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Non-compact and Non-trivial Minimal Sets of a Locally Compact Flow

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Introduction

The structure of the compact minimal sets of a flow is well known [1, p. 38]. However, the literature on the non-compact minimal sets seems rather scanty [1, p. 40].

The purpose of this paper is to investigate the structure of non-compact and non-trivial minimal sets of a locally compact flow, i.e., a dynamical system on a locally compact metric space.

The main results obtained are as follows.

1) The non-compact and non-trivial minimal set of a locally compact flow consists of infinitely many trajectories (Theorem 2).

2) The non-compact and non-trivial minimal set of a locally compact flow consists of the points which are

i) Poisson stable,

ii) positively Poisson stable and negatively receding,

iii) negatively Poisson stable and positively receding,

and they all exist simultaneously (Theorem 3).

3) Poisson stable trajectories in the non-compact and non-trivial minimal set of the locally compact flow are neither pseudorecurrent nor almost recurrent (Theorem 4).

4) The set of all Poisson stable points in the non-compact and non-trivial minimal set of a locally compact flow is dense in the minimal set (Theorem 1).

§1. Standing notations and definitions.

N is the set of all natural numbers. R is the real line. R^+ denotes the set $\{t \in R; t \geqq 0\}$. R^- denotes the set $\{t \in R; t \leqq 0\}$.

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Let X be a metric space with metric d. A flow, or a dynamical system, on X is the triplet (X, R, f), where f is a map of $X \times R$ onto X such that

a) $f(x, 0)=x$ for every $x \in X$,

b) $f(f(x, s), t)=f(x, s+t)$ for every $x \in X$ and every $s, t \in R$,

c) f is continuous on $X \times R$.

The trajectory of $x \in X$ in the flow (X, R, f) is defined to be the set $\{f(x, t); t \in R\}$, which is denoted by $C(x)$.

$f(x, [a, b])$ denotes the arc $\{f(x, t); t \in [a, b]\}$.

A point $x \in X$ is called a rest point of (X, R, f) if $f(x, t)=x$ for every $t \in R$. A point $x \in X$ is called periodic in (X, R, f) if there exists $T \in R-\{0\}$ such that $f(x, t)=f(x, t+T)$ for every $t \in R$.

A set $M \subset X$ is called invariant (positively invariant) (negatively invariant) in (X, R, f) if $f(x, t) \in M$ for any $x \in M$ and any $t \in R (t \in R^+)$ $(t \in R^-)$.

A set $M \subset X$ is called minimal in (X, R, f) if it is non-empty, closed, invariant, and no proper subsets of M have these properties.

$L^+(x)$ denotes the set $\{y \in X$; there exists a sequence $\{t_n\}$ in R with $t_n \rightarrow +\infty$ and $f(x, t_n) \rightarrow y\}$.

$L^-(x)$ denotes the set $\{y \in X$; there exists a sequence $\{t_n\}$ in R with $t_n \rightarrow -\infty$ and $f(x, t_n) \rightarrow y\}$.

$L^+(x)$ $(L^-(x))$ is called the positive (negative) limit set of x.

A point $x \in X$ or the trajectory $C(x)$ is called positively (negatively) receding, if $L^+(x)$ $(L^-(x))$ is empty;

receding, if x is receding both positively and negatively;

positively (negatively) asymptotic, if $L^+(x)$ $(L^-(x))$ is non-empty but $L^+(x) \cap C(x)$ $(L^-(x) \cap C(x))$ is empty;

positively (negatively) Poisson stable, if $L^+(x) \cap C(x)$ $(L^-(x) \cap C(x))$ is non-empty;

Poisson stable, if x is both positively and negatively Poisson stable.

$J^+(x)$ denotes the set $\{y \in X$; there exist a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R such that $x_n \rightarrow x$, $t_n \rightarrow +\infty$, and $f(x_n, t_n) \rightarrow y\}$.

$J^-(x)$ denotes the set $\{y \in X$; there exist a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R such that $x_n \rightarrow x$, $t_n \rightarrow -\infty$, and $f(x_n, t_n) \rightarrow y\}$.

$J^+(x)$ $(J^-(x))$ is called the first positive (negative) prolongational limit set of x.

A point $x \in X$ is called non-wandering, if $x \in J^+(x)$ or $x \in J^-(x)$. ($x \in J^+(x)$ and $x \in J^-(x)$ are equivalent [1, p. 35, Theorem 2.12]).

