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A Note on Traces of Singular Moduli

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Abstract. We generalize Osburn's work ([6]) about a congruence for traces defined in terms of Hauptmoduli associated to certain genus zero groups of higher levels.

1. Introduction

Let $\mathfrak{H}$ denote the complex upper half-plane and $\mathfrak{H}^* := \mathfrak{H} \cup \mathbf{Q} \cup \{\infty\}$. For an integer N (≥ 2), let $\Gamma_0(N)^*$ be the group generated by $\Gamma_0(N)$ and all Atkin-Lehner involutions W_e for $e||N$. There are only finitely many N for which the modular curve $\Gamma_0(N)^*\backslash\mathfrak{H}^*$ has genus zero ([5]). In particular, if we let $\mathfrak{S}$ be the set of such N which are prime, then

$$\mathfrak{S} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}.$$

For each $p \in \mathfrak{S}$, let $j_p^*(\tau)$ be the corresponding Hauptmodul with a Fourier expansion of the form $q^{-1} + O(q)$ where $q := e^{2\pi i\tau}$.

Let $p \in \mathfrak{S}$. For an integer d (≥ 1) such that $-d \equiv \square \pmod{4p}$, let $\mathcal{Q}_d$ be the set of all positive definite integral binary quadratic forms $Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$ of discriminant $-d = b^2 - 4ac$. To each $Q \in \mathcal{Q}_d$, we associate the unique root $\alpha_Q \in \mathfrak{H}$ of $Q(x, 1)$. Consider the set

$$\mathcal{Q}_{d,p} := \{[a, b, c] \in \mathcal{Q}_d \ :\ a \equiv 0 \pmod{p}\},$$

on which $\Gamma_0(p)^*$ acts. We then define the *trace* $t^{(p)}(d)$ by

$$t^{(p)}(d) := \sum_{Q \in \mathcal{Q}_{d,p}/\Gamma_0(p)^*} \frac{1}{\omega_Q} j_p^*(\alpha_Q) \quad (\in \mathbf{Z}),$$

where ω_Q is the number of stabilizers of Q in the transformation group $\pm\Gamma_0(p)^*/\pm 1$ ([4]).

Osburn ([6]) showed the following congruence:

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THEOREM 1.1. *Let $p \in \mathfrak{S}$. If d (≥ 1) is an integer such that $-d \equiv \square \pmod{4p}$ and ℓ $(\neq p)$ is an odd prime which splits in $\mathbf{Q}(\sqrt{-d})$, then*

$$t^{(p)}(\ell^2 d) \equiv 0 \pmod{\ell} .$$

Although this result is true, we think that his proof seems to be unclear. Precisely speaking, let D (≥ 1) be an integer such that $D \equiv \square \pmod{4p}$. In §3 we shall define

$$A_\ell(D,d) := \text{the coefficient of } q^D \text{ in } f_{d,p}(\tau)|T_{1/2,p}(\ell^2) ,$$
$$B_\ell(D,d) := \text{the coefficient of } q^d \text{ in } g_{D,p}(\tau)|T_{3/2,p}(\ell^2) ,$$

where $f_{d,p}(\tau)$ and $g_{D,p}(\tau)$ are certain half integral weight modular forms, and $T_{1/2,p}(\ell^2)$ and $T_{3/2,p}(\ell^2)$ are Hecke operators of weight 1/2 and 3/2, respectively. The key step that is not presented in Osburn's work is the fact $A_\ell(1,d) = -B_\ell(1,d)$ which would be nontrivial at all. In this paper we shall first give a proof of more general statement $A_\ell(D,d) = -B_\ell(D,d)$ (Proposition 3.1), and then further generalize Theorem 1.1 as follows,

$$t^{(p)}(\ell^{2n} d) \equiv 0 \pmod{\ell^n}$$

for all n (≥ 1) (Theorem 3.3).

2. Preliminaries

Let k and N (≥ 1) be integers. If $f(\tau)$ is a function on $\mathfrak{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$, then we define the *slash operator* $[\gamma]_{k+1/2}$ on $f(\tau)$ by

$$f(\tau)|[\gamma]_{k+1/2} := j(\gamma,\tau)^{-2k-1} f(\gamma\tau) ,$$

where

$$j(\gamma,\tau) := \left(\frac{c}{d}\right)\varepsilon_d^{-1}\sqrt{c\tau+d} \quad \text{with } \varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv 3 \pmod 4 . \end{cases}$$

Here, $(\frac{c}{d})$ is the Kronecker symbol and $\sqrt{c\tau+d}$ takes its argument on the interval $(-\pi/2, \pi/2]$.

