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The Factorization of H^p and the Commutators

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Introduction

In [2] and [8], R. Coifman, R. Rochberg, G. Weiss and A. Uchiyama obtained the factorization theorems for $H^p(R^n)$ by the singular integral operators. Recently S. Chanillo [1] obtained the factorization theorems for $H^1(R^n)$ by the fractional integral operators. In this paper we think about the factorization theorems for $H^p(R^n)$, $p<1$, by the fractional integral operators and we apply the results to the boundedness of certain commutator operators.

I would like to express my gratitude to the referee for his most helpful suggestions and finding some errors.

§ 1. The definition and the results.

DEFINITION. We say that K is a Calderón-Zygmund kernel if

$$K \not\equiv 0 , \quad \int_{S^{n-1}} K(x')dx' = 0 ,$$

where dx' is the element of "area" of the sphere $|x|=1$,

$$K(rx)=r^{-n}K(x) \quad \text{when} \quad r>0 \quad \text{and} \quad x \neq 0 ,$$
$$|K(x)-K(y)| \leqq |x-y| \quad \text{when} \quad |x|=|y|=1 ,$$

and define

$$Kf(x)=P.V.\int_{R^n} K(x-y)f(y)dy ,$$
$$K'f(x)=P.V.\int_{R^n} K(y-x)f(y)dy .$$

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DEFINITION. The fractional integral operator I_α is defined as

$$I_\alpha f(x)=\int_{R^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy\ ,\quad 0<\alpha<n\ .$$

DEFINITION. $H^p(R^n)$ is the space defined by Fefferman and Stein [3].

The factorization theorem by singular integral operators is as follows.

THEOREM A. (Coifman, Rochberg and Weiss [2] and Uchiyama [8]). *If* $(n+1)/n>1/p=1/q+1/r\geqq 1$, $q>n/(n+1)$, $r>n/(n+1)$ *and* $f\in H^p(R^n)$, *then*

$$\begin{aligned}C_{q,r}\|f\|_{H^p}&\leqq\inf\left\{\left(\sum_{j=1}^{\infty}(\|g_j\|_{H^q}\|h_j\|_{H^r})^p\right)^{1/p},\ f=\sum_j(h_jKg_j-g_jK'h_j)\right\}\\&\leqq C'_{q,r}\|f\|_{H^p}\ .\end{aligned}$$

This theorem is proved in [2] for $p=1$ and proved in [8] for $p<1$. Our results are following.

THEOREM 1. *If* $(n+1-\alpha)/n>1/p=1/q+1/r-\alpha/n$, $n/\alpha>q>n/(n+1-\alpha)$, $n/\alpha>r>1$, $1>\alpha>0$, $g\in H^q(R^n)\cap\mathcal{S}(R^n)$ *and* $h\in H^r(R^n)\cap\mathcal{S}(R^n)$, *then*

$$\|hI_\alpha g-gI_\alpha h\|_{H^p}\leqq C_{q,r,\alpha}\|g\|_{H^q}\|h\|_{H^r}$$

where $C_{q,r,\alpha}$ *is a positive constant depending only on* q, r, α *and* n.

This theorem is proved in [1] for $p=1$, but we give another proof for $p\leqq 1$.

THEOREM 2. *If* $(n+1)/n>1/p=1/q+1/r-\alpha/n\geqq 1$, $q>n/(n+1)$, $r>1$, $n>\alpha>0$ *and* $f\in H^p(R^n)$, *then there exist* $\{g_j\}_{j=1}^{\infty}\subset H^q(R^n)$ *and* $\{h_j\}_{j=1}^{\infty}\subset H^r(R^n)$ *such that*

$$\begin{gathered}f=\sum_{j=1}^{\infty}(h_jI_\alpha g_j-g_jI_\alpha h_j)\ ,\\(\sum(\|g_j\|_{H^q}\|h_j\|_{H^r})^p)^{1/p}\leqq C_{q,r,\alpha}\|f\|_{H^p}\ .\end{gathered}$$