§2. The structure of non-compact and non-trivial minimal sets.

A minimal set is called trivial if it consists only of one trajectory. A minimal set which is not trivial is called non-trivial. The structure of a trivial minimal set is simple, since it is receding, periodic, or a rest point. On the other hand, the structure of compact minimal sets is completely determined by G. D. Birkhoff [1, p. 38ff]. However, not very much is known about the properties of non-compact and non-trivial minimal sets. Thus we are interested in the study of the properties of these sets.

PROPOSITION 1. *Let X be a metric space, and let $M \subset X$ be a non-trivial minimal set of (X, R, f). Then, every trajectory in M is Poisson stable, positively Poisson stable and negatively receding, or negatively Poisson stable and positively receding.*

PROOF. This proposition is equivalent to the proposition that M contains no trajectories such as receding, positively asymptotic, or negatively asymptotic. Suppose $x \in M$ and $C(x)$ is receding. Then $C(x)$ itself is a closed invariant subset of M. Hence $C(x)=M$ in contradiction with the non-triviality of M. If $C(x)$ is positively asymptotic, then $\emptyset \neq L^+(x) \subset M$, $x \bar{\in} L^+(x)$. Since $L^+(x)$ is a closed invariant set, this contradicts the minimality of M. Hence $C(x)$ cannot be positively asymptotic. Analogously $C(x)$ cannot be negatively asymptotic. Q.E.D.

Hereafter in this section we consider the dynamical system on a locally compact metric space.

LEMMA 1 [2, p. 60]. *Let X be a locally compact metric space. If every point in X is non-wandering in (X, R, f), then the set of all Poisson stable points in (X, R, f) is dense in X.*

We need this lemma for the proof of the following theorem.

THEOREM 1. *Let M be a non-trivial minimal set of (X, R, f), where X is assumed to be locally compact metric space. Then the set of all Poisson stable points in M is dense in M.*

PROOF. Let $x \in M$. Then we have $L^+(x)=M$ or $L^-(x)=M$. Suppose $L^+(x)=M$. Let g be the restriction of f to M. The map g defines a dynamical system (M, R, g). By $L_M^+(x)$ and $J_M^+(x)$ we denote the positive limit set and the positive prolongational limit set of x in (M, R, g) respectively, i.e.,

$L_M^+(x)=\{y \in M$; there exists a sequence $\{t_n\}$ in R with $t_n \to +\infty$ and $g(x, t_n) \to y\}$ and

$J_M^+(x)=\{y \in M$; there exist a sequence $\{x_n\}$ in M and a sequence $\{t_n\}$ in R such that $x_n \to x$, $t_n \to +\infty$, and $g(x_n, t_n) \to y\}$.

$L_M^-(x)$ and $J_M^-(x)$ are also defined analogously.

It is clear that

$$L_M^+(x) \subset J_M^+(x) \subset M. \tag{a}$$

On the other hand we have

$$L_M^+(x) = L^+(x) \tag{b}$$

because of the invariance and closedness of M. Thus we see $L_M^+(x)=J_M^+(x)=M$ from (a) and (b). Hence $x \in J_M^+(x)$. This implies that x is non-wandering in (M, R, g). In the same way as above we can prove that every $x \in M$ such that $L^-(x)=M$ is non-wandering in (M, R, g). On the other hand, M is locally compact, since X is locally compact and M is closed in X. Hence the set of all the points which are Poisson stable in (M, R, g) is dense in M by Lemma 2. Q.E.D.

The trivial minimal set contains only one trajectory. How many trajectories does a non-trivial minimal set contain? An answer to this problem is the following.

THEOREM 2. *Every non-trivial minimal set in the dynamical system on a locally compact metric space contains infinitely many trajectories.*

We need the following lemma to prove this theorem.

LEMMA 2. *Every point in the non-trivial minimal set is neither periodic nor a rest point.*

This lemma is an immediate consequence of the definition of the non-trivial minimal set.

