D2) refer to the behavior of the values of $\{m(T^n A)\}$ or their Cesaro sums. These conditions are closer in nature to an invariant measure that is being sought. The other conditions (H), (W), (S) and (F), which are the strengthened versions of the corresponding conditions in the Recurrence Theorem 2.3, are more geometric in nature. They depend on the class of measures equivalent with m rather than on the measure of the iterates $\{T^n A\}$. In the following Theorem we show by direct arguments that all of these conditions are equivalent. Some of the results and arguments we employ were discussed and used in [5], [6] and [8]. In what follows, for the sake of completeness, we repeat and copy some of these arguments in the next result which we call the Finite Invariant Measures Theorem.

THEOREM 3.2 (Finite Invariant Measures). *Let* T *be a measurable and nonsingular transformation defined on the finite measure space* $(X, \mathcal{B}, m)$. *Then conditions* (D1), (C), (D2), (H), (W), (S), (F) *mentioned above are equivalent to each other. Furthermore, they are necessary and sufficient conditions for the existence of a finite invariant measure* $\mu \sim m$.

To prepare the way for the proof of this Finite Invariant Measures Theorem, we first prove several Propositions and make some observations. For a transformation T on the finite measure space $(X, \mathcal{B}, m)$, let us denote by:

$$\sigma_n(A) = \frac{1}{n}\sum_{i=0}^{n-1} m(T^i A) \quad \text{and} \quad \overline{\sigma}(A) = \limsup_{n\to\infty} \sigma_n(A)\,.$$

It is clear that $\overline{\sigma}$ is a non-negative sub-additive set function defined on $\mathcal{B}$ and is invariant under T. The following Proposition exhibits the additive nature of $\overline{\sigma}$ on finitely equivalent sets.

PROPOSITION 3.3. *Let* T *be a measurable and nonsingular transformation defined on the finite measure space* $(X, \mathcal{B}, m)$. *Let* $A_1, A_2, \ldots, A_s$ *be a finite collection of mutually disjoint sets which are finitely equivalent with each other; namely,* $A_i \approx A_j$, *and* $A_i \cap A_j = \emptyset$ *for* $i \neq j$; $i, j = 1, 2, \ldots, s$. *Then,*

$$\overline{\sigma}(A_i) = \overline{\sigma}(A_j) \quad \text{for } i, j = 1, 2, \ldots, s$$

and

$$\overline{\sigma}\Big(\bigcup_{i=1}^{s} A_i\Big) = \sum_{i=1}^{s} \overline{\sigma}(A_i) = s\overline{\sigma}(A_1)\,.$$

PROOF. Suppose $E \approx F$, then

$$E = \bigcup_{j=1}^{s} E_j (disj)\,, \quad F = \bigcup_{j=1}^{s} F_j (disj) \quad \text{and} \quad T^{p_j} E_j = F_j\,, \quad \text{for } 1 \le j \le s\,.$$

We first observe that for any set $C \in \mathcal{B}$, and any integer $p \in \mathbb{Z}$, $n \geq 1$,

$$\left|\sigma_n(C) - \sigma_n(T^p C)\right| = \left|\frac{1}{n}\sum_{j=0}^{n-1} m(T^j C) - \frac{1}{n}\sum_{j=p}^{p+n-1} m(T^j C)\right| \le \frac{2|p|}{n} m(X)\,.$$

From the above follows that for any sequence $\{n_k\}$ tending to ∞ as $k \to \infty$,

$$\left|\sigma_{n_k}(E) - \sigma_{n_k}(F)\right| = \left|\sum_{i=1}^{s} \sigma_{n_k}(E_i) - \sum_{i=1}^{s} \sigma_{n_k}(F_i)\right|$$

$$\le \sum_{i=1}^{s} \left|\sigma_{n_k}(E_i) - \sigma_{n_k}(T^{p_i} E_i)\right| \le \sum_{i=1}^{s} 2\frac{|p_i|}{n_k} m(X) \to 0 \tag{3.1}$$

as $k \to \infty$.

Next, let $\{n_k\}$ be a sequence of integers such that

$$\overline{\sigma}(E) = \lim_{k\to\infty} \sigma_{n_k}(E)\,;$$

then,

$$\overline{\sigma}(E) = \lim_{k\to\infty} \sigma_{n_k}(E) = \lim_{k\to\infty} \sigma_{n_k}(F) \le \overline{\sigma}(F)\,.$$

By symmetry, we conclude $\overline{\sigma}(E) = \overline{\sigma}(F)$ for any two sets $E \approx F$.

Therefore $\overline{\sigma}(A_i) = \overline{\sigma}(A_j)$ for $i, j = 1, 2, \ldots, s$. We also have from (3.1)

$$\left|\sum_{i=1}^{s} \sigma_{n_k}(A_i) - s\sigma_{n_k}(A_1)\right| \le \sum_{i=1}^{s} \left|\sigma_{n_k}(A_i) - \sigma_{n_k}(A_1)\right| \to 0$$

as $k \to \infty$.

By a similar argument, we conclude $\overline{\sigma}(\bigcup_{i=1}^{s} A_i) = s\overline{\sigma}(A_1)$. □

PROPOSITION 3.4. *Let T be a nonsingular transformation defined on the finite measure space $(X, \mathcal{B}, m)$. Let A be a set of positive measure such that*

$$\liminf_{n\to\infty} m(T^n A) = 0\,. \tag{3.2}$$

Then, for $0 < \varepsilon < m(A)$ there is a set $A' \subset A$, with $m(A') < \varepsilon$, so that $S = A - A'$ is not strongly recurrent.