We denote by $M^{+\cdots+}_{k+1/2}(N)^!$ the infinite dimensional vector space of weakly holomorphic modular forms of weight $k+1/2$ on $\Gamma_0(4N)$ which satisfy the Kohnen plus condition. Namely, the space consists of the functions $f(\tau)$ on $\mathfrak{H}$ such that

(i) $f(\tau)$ is holomorphic on $\mathfrak{H}$ and meromorphic at the cusps,
(ii) $f(\tau)$ is invariant under the action of $[\gamma]_{k+1/2}$ for all $\gamma \in \Gamma_0(4N)$,
(iii) $f(\tau)$ has a Fourier expansion of the form

$$\sum_{(-1)^k n \equiv \square \pmod{4N}} a(n) q^n .$$

Suppose that ℓ is a prime with $\ell \nmid N$. The action of the *Hecke operator* $T_{k+1/2,N}(\ell^2)$ on a form

$$f(\tau) = \sum_{(-1)^k n \equiv \square \pmod{4N}} a(n) q^n \quad \text{in} \quad M^{+\cdots+}_{k+1/2}(N)^!$$

is given by

$$\begin{aligned} f(\tau)|T_{k+1/2,N}(\ell^2) := \ell_k & \sum_{(-1)^k n \equiv \square \pmod{4N}} \\ & \times \left(a(\ell^2 n) + \left(\frac{(-1)^k n}{\ell} \right) \ell^{k-1} a(n) + \ell^{2k-1} a(n/\ell^2) \right) q^n , \end{aligned} \tag{2.1}$$

where

$$\ell_k := \begin{cases} \ell^{1-2k} & \text{if } k \le 0 \\ 1 & \text{otherwise.} \end{cases}$$

Here, $a(n/\ell^2) := 0$ if $\ell^2 \nmid n$. As is well-known, $f(\tau)|T_{k+1/2,N}(\ell^2)$ belongs to $M^{+\cdots+}_{k+1/2}(N)^!$.

PROPOSITION 2.1. *Let $p \in \mathfrak{S}$.*

(i) *For every integer D (≥ 1) such that $D \equiv \square \pmod{4p}$, there is a unique $g_{D,p}$ in $M^{+\cdots+}_{3/2}(p)^!$ with the Fourier expansion*

$$g_{D,p}(\tau) = q^{-D} + \sum_{d \ge 0,\ -d \equiv \square \pmod{4p}} B(D,d) q^d \quad (B(D,d) \in \mathbf{Z}) .$$

(ii) *For every integer d (≥ 0) such that $-d \equiv \square \pmod{4p}$, there is a unique form*

$$f_{d,p}(\tau) = \sum_{D \in \mathbf{Z}} A(D,d) q^D \quad (A(D,d) \in \mathbf{Z})$$

in $M^{+\cdots+}_{1/2}(p)^!$ with a Fourier expansion of the form $q^{-d} + O(q)$. They form a basis of $M^{+\cdots+}_{1/2}(p)^!$.

(iii) *For every integer d (≥ 0) such that $-d \equiv \square \pmod{4p}$ and every integer D (≥ 1) such that $D \equiv \square \pmod{4p}$, we have*

$$A(D,d) = -B(D,d) .$$

(iv) *For every integer d (≥ 1) such that $-d \equiv \square \pmod{4p}$, we get*

$$t^{(p)}(d) = -B(1,d) .$$

PROOF. See [1, Theorem 5.6], [3, §2.2] and [4, Lemma 3.4 and Corollary 3.5]. $\square$

3. Generalization of Theorem 1.1

We first prove the following necessary proposition by adopting Zagier's argument ([7, Theorem 5]).

PROPOSITION 3.1. *Let $p \in \mathfrak{S}$ and ℓ ($\neq p$) be a prime. For each integer d (≥ 0) such that $-d \equiv \square \pmod{4p}$, we define integers $A_\ell(D,d)$ and $B_\ell(D,d)$ in the following manner*:

$$A_\ell(D,d) := \text{the coefficient of } q^D \text{ in } f_{d,p}(\tau)|T_{1/2,p}(\ell^2) \text{ for each integer } D,$$

$$B_\ell(D,d) := \text{the coefficient of } q^d \text{ in } g_{D,p}(\tau)|T_{3/2,p}(\ell^2) \text{ for each integer } D\ (\geq 1)$$
$$\text{such that } D \equiv \square \pmod{4p}.$$

Then we have the relation

$$A_\ell(D,d) = -B_\ell(D,d) \text{ for every integer } D\ (\geq 1) \text{ such that } D \equiv \square \pmod{4p}.$$

PROOF. For a pair of rational numbers a and b, let

$$\delta_{a,b} := \begin{cases} 1 & \text{if } a = b \in \mathbf{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Let d (≥ 0) be a fixed integer such that $-d \equiv \square \pmod{4p}$. It follows from the defining property of $f_{d,p}(\tau)$, namely,

$$A(D,d) = \delta_{D,-d} \quad \text{if } D \leq 0$$

that if $D \leq 0$, then

$$\begin{aligned} A_\ell(D,d) &= \ell A(\ell^2 D, d) + \left(\frac{D}{\ell}\right) A(D,d) + A(D/\ell^2, d) \text{ by the definition (2.1)} \\ &= \ell\delta_{\ell^2 D,-d} + \left(\frac{D}{\ell}\right)\delta_{D,-d} + \delta_{D/\ell^2,-d} \\ &= \ell\delta_{D,-d/\ell^2} + \left(\frac{D}{\ell}\right)\delta_{D,-d} + \delta_{D,-d\ell^2}\,. \end{aligned}$$