PROOF OF THEOREM 2. Let X be a locally compact metric space. M is assumed to be a non-trivial minimal set of (X, R, f). Suppose M consists of finitely many trajectories, say $C(x_1), \cdots, C(x_n)$. Then we have

$$M = L^+(x_i) \cup L^-(x_i) \tag{1}$$

for every $i \in \{1, \cdots, n\}$. Choose any $p_0 \in M$. p_0 belongs to $L^+(x_1) \cup L^-(x_1)$ by (1). Take any $\varepsilon > 0$ such that $\overline{S(p_0, \varepsilon)}$ is compact, where $\overline{S(p_0, \varepsilon)}$ is the closure of an open ball $S(p_0, \varepsilon)$ of radius ε and center p_0. Then there exists $t_1 \in R$ such that $|t_1|>1$ and $f(x_1, t_1) \in S(p_0, \varepsilon)$. For brevity, we denote $f(x_1, t_1)$ by p_1. Then

$$p_1 \in S(p_0, \varepsilon) . \tag{2}$$

Since x_1 is neither periodic nor a rest point by Lemma 2, we have

$$p_1 \bar{\in} f(x, [-1, 1]) .$$

On the other hand it is valid that

$$p_1 = f(x_1, t_1) \bar{\in} f(x_i, [-1, 1])$$

for every $i \neq 1$. Thus we have

$$p_1 \bar{\in} \bigcup_{i=0}^{n} f(x_i, [-1, 1]) . \tag{3}$$

There exists $\alpha>0$ such that $\alpha<(\varepsilon/2)$ and

$$S(p_1, \alpha) \subset S(p_0, \varepsilon)$$

by (2). On the other hand there exists $\beta>0$ such that $\beta<(\varepsilon/2)$ and

$$\overline{S(p_1, \beta)} \cap \left(\bigcup_{i=0}^{n} f(x_i, [-1, 1]) \right) = \emptyset$$

by (3). Let $\varepsilon_1 = \min\{\alpha, \beta\}$. Then we obtain the following:

$$\begin{cases} S(p_1, \varepsilon_1) \subset S(p_0, \varepsilon) , \\ \overline{S(p_1, \varepsilon_1)} \cap \left(\bigcup_{i=1}^{n} f(x_i, [-1, 1]) \right) = \emptyset , \\ \varepsilon_1 < \dfrac{\varepsilon}{2} . \end{cases} \tag{4}$$

Since we have

$$p_1 \in C(x_1) \subset L^+(x_1) \cup L^-(x_1) = M ,$$

there exists t_2 such that

$$|t_2|>2 \quad \text{and} \quad f(x_1, t_2) \in S(p_1, \varepsilon_1) .$$

Let $p_2 = f(x_1, t_2)$. As $C(x_1)$ is neither periodic nor a rest point, we have

$$p_2 = f(x_1, t_2) \bar{\in} f(x_1, [-2, 2])$$

and

$$p_2=f(x_1, t_2)\,\bar{\in}\, f(x_i, [-2, 2])$$

for every $i\neq 1$, so that

$$p_2\,\bar{\in}\,\bigcup_{i=1}^{*} f(x_i, [-2, 2])\,.$$

Applying the procedure to obtain ε_1 which satisfies (4), we obtain $\varepsilon_2>0$ such that

$$\begin{aligned}&S(p_2, \varepsilon_2)\subset S(p_1, \varepsilon_1)\,,\\&\overline{S(p_2, \varepsilon_2)}\cap\Big(\bigcup_{i=1}^{*} f(x_i, [-2, 2])\Big)=\varnothing\,,\\&\varepsilon_2<\frac{\varepsilon_1}{2}\,.\end{aligned}$$

Iterating this procedure we obtain a sequence $\{\varepsilon_k\}$ in R such that for every $k\in N$

$$\begin{aligned}&S(p_k, \varepsilon_k)\subset S(p_{k-1}, \varepsilon_{k-1})\,,\\&\overline{S(p_k, \varepsilon_k)}\cap\Big(\bigcup_{i=1}^{*} f(x_i, [-k, k])\Big)=\varnothing\,,\\&0<\varepsilon_k<\frac{\varepsilon_{k-1}}{2}\,,\end{aligned}$$

where $p_k=f(x_1, t_k)$ and $|t_k|>k$. Thus we have a nested sequence

$$\overline{S(p_0, \varepsilon)}\supset\overline{S(p_1, \varepsilon_1)}\cdots\supset\overline{S(p_k, \varepsilon_k)}\supset\cdots\,,$$

where every $\overline{S(p_k, \varepsilon_k)}$ is non-empty and $\overline{S(p_0, \varepsilon)}$ is compact. Hence $\bigcap_{k=1}^{\infty}\overline{S(p_k, \varepsilon_k)}\neq\varnothing$. Take any $q\in\bigcap_{k=1}^{\infty}\overline{S(p_k, \varepsilon_k)}$, then we have