PROOF. Let A and $\varepsilon > 0$ as above, be given. For $k \geq 1$, let $\varepsilon_k = \frac{\varepsilon}{k2^k}$. For each $k \geq 1$ we choose an integer $n_k > 0$ such that $m\left(T^{n_k - i} A \cap A\right) < \varepsilon_k$ for $0 \leq i \leq k-1$. This is possible, since A satisfies (3.2), $m(X) < \infty$, and T is nonsingular, which implies that each of the measures $m_{-i}(A) = m(T^{-i}A)$ is absolutely continuous with respect to m. Let us put

$$A' = \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{k-1} T^{n_k - i} A \cap A .$$

Then,

$$m(A') \leq \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} m(T^{n_k - i} A \cap A) < \sum_{k=1}^{\infty} k\varepsilon_k = \varepsilon .$$

Let $S = A - A'$; then, it is easy to see that $m(S) \geq m(A) - \varepsilon > 0$ and

$$T^{n_k - i} S \cap S \subset T^{n_k - i} A \cap (A - A') = \emptyset \quad \text{for } 1 \leq i \leq k-1 , \quad k \geq 1 .$$

This says that for each $k \geq 1$ there is an integer $n_k > 0$ such that $T^{n_k} S \cap \left(\bigcup_{i=0}^{k-1} T^i S\right) = \emptyset$. □

PROPOSITION 3.5. *Let T be a measurable and nonsingular transformation defined on the finite measure space $(X, \mathcal{B}, m)$. If A is a compressible set of positive measure (i.e., $A \sim B$ for some subset B of A and $m(A - B) > 0$), then there is a sequence of mutually disjoint sets $\{D_n : n \geq 1\}$ in A, such that $D_i \approx D_j$ for $i \neq j$; $i, j = 1, 2, \ldots$, and $m(D_1) > 0$.*

PROOF. By assumption, $A = B \cup C$ (disj.) with $m(C) > 0$ and $A \sim B$. Using the countable equivalence

$$A = \bigcup_{i=1}^{\infty} A_i (disj) , \quad B = \bigcup_{i=1}^{\infty} T^{n_i} A_i (disj) ,$$

we obtain a decomposition $B = B_2 \cup C_2$ with $B \sim B_2$ and $C \sim C_2$ (and so $m(C_2) > 0$).

To see this, we observe that,

$$C = \bigcup_{i=1}^{\infty} (A_i \cap C)(disj) , \quad \text{and also } B = \bigcup_{i=1}^{\infty} (A_i \cap B)(disj) .$$

Define

$$C_2 = \bigcup_{i=1}^{\infty} T^{n_i}(A_i \cap C) \quad \text{and} \quad B_2 = \bigcup_{i=1}^{\infty} T^{n_i}(A_i \cap B) .$$

Clearly C_2 and B_2 satisfy the desired properties.

This same argument applies to $B = B_2 \cup C_2(disj)$ and we obtain $B_2 = B_3 \cup C_3(disj)$ with $B_2 \sim B_3$ and $C_2 \sim C_3$. Continuing by induction (and denoting $C_1 = C$) we obtain a sequence C_i of countably equivalent sets. In addition, the sets C_i are mutually disjoint because each $C_{i+1} \subset B_i$ and $C_i \cap B_i = \emptyset$.

As $\sim$ is an equivalence relation, it follows that $C_1 \sim C_n$ for all $n > 1$. By carefully truncating each of these countable equivalences we can obtain the sets D_n.

To do this, fix $\varepsilon > 0$ be such that $m(C_1) - \varepsilon > 0$. For each $k \geq 2$, since $C_1 \sim C_k$, it follows that

$$C_1 = \bigcup_{i=1}^{\infty} E_{i,k}(disj), \quad C_k = \bigcup_{i=1}^{\infty} T^{n_{i,k}}(E_{i,k})(disj).$$

Since $m(X) < \infty$, for each $k \geq 2$, there is an $N_k > 0$ such that

$$m\left(\bigcup_{i=N_k}^{\infty} E_{i,k}\right) < \frac{\varepsilon}{2^{k-1}}.$$

We let $D_1 = C_1 - \bigcup_{k=2}^{\infty} \bigcup_{i=N_k}^{\infty} E_{i,k}$. Then,

$$m(D_1) \geq m(C_1) - \sum_{k=2}^{\infty} \frac{\varepsilon}{2^{k-1}} = m(C_1) - \varepsilon > 0.$$

Next, for $k \geq 2$ we define

$$D_k = \bigcup_{i=1}^{N_k - 1} T^{n_{i,k}}\left(E_{i,k} \cap D_1\right). \tag{3.3}$$

Equation (3.3) and the fact that $D_1 = \bigcup_{i=1}^{N_k-1}\left(E_{i,k} \cap D_1\right)$ imply $D_1 \approx D_k$ for $k \geq 2$. Moreover, (3.3) implies $D_k \subset C_k$ for $k \geq 2$, and therefore, $D_i \cap D_j = \emptyset$ for $i \neq j$; $i, j = 1, 2, \ldots$. □

PROPOSITION 3.6. *Condition* (D1) *implies condition* (F).