As a result of these theorems, we get

COROLLARY 1. *If* $(n+1-\alpha)/n>1/p=1/q+1/r-\alpha/n\geqq 1$, $n/\alpha>q>n/(n+1-\alpha)$, $n/\alpha>r>1$, $1>\alpha>0$ *and* $f\in H^p(R^n)$, *then*

$$\begin{aligned}C_{q,r,\alpha}\|f\|_{H^p}&\leqq\inf\left\{\left(\sum_j(\|g_j\|_{H^q}\|h_j\|_{H^r})^p\right)^{1/p};\ f=\sum_j(h_jI_\alpha g_j-g_jI_\alpha h_j)\right\}\\&\leqq C'_{q,r,\alpha}\|f\|_{H^p}\ .\end{aligned}$$

§2. The basic lemmas.

DEFINITION. We define some maximal functions. Let

$$A^*_{\varepsilon,\delta}(x_0)=\left\{\Phi;\ |\Phi(x)|\leqq\frac{\varepsilon^\delta}{(\varepsilon+|x-x_0|)^{n+\delta}}\ ,\right.$$

$$\left.|\Phi(x)-\Phi(y)|\leqq\frac{|x-y|\cdot\varepsilon^{\delta-1}}{(\varepsilon+|x-x_0|)^{n+\delta}}\quad\text{if}\quad 2|x-y|\leqq|x-x_0|\right\}\ ,$$

$$A^+_\varepsilon(x_0)=\{\Phi;\ \operatorname{supp}\Phi\subset\{x;\ |x-x_0|<\varepsilon\},\ |\Phi(x)|\leqq\varepsilon^{-n}\ ,$$

$$|\Phi(x)-\Phi(y)|\leqq\varepsilon^{-n-1}|x-y|\}\ ,$$

and we define

$$f^{*(\delta)}(x_0)=\sup_{\substack{\varepsilon>0\\ \Phi\in A^*_{\varepsilon,\delta}(x_0)}}\left|\int f(x)\Phi(x)dx\right|\ ,$$

$$f^+(x_0)=\sup_{\substack{\varepsilon>0\\ \Phi\in A^+_\varepsilon(x_0)}}\left|\int f(x)\Phi(x)dx\right|\ .$$

Let $\varphi\in C^\infty(R^n)$ be a fixed function such that

$$\operatorname{supp}\varphi\subset\{x;\ |x|<1\}\ ,\quad \int\varphi(x)dx\neq 0\ ,\quad |\varphi(x)|\leqq 1$$

and $|\varphi(x)-\varphi(y)|\leqq|x-y|$, then we define

$$f^{+(\varphi)}(x_0)=\sup_{\varepsilon>0}\left|\int f(x)\varphi_\varepsilon(x_0-x)dx\right|\ ,$$

where $\varphi_\varepsilon(x)=(1/\varepsilon^n)\varphi(x/\varepsilon)$.

LEMMA 1 (Fefferman and Stein [3]). *If* $1\geqq p>n/(n+\delta)$ *and* $1\geqq\delta>0$, *then*

$$\|f\|_{H^p}\approx\|f^{+(\varphi)}\|_{L^p}\approx\|f^+\|_{L^p}\approx\|f^{*(\delta)}\|_{L^p}\ .$$

If $p>1$, *then* $\|f\|_{H^p}\approx\|f\|_{L^p}$.

This is essentially proved in [3], §10; a precise proof can be obtained by using the "atomic" theory (see Latter [4]).

LEMMA 2. *Let*

$$I^*_\alpha f(x)=\sup_{\varepsilon>0}\left|\int\frac{1}{|x-y|^{n-\alpha}}\psi\left(\frac{|x-y|}{\varepsilon}\right)f(y)dy\right|$$

where $\psi\in C^\infty(0,\infty)$ *is a fixed nonnegative function such that* $\psi(t)=0$ *on* $(0,1)$ *and* $\psi(t)=1$ *on* $(2,\infty)$, *then*

$$\|I_\alpha^* f\|_{L^q} \leqq C_{p,\alpha} \|f\|_{H^p}$$

where $1/q = 1/p - \alpha/n$, $n/\alpha > p > n/(n+1)$ *and* $n > \alpha > 0$.