$$q\in L^+(x_1)\cup L^-(x_1)=M\,,$$

the proof of which is as follows. By the assumption it is valid that $d(p_k, q)\leqq\varepsilon_k$ for every $k\in N$. On the other hand $p_k=f(x_1, t_k)$ and $|t_k|>k$ hold for every $k\in N$. Hence we have

$$|t_k|\longrightarrow+\infty\quad(k\longrightarrow+\infty)$$

and

$$f(x_1, t_k)\longrightarrow q\quad(k\longrightarrow+\infty)\,,$$

so that

$$q \in L^+(x_1) \cup L^-(x_1) = M .$$

However, since

$$\overline{S(p_k, \varepsilon_k)} \cap \left(\bigcup_{i=1}^{n} f(x_i, [-k, k]) \right) = \varnothing$$

holds for every $k \in N$, we have

$$q \bar{\in} \bigcup_{i=1}^{n} f(x_i, [-k, k])$$

for every $k \in N$, which implies that

$$q \bar{\in} f(x_i, [-k, k])$$

for every $i \in \{1, \cdots, n\}$ and every $k \in N$, so that $q \bar{\in} f(x_i, R)$ for every $i \in \{1, \cdots, n\}$. Hence we have

$$q \in M - \left(\bigcup_{i=1}^{n} f(x_i, R) \right) ,$$

which is a contradiction. Q.E.D.

We see from Proposion 1 and Theorem 1 that there are four possible cases for the non-trivial minimal sets of the dynamical system on a locally compact metric space: the non-trivial minimal set of such a flow consists of

1) Poisson stable points,

2) Poisson stable points and the points which are positively Poisson stable and negatively receding,

3) Poisson stable points and the points which are negatively Poisson stable and positively receding,

4) Poisson stable points, the points which are positively Poison stable and negatively receding, and the points which are negatively Poisson stable and positively receding.

We shall show that 4) is the only possible case. For that purpose we introduce some concepts.

DEFINITION 1 [2, p. 65]. Let (Y, R, g) be a dynamical system. A set $M \subset Y$ is said to be positively (negatively) minimal if M is closed, positively (negatively) invariant, and contains no non-empty proper subsets with these properties.

The following lemmas are known.

LEMMA 3 [2, p. 67]. *Let (Y, R, g) be a dynamical system. For any $M \subset Y$ the following are pairwise equivalent:*

1) *M is positively (negatively) minimal,*
2) *$\overline{C^+(x)} = M$ $(\overline{C^-(x)} = M)$ for every $x \in M$,*
3) *$L^+(x) = M$ $(L^-(x) = M)$ for every $x \in M$.*

LEMMA 4 [3, p. 48]. *Let Y be a locally compact metric space. A subset of Y is positively (or negatively) minimal in a dynamical system (Y, R, g) if and only if it is compact and minimal in (Y, R, g).*

By the aid of these lemmas we can determine the structure of the non-compact and non-trivial minimal set in the locally compact flow.

THEOREM 3. *Let X be a locally compact metric space. Then the non-trivial and non-compact minimal set of a dynamical system (X, R, f) always contains the trajectories which are Poisson stable, positively Poisson stable and negatively receding, and negatively Poisson stable and positively receding simultaneously and no trajectories of other types.*

PROOF. Let M be a non-compact and non-trivial minimal set in (X, R, f). Since X is locally compact, M is neither positively minimal nor negatively minimal by Lemma 4, so that there exist $u, v \in M$ such that

$$L^+(u) \subset M \quad \text{and} \quad L^+(u) \neq M\,,$$
$$L^-(v) \subset M \quad \text{and} \quad L^-(v) \neq M\,,$$

by Lemma 3. Since M is minimal, both $L^+(u)$ and $L^-(v)$ must be empty. But then we have

$$L^-(u) = M \quad \text{and} \quad L^+(v) = M\,, \tag{5}$$

the proof of which is as follows: M contains no receding points by Proposition 1, so that $L^-(u)$ and $L^+(v)$ are both non-empty and contained in M, and so (5) is valid by the minimality of M.

Thus the trajectory $C(u)$ $(C(v))$ is negatively (positively) Poisson stable and positively (negatively) receding. On the other hand, M contains Poisson stable trajectories by Theorem 1. No trajectories except these are contained in M by Proposition 1. Q.E.D.