PROOF. Assume that condition (F) is not true. Then there is a measurable function $f > 0$, a sequence of integers $\{n_i : i \geq 0\}$ and a set A with $m(A) > 0$ such that

$$\sum_{i=0}^{\infty} f(T^{n_i}x) < \infty \quad \text{for } x \in A. \tag{3.4}$$

Since $m(X) < \infty$, for each $\varepsilon > 0$ there is a $\delta > 0$ and a set B such that $m(B) < \varepsilon$ and $f(x) \geq \delta$ for $x \in X - B$. Equation (3.4) implies: for $x \in A$, $T^{n_i}x \in X - B$ for finitely many i only. In other words, for almost all $x \in A$ there is an integer $N = N(x) > 0$ such that $T^{n_i}x \in B$ for $i \geq N$.

For $k \geq 1$ let $A(k) = \{x \in A : N(x) \geq k\}$. Then, $A = \bigcup_{k=1}^{\infty} A(k)$, and $m(A) > 0$ implies for any fixed η, $0 < \eta < m(A)$, there is $k_1 > 0$ such that $m(A - A_1) \leq \eta/2$, where $A_1 = A(k_1)$. We see that for $x \in A_1$, $T^{n_i}x \in B$ for all $i \geq k_1$; therefore, $m(T^{n_i}A_1) \leq \varepsilon$ holds for all $i \geq k_1$.

From the above discussion we conclude: for $\varepsilon > 0$ and $\eta > 0$ there is a subset $A_1 \subset A$ such that $\limsup_{i\to\infty} m(T^{n_i}A_1) \leq \varepsilon$ and $m(A - A_1) \leq \eta/2$.

Next, we repeat the above argument, and using induction, choose sets $A \supset A_1 \supset A_2 \supset \cdots$ as follows: For $p \geq 1$ let $\varepsilon_p = \varepsilon/p$ and $\eta_p = \eta/2^p$. Assume we have chosen the sets $A \supset A_1 \supset \cdots \supset A_p$ for some $p \geq 1$ that satisfy

$$\limsup_{i\to\infty} m(T^{n_i}A_p) \leq \varepsilon_p \quad \text{and} \quad m(A_{p-1} - A_p) \leq \eta_p\,. \tag{3.5}$$

We use the same argument as above and obtain a set $A_{p+1} \subset A_p$ that satisfies (3.5) with p replaced by $p+1$.

Finally, we let $A' = \bigcap_{p=1}^{\infty} A_p$. It follows that

$$\liminf_{n\to\infty} m(T^n A') \leq \limsup_{i\to\infty} m(T^{n_i}A') \leq \limsup_{i\to\infty} m(T^{n_i}A_p) \leq \varepsilon_p$$

for each p. Then we have

$$\liminf_{n\to\infty} m(T^n A') = 0\,,$$

while

$$m(A') \geq m\left(A - \bigcup_{p=1}^{\infty}\left(A_p - A_{p+1}\right)\right) \geq m(A) - \sum_{p=1}^{\infty} \eta/2^p = m(A) - \eta > 0\,.$$

This is a contradiction to condition (D1). □

For each integer k, denote by $\omega_k(x)$ the Radon-Nikodym derivative of m_k with respect to m, where $m_k(A) = m(T^k A)$ for $A \in \mathcal{B}$. Since for each k, T^k is a nonsingular transformation, $\omega_k(x)$ is positive, and since $T^k T^n = T^{k+n}$, it follows that

$$\omega_{k+n}(x) = \omega_k(T^n(x))\omega_n(x) = \omega_n(T^k(x))\omega_k(x)$$

holds for every pair of integers k and n.

Define a linear operator V on the Hilbert space $\mathcal{H} = L^2(X, \mathcal{B}, m)$ by setting $Vf(x) = f(Tx)(\omega_1(x))^{\frac{1}{2}}$ for every $f \in \mathcal{H}$. Then, V is a unitary operator on $\mathcal{H}$, and it is easy to verify that $V^k f(x) = f(T^k(x))(\omega_k(x))^{\frac{1}{2}}$ holds for each k.

Let $\mathcal{C}$ be the closed convex hull of the set $\{V^k\mathbf{1} : k = 0, 1, 2, \ldots\}$ in $\mathcal{H}$, where $\mathbf{1}$ denotes the constant function 1. Note that $V^k\mathbf{1}(x) = (\omega_k(x))^{\frac{1}{2}}$.

It is clear from the definition that the closed convex set $\mathcal{C}$ is mapped into itself by the unitary operator V. It is well-known and easy to prove that in any Hilbert space a closed convex subset contains a unique element having the smallest norm. Hence, there exists a unique element g_0 in the closed convex subset $\mathcal{C}$ of $\mathcal{H}$ satisfying the property that $\|g_0\| \le \|f\|$ holds for every $f \in \mathcal{C}$ where $\| \ \|$ denotes the L^2-norm of the space $\mathcal{H}$.

The element Vg_0 belongs to the convex set $\mathcal{C}$ and since V is unitary, we have $\|Vg_0\| = \|g_0\|$. From the uniqueness of g_0, it then follows that $Vg_0 = g_0$, i.e., g_0 is a fixed point of the linear operator V.

We use this function g_0 in the proof of our next proposition.