This is essentially proved in Taibleson and Weiss [6], p. 102.

LEMMA 3. *Let*

$$M_{\alpha,p} f(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-\alpha p/n}} \int_Q |f(y)|^p dy \right)^{1/p}$$

where Q *is a cube centered at* x, *then*

$$\|M_{\alpha,p} f\|_{L^s} \leqq C_{r,p,\alpha} \|f\|_{L^r}$$

where $1/s = 1/r - \alpha/n$ *and* $n/\alpha > r > p > 0$.

This is an analogy of the Hardy-Littlewood maximal theorem and proved in [1], Lemma 2 and [5].

LEMMA 4. *Let* $n > \alpha > 0$, *then for any* $\varepsilon > 0$, *we have*

$$\varepsilon \cdot \int_{|x-y|>\varepsilon} \frac{|f(y)|}{|x-y|^{n-\alpha+1}} dy \leqq C_\alpha \cdot M_{\alpha,1} f(x) .$$

LEMMA 5. *If* $1 \geqq p > n/(n+1)$, $u \in L^1(R^n)$, $\int u(x)dx = 0$ *and* $\operatorname{supp} u \subset \{x;\ |x - x_0| < t\}$, *then* $\|u\|_{H^p}^p \leqq C_p \int_{|x-x_0|<2t} u^+(x)^p dx$.

This is proved in [8], Lemma 6.

LEMMA 6. *Let* $\zeta(x, y)$ *be a function defined on* $R^n \times R^n$ *such that*

$$|\zeta(x, y)| \leqq |x-y|^{-n+\alpha+\delta} ,$$
$$|\zeta(x, y) - \zeta(x, z)| \leqq |y-z|^\delta \cdot |x-y|^{-n+\alpha} \quad \text{if} \quad 2|y-z| < |x-y| .$$

Let $u \in L^2(R^n)$, $\operatorname{supp} u \subset \{x;\ |x-x_0| < t\}$,

$$v(x) = \int \zeta(x, y) u(y) dy .$$

Then

$$\left(\int_{|x-x_0|<t} |v|^{s_2} dx \right)^{1/s_2} \leqq C_{s_1,\alpha,\delta} \left(\int_{|x-x_0|<2t} u^{+s_1} dx \right)^{1/s_1}$$

where $1/s_2 = 1/s_1 - (\alpha+\delta)/n$, $n/(\alpha+\delta) > s_1 > n/(n+\delta)$, $n > \alpha + \delta > 0$, $\alpha \geqq 0$ *and* $1 \geqq \delta > 0$.

This lemma is proved by the same argument in [8], Lemma 9. In [8], this is proved where $\alpha=0$, $\delta=1$ and $n\geqq 2$.

§ 3. Proof of Theorem 1.

We estimate $(hI_\alpha g-gI_\alpha h)^{+(\varphi)}$ where φ is the function defined in §2. Let $x\in R^n$ be fixed and let

$$\begin{aligned}&\int\varphi_\varepsilon(x-y)(h(y)\cdot I_\alpha g(y)-g(y)\cdot I_\alpha h(y))dy\\&\quad=\int h(y)\Big(\int\frac{1}{|y-z|^{n-\alpha}}(\varphi_\varepsilon(x-y)-\varphi_\varepsilon(x-z))g(z)dz\Big)dy\\&\quad=\int h(y)T(y)dy\ .\end{aligned}$$

First we shall estimate $T(y)$ for $|x-y|\geqq 10\varepsilon$. Then write

$$\begin{aligned}T(y)&=-\frac{1}{|x-y|^{n-\alpha}}\int\varphi_\varepsilon(x-z)g(z)dz\\&\quad+\int\Big(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|y-z|^{n-\alpha}}\Big)\varphi_\varepsilon(x-z)g(z)dz\\&=\eta_1(y)+\int\zeta_2(y,z)g(z)dz=\eta_1(y)+\eta_2(y)\ ,\quad |x-y|\geqq 10\varepsilon\ .\end{aligned}$$

Then we have

(i) $$|\eta_1(y)|\leqq\frac{1}{|x-y|^{n-\alpha}}g^{+(\varphi)}(x),\ |x-y|\geqq 10\varepsilon\ .$$

(ii) $$|\eta_2(y)|\leqq C\frac{\varepsilon}{|x-y|^{n-\alpha+1}}g^{+}(x),\ |x-y|\geqq 10\varepsilon\ .$$

We shall prove (ii).