We give here an example of non-trivial and non-compact minimal sets of a locally compact flow.

EXAMPLE 1 [1, p. 33]. Consider a planar differential system

$$(6) \qquad u' = g(u, v) , \qquad v' = \alpha g(u, v) \qquad \left(' = \frac{d}{dt}\right)$$

where α is a positive irrational number, and g is a function from R^2 into R such that

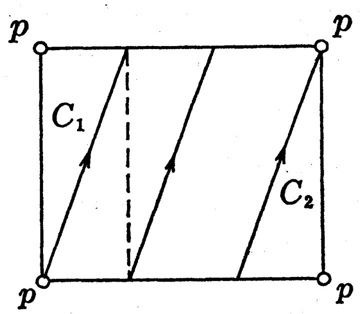
$$g(u, v) = g(u+1, v+1) = g(u+1, v) = g(u, v+1)$$

for every $u, v \in R$, and

$$g(u, v) > 0 \qquad ((u, v) \bar{\in} \{(0, 0), (0, 1), (1, 0), (1, 1)\}) ,$$
$$g(0, 0) = g(0, 1) = g(1, 0) = g(1, 1) = 0 .$$

Farther, assume that g has the properties assuring the existence and uniqueness of the solutions of the initial value problem for (6). Then (6) defines a dynamical system on the 2-dimensional torus T^2. This dynamical system has exactly one rest point p, exactly one trajectory C_1 which is positively Poisson stable and negatively asymptotic to p, and exactly one trajectory C_2 which is negatively Poisson stable and positively asymptotic to p. The other trajectories are all Poisson stable.

Restricting the dynamical system on $T^2 - \{p\}$, we obtain a dynamical system on a locally compact metric space, since $T^2 - \{p\}$ is obviously locally compact. This dynamical system has only one minimal set, which is $T^2 - \{p\}$. It is clear that $T^2 - \{p\}$ is non-trivial and non-compact. Then C_1 and C_2 turn out to be positively Poisson stable and negatively receding, and negatively Poisson stable and positively receding, respectively. Other trajectories are all Poisson stable.



The non-compact and non-trivial minimal set of the locally compact flow contains Poisson stable trajectories (see Theorem 3). To examine the properties of these trajectories, we introduce Shcherbakov's classification of Poisson stable trajectories.

It is known that a trajectory $C(x)$ of a dynamical system (X, R, f)

is positively Poisson stable if and only if for any $\varepsilon>0$, any $t_0 \in R$ and any $y \in C(x)$ there exists $T>0$ such that

$$d(f(y, [t_0, t_0+T]), y)<\varepsilon$$

(see [4, p. 75]). Generally, T depends on ε, t_0 and y. According to the mode of the dependence of T on these values, B. A. Shcherbakov classified the Poisson stable trajectories as follows [4, p. 75 ff.], [5].

1. T is independent of ε. In this case $C(x)$ is a rest point or periodic.
2. T is independent of t_0. In this case $C(x)$ is almost recurrent.
3. T is independent of y. In this case $C(x)$ is defined to be pseudorecurrent.
4. T is independent of t_0 and y. In this case $C(x)$ is recurrent.

Here the definitions of the almost recurrence and recurrence of trajectories are as follows. The trajectory $C(p)$ of a dynamical system (X, R, f) is said to approximate uniformly the set $Q \subset X$ if for any $\varepsilon>0$ there exists $T>0$ such that

$$Q \subset S(f(p, t, t+T), \varepsilon)$$

for any $t \in R$. Then $C(p)$ is said to be:

almost recurrent if it approximates $\{p\}$ uniformly;

recurrent if it approximates itself uniformly.

It is known that every trajectory which is almost recurrent or pseudorecurrent is Poisson stable [4, p. 74 ff.], and that the closure of the trajectory which is almost recurrent (pseudorecurrent) contains only the trajectories which are almost recurrent (pseudorecurrent) [4, p. 77]. Since a non-trivial and non-compact minimal set of a locally compact flow always contains the trajectory which is not Poisson stable (Theorem 3), we have

THEOREM 4. *Poisson stable trajectories in the non-trivial and non-compact minimal set in a locally compact flow are neither pseudo-recurrent nor almost recurrent.*

COROLLARY 4.1. *No Poisson stable trajectories in the non-trivial and non-compact minimal set of a locally compact flow are recurrent.*

PROOF. The recurrence implies the almost recurrence.

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