PROPOSITION 3.7. *Let T be a bi-measurable, nonsingular transformation on $(X, \mathcal{B}, m)$. Then, Dowker's condition* (D1) (*and hence any of the equivalent conditions introduced above*) *is sufficient for the existence of a finite T-invariant measure on $(X, \mathcal{B})$ equivalent to m.*

PROOF. Let g_0 be the fixed point of V mentioned in the paragraph above, and define a measure μ on $(X, \mathcal{B})$ by setting

$$\mu(A) = \int_A (g_0(x))^2 dm \quad \text{for } A \in \mathcal{B}.$$

Since the function $g_0(x)$ is square integrable, it is clear that μ is a finite measure on $(X, \mathcal{B})$, absolutely continuous with respect to m. Furthermore, since $Vg_0 = g_0$, we have for any $A \in \mathcal{B}$,

$$\begin{aligned}\mu(TA) &= \int_{TA} (g_0(x))^2 dm = \int_A (g_0(Tx))^2 \omega_1(x) dm \\ &= \int_A (Vg_0(x))^2 dm = \int_A (g_0(x))^2 dm = \mu(A),\end{aligned}$$

which shows that μ is a T-invariant measure.

To prove the sufficiency of the condition (D1) it remains to show that the condition (D1) implies that the measure m is absolutely continuous with respect to μ. For this purpose, we prove the following Lemma:

LEMMA 3.8. *For a set $A \in \mathcal{B}$, let $\delta(A) = \inf_{n \ge 0}\{\int_A (\omega_n(x))^{\frac{1}{2}} dm\}$, and consider the following condition on the transformation T:*

(D1)* *If $m(A) > 0$ then $\delta(A) > 0$.*

Then, the conditions (D1) *and* (D1)* *are mutually equivalent.*

PROOF. First note that because T^n is nonsingular for each n, we may replace the $\liminf_{n\to\infty}$ in the statement of condition (D1) by $\inf_{n\ge 0}$, whenever convenient.

Using the Cauchy-Schwarz inequality it is easy to see that (D1)* implies (D1). Conversely, suppose condition (D1) holds but there exists a set A with $m(A) > 0$ and $\delta(A) = 0$. Then, we can get an infinite sequence of integers $\{n_k\}$ for which

$$\int_A \sum_k (\omega_{n_k}(x))^{\frac{1}{2}} dm = \sum_k \int_A (\omega_{n_k}(x))^{\frac{1}{2}} dm < \infty,$$

and therefore, $\sum_k (\omega_{n_k}(x))^{\frac{1}{2}} < \infty$ a. e. on A. But this in turn implies that $\sum_k \omega_{n_k}(x) < \infty$ a.e. on A, from which we conclude that there exists a subset A' of A with $m(A') > 0$ such that

$$\sum_k m(T^{n_k} A') = \int_{A'} \sum_k \omega_{n_k}(x) dm < \infty .$$

From this it follows $m(T^{n_k} A') \to 0$ as $k \to \infty$, and therefore, $0 = \liminf_{n\to\infty} m(T^n A')$. This contradicts (D1) since $m(A') > 0$. □

To complete the proof of the Proposition, let us suppose that there is a set $A \in \mathcal{B}$ for which $\mu(A) = 0$, but $m(A) > 0$. Then, by (D1)*, $\delta(A) > 0$. From the definition of the closed convex set $\mathcal{C}$ there exists for any ε, $0 < \varepsilon < \delta(A)$, a finite set $\{\alpha_j\}$ of positive numbers such that $\sum_j \alpha_j = 1$ and a finite set of non-negative integers $\{k_j\}$ for which the inequality below holds:

$$\|g_0 - \sum_j \alpha_j V^{k_j} \mathbf{1}\| < \frac{\varepsilon}{(m(X))^{\frac{1}{2}}} .$$

Note that from the Cauchy-Schwarz inequality, follows the fact that

$$\text{if } \mu(A) = 0 \quad \text{then } \int_A g_0(x) dm = 0 .$$

Furthermore, we have

$$\int_A |g_0(x) - \sum_j \alpha_j (\omega_{k_j}(x))^{\frac{1}{2}}| dm \le \|g_0 - \sum_j \alpha_j V^{k_j} \mathbf{1}\| (m(X))^{\frac{1}{2}} < \varepsilon ,$$

and

$$\sum_j \alpha_j \int_A (\omega_{k_j}(x))^{\frac{1}{2}} dm \le \int_A g_0(x) dm + \int_A |g_0(x) - \sum_j \alpha_j (\omega_{k_j}(x))^{\frac{1}{2}}| dm \le \varepsilon .$$

Putting all the above together; since $\sum_j \alpha_j = 1$, we obtain

$$\varepsilon \ge \sum_j \alpha_j \int_A (\omega_{k_j}(x))^{\frac{1}{2}} dm \ge \sum_j \alpha_j \delta(A) = \delta(A) > \varepsilon ,$$

a contradiction. This shows that $\mu(A) = 0$ must imply $m(A) = 0$, and hence $m \sim \mu$. □

PROOF OF THEOREM 3.2. The implications **(D1)** $\Rightarrow$ **(C)** $\Rightarrow$ **(D2)** are obvious.

(D2) $\Rightarrow$ **(H):** Assume condition (H) is not true.

Then, there are sets A and B such that $A \sim B$, $B \subset A$, and $m(A - B) > 0$. From Proposition 3.5 it follows that there is a sequence of mutually disjoint sets $\{D_n : n \ge 1\}$ such that $m(D_1) > 0$ and $D_1 \approx D_k$ for $k \ge 2$. Then, for any integer $s > 0$, we have:

$\overline{\sigma}(D_1 \cup D_2 \cup \cdots \cup D_s) = s\overline{\sigma}(D_1)$. This implies $s\overline{\sigma}(D_1) \leq m(X)$. Since this is true for any integer $s > 0$ we conclude that $\overline{\sigma}(D_1) = 0$. This contradicts condition (D2).