(1) $$\begin{aligned}|\zeta_2(y,z)|&\leqq C\frac{|x-z|}{|x-y|^{n-\alpha+1}}\frac{1}{\varepsilon^n}\chi_{\{|x-z|<\varepsilon\}}(z)\\&\leqq C\frac{\varepsilon}{|x-y|^{n-\alpha+1}}\frac{1}{\varepsilon^n}\chi_{\{|x-z|<\varepsilon\}}(z)\ ,\end{aligned}$$

where χ_E is the characteristic function of a measurable set E. To estimate $|\zeta_2(y,z_1)-\zeta_2(y,z_2)|$ we may assume that $|x-z_1|<\varepsilon$ or $|x-z_2|<\varepsilon$, so we have

(2) $$\begin{aligned}&|\zeta_2(y,z_1)-\zeta_2(y,z_2)|\\&\quad\leqq\Big|\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|y-z_1|^{n-\alpha}}\Big|\,|\varphi_\varepsilon(x-z_1)-\varphi_\varepsilon(x-z_2)|\end{aligned}$$

$$+\left|\frac{1}{|y-z_1|^{n-\alpha}}-\frac{1}{|y-z_2|^{n-\alpha}}\right||\varphi_\varepsilon(x-z_2)|$$
$$\leqq C\frac{\varepsilon}{|x-y|^{n-\alpha+1}}\frac{|z_1-z_2|}{\varepsilon^{n+1}}\,.$$

By (1) and (2), $C\zeta_2(y,z)|x-y|^{n-\alpha+1}/\varepsilon\in A_\varepsilon^+(x)$. So we have (ii).

Next we shall estimate $T(y)$ for $|x-y|<10\varepsilon$. Then set

$$\begin{aligned}T(y)&=\varphi_\varepsilon(x-y)\int\frac{1}{|x-z|^{n-\alpha}}\psi\Big(\frac{|x-z|}{200\varepsilon}\Big)g(z)dz\\&\quad+\varphi_\varepsilon(x-y)\int\Big(\frac{1}{|y-z|^{n-\alpha}}-\frac{1}{|x-z|^{n-\alpha}}\Big)\psi\Big(\frac{|x-z|}{200\varepsilon}\Big)g(z)dz\\&\quad+\int\frac{1}{|y-z|^{n-\alpha}}(\varphi_\varepsilon(x-y)-\varphi_\varepsilon(x-z))\Big(1-\psi\Big(\frac{|x-z|}{200\varepsilon}\Big)\Big)g(z)dz\\&=\eta_3(y)+\varphi_\varepsilon(x-y)\int\zeta_4(y,z)g(z)dz\\&\quad+\int\zeta_5(y,z)g(z)\Big(1-\psi\Big(\frac{|x-z|}{400\varepsilon}\Big)\Big)dz\cdot\chi_{\{|x-y|<10\varepsilon\}}(y)\\&=\eta_3(y)+\eta_4(y)+\eta_5(y)\,,\quad|x-y|<10\varepsilon\,,\end{aligned}$$

where

$$\zeta_5(y,z)=\frac{1}{|y-z|^{n-\alpha}}(\varphi_\varepsilon(x-y)-\varphi_\varepsilon(x-z))\Big(1-\psi\Big(\frac{|x-z|}{200\varepsilon}\Big)\Big)$$

and ψ is the function defined in Lemma 2. Then we have

(iii) $$|\eta_3(y)|\leqq|\varphi_\varepsilon(x-y)|I_\alpha^*g(x)\,,$$

(iv) $$|\eta_4(y)|\leqq C|\varphi_\varepsilon(x-y)|\varepsilon^\alpha g^{*(1-\alpha)}(x)\,,$$

(v) $$\Big(\int_{|x-y|<10\varepsilon}|\eta_5(y)|^{s_2}dy\Big)^{1/s_2}\leqq C\varepsilon^{-n-\delta}\Big(\int_{|x-y|<800\varepsilon}g^+(y)^{s_1}dy\Big)^{1/s_1}\,,$$

where $1/s_2=1/s_1-(\alpha+\delta)/n$, $n/(\alpha+\delta)>s_1>n/(n+\delta)$, $n>\alpha+\delta$ and $1\geqq\delta>0$.