Hence the principal part of $f_{d,p}(\tau)|T_{1/2,p}(\ell^2)$ at infinity is

$$\ell q^{-d/\ell^2} + \left(\frac{-d}{\ell}\right) q^{-d} + q^{-d\ell^2}\,,$$

where the first term should be omitted unless $-d/\ell^2$ is an integer. Therefore we achieve

$$f_{d,p}(\tau)|T_{1/2,p}(\ell^2) = \ell f_{d/\ell^2,p}(\tau) + \left(\frac{-d}{\ell}\right) f_{d,p}(\tau) + f_{d\ell^2,p}(\tau) \text{ by Proposition 2.1(ii).} \quad (3.1)$$

And, for every integer D (≥ 1) such that $D \equiv \square \pmod{4p}$ we derive that

$$A_\ell(D,d) = \ell A(D, d/\ell^2) + \left(\frac{-d}{\ell}\right) A(D,d) + A(D, d\ell^2) \text{ by (3.1)}$$

$$= -\ell B(D, d/\ell^2) - \left(\frac{-d}{\ell}\right) B(D, d) - B(D, d\ell^2) \text{ by Proposition 2.1(iii)}$$
$$= -B_\ell(D, d) \text{ by the definition (2.1)}.$$

□

On the other hand, we apply Jenkins' idea ([2]) to develop a formula for the coefficient $B(D, \ell^{2n} d)$.

PROPOSITION 3.2. *Let $p \in \mathfrak{S}$ and ℓ ($\neq p$) be a prime. If d (≥ 0) and D (≥ 1) are integers such that $-d \equiv \square \pmod{4p}$ and $D \equiv \square \pmod{4p}$, then*

$$B(D, \ell^{2n} d) = \ell^n B(\ell^{2n} D, d) + \sum_{t=0}^{n-1} \left(\frac{D}{\ell}\right)^{n-t-1} (B(D/\ell^2, \ell^{2t} d) - \ell^{t+1} B(\ell^{2t} D, d/\ell^2))$$
$$+ \sum_{t=0}^{n-1} \left(\frac{D}{\ell}\right)^{n-t-1} \left(\left(\frac{D}{\ell}\right) - \left(\frac{-d}{\ell}\right)\right) \ell^t B(\ell^{2t} D, d)$$

for all n (≥ 1).

PROOF. From the definition (2.1), we have

$$A_\ell(D, d) = \ell A(\ell^2 D, d) + \left(\frac{D}{\ell}\right) A(D, d) + A(D/\ell^2, d), \tag{3.2}$$

$$B_\ell(D, d) = \ell B(D, d/\ell^2) + \left(\frac{-d}{\ell}\right) B(D, d) + B(D, d\ell^2). \tag{3.3}$$

Combining Proposition 3.1 with (3.2), we get

$$B_\ell(D, d) = \ell B(\ell^2 D, d) + \left(\frac{D}{\ell}\right) B(D, d) + B(D/\ell^2, d). \tag{3.4}$$

We then derive from (3.3) and (3.4) that

$$\begin{aligned} B(D, \ell^2 d) &= \ell B(\ell^2 D, d) \\ &\quad + \left(\frac{D}{\ell}\right) B(D, d) + B(D/\ell^2, d) - \ell B(D, d/\ell^2) - \left(\frac{-d}{\ell}\right) B(D, d). \end{aligned} \tag{3.5}$$

The remaining part of the proof is exactly the same as that of [2] Theorem 1.1. Indeed, one can readily prove the proposition by using induction on n and applying only (3.5). □

Now, we are ready to prove our main theorem which would be a generalization of Osburn's result.

THEOREM 3.3. *With the same notations as in Theorem 1.1, we have*

$$t^{(p)}(\ell^{2n} d) \equiv 0 \pmod{\ell^n}$$

for all n (≥ 1).

PROOF. We achieve that

$$
\begin{aligned}
t^{(p)}(\ell^{2n}d) &= -B(1,\ell^{2n}d) \text{ by Proposition 2.1(iv)}\\
&= -\ell^n B(\ell^{2n},d) - \sum_{t=0}^{n-1}\left(\frac{1}{\ell}\right)^{n-t-1}(B(1/\ell^2,\ell^{2t}d) - \ell^{t+1}B(\ell^{2t},d/\ell^2))\\
&\quad -\sum_{t=0}^{n-1}\left(\frac{1}{\ell}\right)^{n-t-1}\left(\left(\frac{1}{\ell}\right)-\left(\frac{-d}{\ell}\right)\right)\ell^t B(\ell^{2t},d) \text{ by Proposition 3.2}\\
&= -\ell^n B(\ell^{2n},d) \text{ by the facts that } 1/\ell^2 \text{ and } d/\ell^2 \text{ are not integers, and } \left(\frac{-d}{\ell}\right)=1\\
&\equiv 0 \pmod{\ell^n},
\end{aligned}
$$

as desired. □

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