(H) $\Rightarrow$ (W): Suppose that condition (W) is not true. Then, there is a weakly wandering set C of positive measure, and a sequence of integers $\{n_i : i \geq 1\}$ satisfying $T^{n_i}C \cap T^{n_j}C = \emptyset$ for $i \neq j$. We let $A = \bigcup_{i=1}^{\infty} A_i$ where $A_i = T^{n_i}C$ for $i \geq 1$ and $B = \bigcup_{i=1}^{\infty} B_i$ where $B_i = T^{n_{i+1}}C$ for $i \geq 1$. Then, $B_i = T^{n_{i+1}-n_i}A_i$ and $B \sim A$, $B \subset A$, and $m(A - B) = m(T^{n_1}C) > 0$; i.e., A is compressible. Thus condition (H) implies condition (W).

(W) $\Rightarrow$ (S): Suppose condition (S) does not hold. Then, there is a measurable set A of positive measure so that, for every integer n_k there is an integer n_{k+1} such that

$$m\left(T^{n_{k+1}}A \cap \bigcup_{i=0}^{n_k} T^i A\right) = 0 .$$

Let $N = \bigcup_{k=1}^{\infty} \left(T^{n_{k+1}}A \cap \bigcup_{i=0}^{n_k} T^i A\right)$. Then $m(N) = 0$, and the set $B = A - N$ is a weakly wandering set under the sequence $\{n_k : k \geq 1\}$. This proves (W) $\Rightarrow$ (S).

Proposition 3.4 proves the implication **(S) $\Rightarrow$ (D1)**, and therefore completes the proof that conditions (D1), (C), (D2), (W), and (S) are all equivalent to each other.

Proposition 3.6 is the implication **(D1) $\Rightarrow$ (F)**. It remains to show condition (F) implies condition (W).

(F) $\Rightarrow$ (W): Again, proceed by assuming (W) is not satisfied. Then there is a sequence of integers $\{n_i : i \geq 0\}$ and a set C of positive measure, such that $T^{n_i}C \cap T^{n_j}C = \emptyset$ for $i \neq j$. We let

$$f(x) = \begin{cases} 1 & \text{if } x \in X - \bigcup_{i=0}^{\infty} T^{n_i}C \\ 1/2^i & \text{if } x \in T^{n_i}C \quad \text{for } i > 0 . \end{cases}$$

Then, for the above sequence $\{n_i\}$ and for $x \in C$ we have

$$\sum_{i=0}^{\infty} f(T^{n_i}x) = \sum_{i=0}^{\infty} 1/2^i < \infty ,$$

which is a contradiction to condition (F). This proves that all the conditions mentioned in the theorem are mutually equivalent.

Finally, we note that if there exists a finite invariant measure μ, $\mu \sim m$, then condition (W) is clearly true. This shows that any of the conditions of the theorem is necessary for the existence of such a measure. We finish the proof by noting that Proposition 3.7 shows that these conditions are also sufficient for the existence of finite invariant measure equivalent to m. □

4. Non-existence of a finite invariant measure $\mu \sim m$

In this section we study properties of transformations T that do not preserve a finite invariant measure μ equivalent to m.

DEFINITION 4.1. Let T be a measurable and nonsingular transformation defined on the finite measure space $(X, \mathcal{B}, m)$.

- T is *ergodic* if whenever $TA = A$, then either $m(A) = 0$ or $m(X - A) = 0$.
- An infinite set of integers $\{n_i : i \geq 0\}$ is an *Exhaustive Weakly Wandering (eww) sequence* for the transformation T with the set A if $m(A) > 0$, $T^{n_i}A \cap T^{n_j}A = \emptyset$ for $i, j = 0, 1, 2, \ldots;\ i > j$ and $X = \bigcup_{i=0}^{\infty} T^{n_i}A$.
- An infinite set of integers $\{n_i : i \geq 0\}$ is a *Strongly Weakly Wandering (sww) sequence* for the transformation T with the set A if $m(A) > 0$ and

$$T^{n_i - n_k + k}A \cap T^{n_j - n_l + l}A = \emptyset$$

  for $i, j, k, l = 0, 1, 2, \ldots;\ i > j$ and whenever one of the indices $\{i, j, k, l\}$ is larger than all the others, or $i = l > \max\{j, k\}$.
- For any sequence of integers $\{n_i : i \geq 1\}$ and any set $A \in \mathcal{B}$ we consider the following sequence of sets related to A and the sequence $\{n_i\}$:
  Set $n_0 = 0$ and let $A_0 = A$, $A_1 = TA - \bigcup_{r=0}^{\infty} T^{n_r}\left(T^{-n_0}A_0\right)$,
  $A_2 = T^2A - \bigcup_{r=0}^{\infty} T^{n_r}\left(T^{-n_0}A_0 \cup T^{-n_1}A_1\right)$. Inductively for $p > 2$ we set,

$$A_p = T^pA - \bigcup_{r=0}^{\infty} T^{n_r}\left(T^{-n_0}A_0 \cup T^{-n_1}A_1 \cup \cdots \cup T^{-n_{p-1}}A_{p-1}\right). \tag{4.1}$$

  Let

$$W = \bigcup_{p=0}^{\infty} T^{-n_p}A_p\,. \tag{4.2}$$

  We call the set W as constructed above *the derived set from the sequence* $\{n_i : i \geq 1\}$ *and the set* $A \in \mathcal{B}$.