We shall prove (iv).

$$\begin{aligned}|\zeta_4(y,z)|&\leqq C\frac{|x-y|}{|x-z|^{n-\alpha+1}}\chi_{\{|x-z|\geqq200\varepsilon\}}(z)\\&\leqq C\cdot\varepsilon^\alpha\frac{\varepsilon^{1-\alpha}}{(\varepsilon+|x-z|)^{n-\alpha+1}}\,.\end{aligned}$$

If $2|z_1-z_2|<|x-z_1|$, then

$$\begin{aligned}|\zeta_4(y, z_1)-\zeta_4(y, z_2)| &\leqq \left|\frac{1}{|x-z_1|^{n-\alpha}}-\frac{1}{|x-z_2|^{n-\alpha}}\right| \psi\left(\frac{|x-z_1|}{200\varepsilon}\right) \\ &\quad+\left|\frac{1}{|y-z_1|^{n-\alpha}}-\frac{1}{|y-z_2|^{n-\alpha}}\right| \psi\left(\frac{|x-z_1|}{200\varepsilon}\right) \\ &\quad+\left|\frac{1}{|x-z_2|^{n-\alpha}}-\frac{1}{|x-z_2|^{n-\alpha}}\right| \left|\psi\left(\frac{|x-z_1|}{200\varepsilon}\right)-\psi\left(\frac{|x-z_2|}{200\varepsilon}\right)\right| \\ &\leqq C\left(\frac{|z_1-z_2|}{|x-z_1|^{n-\alpha+1}}\chi_{\{|x-z_1|\geqq 200\varepsilon\}}(z_1)\right. \\ &\quad\left.+\frac{|z_1-z_2|}{|y-z_1|^{n-\alpha+1}}\chi_{\{|x-z_1|\geqq 200\varepsilon\}}(z_1)+\frac{|x-y|}{|x-z_2|^{n-\alpha+1}}\frac{|z_1-z_2|}{\varepsilon}\right) \\ &\leqq C\cdot\varepsilon^{\alpha}\frac{\varepsilon^{-\alpha}|z_1-z_2|}{(\varepsilon+|x-z_1|)^{n-\alpha+1}}\,.\end{aligned}$$

So we have $C\cdot\varepsilon^{-\alpha}\cdot\zeta_4\in A^*_{\varepsilon,1-\alpha}(x)$ and we obtain (iv).

We shall prove (v).

$$|\zeta_5(y, z)|\leqq C\frac{1}{|y-z|^{n-\alpha}}\frac{|y-z|^{\delta}}{\varepsilon^{n+\delta}} \leqq C\frac{1}{\varepsilon^{n+\delta}}\frac{1}{|y-z|^{n-\alpha-\delta}} \tag{3}$$