The following simple proposition follows from the definitions.

PROPOSITION 4.2. *Let $\{n_i : i \geq 1\}$ be a Strongly Weakly Wandering sequence for the transformation T with the set $A \in \mathcal{B}$; then, the derived set W from the sequence $\{n_i\}$ and the set A have the following property:*

$$\bigcup_{r=0}^{\infty} T^{n_r}W(disj) \supset \bigcup_{p=1}^{\infty} T^pA$$

PROOF. Set $n_0 = 0$. From equations (4.1) and (4.2) in the definition of the derived set, it follows that for all $p > 0$,

$$\begin{aligned}\bigcup_{r=0}^{\infty} T^{n_r}W &\supset A_p \cup \bigcup_{r=0}^{\infty} T^{n_r}\left(T^{-n_0}A_0 \cup T^{-n_1}A_1 \cup \cdots \cup T^{-n_{p-1}}A_{p-1}\right)\\ &\supset T^pA.\end{aligned}$$

This implies $\bigcup_{r=0}^{\infty} T^{n_r} W \supset \bigcup_{p=1}^{\infty} T^p A$.

It remains to show that $T^{n_i} W \cap T^{n_j} W = \emptyset$ for $i, j = 0, 1, 2, \ldots; i > j$. For this it is sufficient to show that

$$T^{n_i - n_k} A_k \cap T^{n_j - n_l} A_l = \emptyset \quad \text{for } i, j, k, l = 0, 1, 2, \ldots; i > j\,. \tag{4.3}$$

It is clear from (4.1) that

$$A_p \cap T^{n_r - n_s} A_s = \emptyset \quad \text{if } p > s\,. \tag{4.4}$$

If $i = k > \max\{j, l\}$ then (4.3) follows from (4.4). In all the other cases, we note that $A_k \subset T^k A$ for $k \geq 0$, and (4.3) follows from the properties defining the sww sequence $\{n_i\}$. This completes the proof. □

In the next proposition we show some useful properties of an ergodic transformation T.

PROPOSITION 4.3. *Let T be an ergodic nonsingular transformation of* $(X, \mathcal{B}, m)$. *Then*, T *satisfies the following properties*:

(1E) *If $T^i A \cap T^j A = \emptyset$ for all $i \neq j$ then $m(A) = 0$* (*i.e.*, *T does not accept wandering sets*).

(2E) *If $TA \subset A$ then $m(A - TA) = 0$.*

(3E) *If A is any set of positive measure*, *then for any integer $k \geq 0$ we have $\bigcup_{n=k}^{\infty} T^n A = X$.*

(4E) *If f is measurable and T-invariant* (*i.e.* $f(Tx) = f(x)$ *a.e.*) *then $f(x) \equiv c$ a.e. for some constant c.*

(5E) *If m and μ are equivalent T-invariant measures then $m = c\mu$ for some non-zero constant c.*

PROOF. Suppose (1E) is not true, and that A is a wandering set of positive measure: i.e., $T^i A \cap T^j A = \emptyset$ for $i \neq j$. Let $B \subset A$ such that $0 < m(B) < m(A)$. Then

$$B^* = \bigcup_{n=-\infty}^{\infty} T^n B$$

is a T-invariant set of positive measure with $m(A - B^*) > 0$, contradicting that T is ergodic.

If (2E) is not true, then there exists a set A with $TA \subset A$ and $m(A - TA) > 0$. But, then the set $B = A - TA$ would be a wandering set contradicting property (1E).

For any $k \geq 0$ and any set A of positive measure, let $A^* = \bigcup_{n=k}^{\infty} T^n A$. Then property (2E) implies that $TA^* = A^*$. The ergodicity of T implies that $A^* = X$. This shows (3E).

Assume (4E) is not true and that f is a T-invariant function (i.e., $f(Tx) = f(x)$ a.e.), and c is a constant so that the sets $A = \{x : f(x) > c\}$ and $B = \{x : f(x) < c\}$ both have positive measure. It follows that $TA = A$, and $TB = B$, and $A \cap B = \emptyset$, again contradicting the ergodicity of T.

Suppose $m \sim \mu$, and both m and μ are T-invariant. Let $f(x)$ be the Radon-Nikodym derivative of m with respect to μ. Then, for every $A \in \mathcal{B}$,

$$\int_A f(Tx)d\mu(x) = \int_A f(Tx)d\mu(Tx) = \int_{TA} f(x)d\mu(x)$$
$$= m(TA) = m(A) = \int_A f(x)d\mu(x)$$

which implies that $f(Tx) = f(x)$. (4E) says $f(x) \equiv c$ for some constant c proving (5E). □

Ergodic transformations are the basic building blocks for nonsingular transformations, and in the literature there are many examples of ergodic transformations: some preserving a finite measure equivalent to m, others preserving a σ-finite infinite measure equivalent to m, and some preserving no σ-finite measure equivalent to m. From Proposition 4.3 property (1E) it follows that ergodic transformations are recurrent. Property (4E) implies that for an ergodic transformation that preserves an infinite measure m there does not exist a finite invariant measure μ, $\mu \sim m$; in particular, none of the conditions (1R)–(6R) of the Recurrence Theorem 2.3 is sufficient for the existence of a finite invariant measure μ equivalent to m. The Finite Invariant Measures Theorem 3.2 then implies that if an ergodic transformation T preserves an infinite measure then it necessarily accepts (ww) sequences. This is in sharp contrast to the misconception that an ergodic transformation in general has some sort of mixing character. It turns out that ergodic transformations that do not preserve a finite measure μ equivalent to m admit even more interesting sequences than the weakly wandering sequences, as we exhibit below. We note for a given transformation T, it is easier to verify the "disjointness" condition in the definition of (sww) sequences than to verify the "exhaustive" condition of (eww) sequences for T. The next result shows that for an ergodic transformation T, all (sww) sequences are (eww).

PROPOSITION 4.4. *Let T be an ergodic transformation. Then every Strongly Weakly Wandering sequence for T is an Exhaustive Weakly Wandering sequence for T.*

PROOF. The proof follows directly from Proposition 4.2 and property (3E) of Proposition 4.3. □

In the next two propositions we show the existence of (sww) sequences for any transformation that does not preserve a finite invariant measure μ equivalent to m.

PROPOSITION 4.5. *For any measurable set A of positive measure, let $A^p = \bigcup_{s=-p}^{p} T^s A$ for $p \geq 1$. Suppose that for some sequence of integers $\{c_i : i \geq 1\}$*

$$\liminf_{i\to\infty} \left[m(T^{c_i}A^p) + m(T^{2c_i}A^p)\right] = 0 \quad \textit{for all} \;\; p \geq 1\,. \tag{4.5}$$

Then there is a subsequence $\{n_i\}$ of $\{c_i\}$ with the property that every infinite subset of the sequence $\{n_i : i \geq 1\}$ is a Strongly Weakly Wandering sequence for the transformation T.

PROOF. Let $\varepsilon > 0$ be such that $0 < \varepsilon < m(A)$. We let $c_0 = 0$, $p_1 = 1$, $A^{p_1} = \bigcup_{s=-p_1}^{p_1} T^s A$, and for $i \geq 1$ we let $\varepsilon_i = \varepsilon/2^i$. We use (4.5) and choose $n_1 = c_{i_1}$ so that $m\left(T^{n_1} A^{p_1}\right) + m\left(T^{2n_1} A^{p_1}\right) < \varepsilon_1$.

Next, we let $p_2 = 2|n_1| + p_1 + 2$, and $A^{p_2} = \bigcup_{s=-p_2}^{p_2} T^s A$. We use (4.5) and choose $n_2 = c_{i_2}$ such that $m\left(T^{n_2} A^{p_2}\right) + m\left(T^{2n_2} A^{p_2}\right) < \varepsilon_2$.

We continue by induction. Assume that the integers $n_1, n_2, \ldots, n_{k-1}$ and $p_1, p_2, \ldots, p_{k-1}$, have been chosen. We set $p_k = 2|n_{k-1}| + p_{k-1} + k$, and

$$A^{p_k} = \bigcup_{s=-p_k}^{p_k} T^s A \,.$$

We use (4.5) and choose $n_k = c_{i_k}$ so that

$$m\left(T^{n_k} A^{p_k}\right) + m\left(T^{2n_k} A^{p_k}\right) < \varepsilon_k \,.$$

Finally, we let

$$A_0 = A - \bigcup_{s=1}^{\infty} \left(T^{n_s} A^{p_s} \cup T^{2n_s} A^{p_s}\right).$$

We note that

$$T^{n_s} A^{p_s} \cap A_0 = \emptyset \quad \text{and} \quad T^{2n_s} A^{p_s} \cap A_0 = \emptyset \quad \text{for all } s \geq 1 \,. \tag{4.6}$$

We have

$$\begin{aligned} m(A_0) &\geq m(A) - m\left[\bigcup_{s=1}^{\infty} \left(T^{n_s} A^{p_s} \cup T^{2n_s} A^{p_s}\right)\right] \\ &\geq m(A) - \sum_{s=1}^{\infty} \left[m\left(T^{n_s} A^{p_s}\right) + m\left(T^{2n_s} A^{p_s}\right)\right] > m(A) - \varepsilon > 0 \,. \end{aligned}$$

Next, we show that any infinite subset of the sequence $\{n_i : i \geq 1\}$ is an (sww) sequence for the transformation T with the same set A_0. We examine when the following is true:

$$T^{n_i - n_k + k'} A_0 \cap T^{n_j - n_l + l'} A_0 = \emptyset \tag{4.7}$$

$$\text{for } i, j, k, l \geq 0 \,, \quad 0 \leq k' \leq k \,, \quad 0 \leq l' \leq l \,, \quad i > j \,.$$

If one of the indices, say k, is larger than all the others, then

$$T^{n_k - k' - n_i + n_j - n_l + l'} A_0 \subset T^{n_k} A^{p_k}$$

and (4.6) implies (4.7).

If $i = l > \max\{j, k\}$, then $T^{2n_i - n_k + k' - n_j - l'} A_0 \subset T^{2n_i} A^{p_i}$ and again (4.6) implies (4.7).

Let us now consider any infinite subsequence of the sequence $\{n_i : i \geq 1\}$ by eliminating members from $\{n_1, n_2, n_3, \ldots\}$. Let $n_0 = 0$, and after re-indexing the new sequence in the order obtained, let us denote it by the same symbols $\{n_i : i \geq 0\}$. From (4.7) it follows that for this sequence $\{n_i\}$ the following holds:

$$T^{n_i - n_k + k} A_0 \cap T^{n_j - n_l + l} A_0 = \emptyset \quad \text{for } \ i, j, k, l = 0, 1, 2, \dots; i > j$$

whenever one of the indices $\{i, j, k, l\}$ is larger than all the others, or $i = k > \max\{j, l\}$.

This completes the proof of the Proposition. $\square$

PROPOSITION 4.6. *Suppose for a measurable set A of positive measure*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} m(T^i A) = 0\,.$$

Let $A^p = \bigcup_{j=-p}^{p} T^j A$ for $p \geq 0$. It follows that

$$\liminf_{n\to\infty} \big[m(T^n A^p) + m(T^{2n} A^p)\big] = 0 \quad \textit{for all} \ \ p \geq 1\,.$$

PROOF. The set function

$$\overline{\sigma}(B) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} m(T^i B)$$

is sub-additive and $\overline{\sigma}(T^k B) = \overline{\sigma}(B)$ for $k \in \mathbb{Z}$. Let $A^p = \bigcup_{j=-p}^{p} T^j A$ for $p \geq 0$. Then,

$$\overline{\sigma}(A^p) = \overline{\sigma}\Big(\bigcup_{j=-p}^{p} T^j A\Big) \leq \sum_{j=-p}^{p} \overline{\sigma}(T^j A) = (2p+1)\overline{\sigma}(A)\,.$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} m(T^i A) = 0 \quad \text{for } \ A \in \mathcal{B}$$

implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} m(T^i A^p) = 0 \quad \text{for any } \ p \geq 0. \tag{4.8}$$

The inequality

$$\frac{1}{n} \sum_{i=0}^{n-1} \big[m(T^i A^p) + m(T^{2i} A^p)\big] \leq \frac{1}{2n} \sum_{i=0}^{2n-1} 4m(T^i A^p)$$

together with (4.8) imply

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \big[m(T^i A^p) + m(T^{2i} A^p)\big] = 0$$

and this implies

$$\liminf_{n\to\infty}\left[m(T^n A^p) + m(T^{2n} A^p)\right] = 0 .$$

This completes the proof. □

The next theorem which generalizes a result of Krengel-Jones [11], follows from the two previous propositions.

THEOREM 4.7. *Let T be an ergodic transformation defined on the finite measure space $(X, \mathcal{B}, m)$. Suppose there is no finite invariant measure μ equivalent to m. Then T possesses Exhaustive Weakly Wandering sequences $\{n_i\}$ with the property that every subsequence $\{n_i'\}$ of the sequence $\{n_i\}$ is again an Exhaustive Weakly Wandering sequence for the transformation T.*

PROOF. Since T does not preserve a finite invariant measure μ equivalent to m, Theorem 3.2 implies that there is a set A, of positive measure and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} m(T^i A) = 0 .$$

The two previous Propositions (4.5 and 4.6) show that there is an (sww) sequence $\{n_i\}$ for the transformation T with the property that every subsequence $\{n_i'\}$ of the sequence $\{n_i\}$ is again an (sww) sequence. Proposition 4.4 then concludes the proof of the Theorem. □

We conclude with an interesting observation that says when a (ww) sequence contains an (eww) sequence.

PROPOSITION 4.8. *Let the sequence $\{n_i\}$ be a Weakly Wandering sequence for the ergodic transformation T. If $\{2n_i\}$ is also a Weakly Wandering sequence for T then $\{n_i\}$ contains an Exhaustive Weakly Wandering subsequence $\{n_i'\}$ with the property that every subsequence of the sequence $\{n_i'\}$ is again an Exhaustive Weakly Wandering sequence for the transformation T.*

PROOF. Let C be a (ww) set under a (ww) sequence $\{n_i\}$ for T, and let $C^p = \bigcup_{s=-p}^{p} T^s C$ for $p \geq 0$. Since for any integer $s \in \mathbb{Z}$, the sets $T^{n_i}\,(T^s C)$ are mutually disjoint for $i \geq 0$, and $m(X) < \infty$, we have $\lim_{i\to\infty} m\left[T^{n_i}\left(T^s C\right)\right] = 0$ for any $s \in \mathbb{Z}$. Then,

$$\begin{aligned}\lim_{i\to\infty} m\big(T^{n_i} C^p\big) &= \lim_{i\to\infty} m\Bigg(T^{n_i} \bigcup_{s=-p}^{p} T^s C\Bigg)\\ &\leq \sum_{s=-p}^{p} \lim_{i\to\infty} m\left(T^{n_i}\left(T^s C\right)\right) = 0 .\end{aligned}$$

Similarly, since $\{2n_i\}$ is a (ww) sequence for T, then there is a set D, $m(D) > 0$, such that if $D^p = \bigcup_{s=-p}^{p} T^s D$ for $p \geq 0$, then $\lim_{i\to\infty} m\left(T^{2n_i} D^p\right) = 0$.

T ergodic implies: for some positive integer k, $A = T^k C \cap D$ has positive measure and that both sequences $\{n_i\}$ and $\{2n_i\}$ are (ww) sequences for the transformation T with the same (ww) set A. From the above discussion we conclude that for the sets $A^p = \bigcup_{s=-p}^{p} T^s A$ for $p \geq 0$,

$$\lim_{i\to\infty} [m(T^{n_i} A^p) + m(T^{2n_i} A^p)] = \lim_{i\to\infty} m(T^{n_i} A^p) + \lim_{i\to\infty} m(T^{2n_i} A^p) = 0\,.$$

Proposition 4.5 then completes the proof. □

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