where $n>\alpha+\delta$ and $1\geqq\delta>0$. If $2|z_1-z_2|<|y-z_1|$, then

$$\begin{aligned}&|\zeta_5(y, z_1)-\zeta_5(y, z_2)| \\ &\leqq C\left(\frac{1}{|y-z|_1^{n-\alpha}}|\varphi_\varepsilon(x-z_1)-\varphi_\varepsilon(x-z_2)|\left|1-\psi\left(\frac{|x-z_1|}{200\varepsilon}\right)\right|\right. \\ &\quad+\left|\frac{1}{|y-z_1|^{n-\alpha}}-\frac{1}{|y-z_2|^{n-\alpha}}\right| |\varphi_\varepsilon(x-y)-\varphi_\varepsilon(x-z_2)| \\ &\quad\left.+\frac{1}{|y-z_2|^{n-\alpha}}|\varphi_\varepsilon(x-y)-\varphi_\varepsilon(x-z_2)|\left|\psi\left(\frac{|x-z_1|}{200\varepsilon}\right)-\psi\left(\frac{|x-z_2|}{200\varepsilon}\right)\right|\right) \\ &\leqq C\left(\frac{1}{|y-z_1|^{n-\alpha}}\frac{|z_1-z_2|^{\delta}}{\varepsilon^{n+\delta}}+\frac{|z_1-z_2|}{|y-z_1|^{n-\alpha+1}}\frac{|y-z_2|^{\delta}}{\varepsilon^{n+\delta}}\right. \\ &\quad\left.+\frac{1}{|y-z_2|^{n-\alpha}}\frac{1}{\varepsilon^{n}}\frac{|z_1-z_2|^{\delta}}{\varepsilon^{\delta}}\right) \\ &\leqq C\frac{1}{\varepsilon^{n+\delta}}\frac{|z_1-z_2|^{\delta}}{|y-z_1|^{n-\alpha}}\,.\end{aligned} \tag{4}$$

By (3), (4) and Lemma 6, we have

$$\left(\int_{|x-y|<10\varepsilon}|\eta_5(y)|^{s_2}dy\right)^{1/s_2}$$
$$\leqq C_{s_1,\alpha}\varepsilon^{-n-\delta}\left(\int_{|x-y|<800\varepsilon}\left(g(\cdot)\left(1-\psi\left(\frac{|x-\cdot|}{400\varepsilon}\right)\right)\right)^+(y)^{s_1}dy\right)^{1/s_1},$$

where $1/s_2=1/s_1-(\alpha+\delta)/n$, $n/(\alpha+\delta)>s_1>n/(n+\delta)$, $n>\alpha+\delta$ and $1\geqq\delta>0$. Since

$$\left(g(\cdot)\left(1-\psi\left(\frac{|x-\cdot|}{400\varepsilon}\right)\right)\right)^+(y)\leqq Cg^+(y)$$

for $|x-y|<800\varepsilon$, we have (v).

By (i)$\sim$(v), we have

$$\left|\int h(y)T(y)dy\right|\leqq\int|h\eta_1|+\int|h\eta_2|+\int|h\eta_3|+\int|h\eta_4|+\int|h\eta_5|$$
$$=J_1(x)+J_2(x)+J_3(x)+J_4(x)+J_5(x),$$
$$J_1(x)\leqq I_\alpha(|h|)(x)\cdot g^{+(\varphi)}(x),$$
$$J_2(x)\leqq CM_{\alpha,1}(h)(x)\cdot g^+(x),$$
$$J_3(x)\leqq CM_{0,1}(h)(x)\cdot I_\alpha^* g(x),$$
$$J_4(x)\leqq CM_{\alpha,1}(h)(x)\cdot g^{*(1-\alpha)}(x),$$
$$J_5(x)\leqq CM_{b,s_2'}(h)(x)\cdot M_{a,s_1}(g^+)(x),$$

where $1/s_2=1/s_1-(\alpha+\delta)/n$, $s_2>1$, $1/s_2+1/s_2'=1$ and $a+b=\alpha$.

So,

$$(h\cdot I_\alpha g-g\cdot I_\alpha h)^{+(\varphi)}(x)\leqq C(I_\alpha(|h|)(x)\cdot g^{+(\varphi)}(x)+M_{\alpha,1}(h)(x)\cdot g^+(x)$$
$$+M_{0,1}(h)(x)\cdot I_\alpha^* g(x)+M_{\alpha,1}(h)(x)\cdot g^{*(1-\alpha)}(x)$$
$$+M_{b,s_2'}(h)(x)\cdot M_{a,s_1}(g^+)(x)).$$

By Lemma 1 and Lemma 2,

$$\|I_\alpha(|h|)\cdot g^{+(\varphi)}\|_{L^p}\leqq\|I_\alpha(|h|)\|_{L^{\tilde r}}\|g^{+(\varphi)}\|_{L^q}$$
$$\leqq C\|h\|_{L^r}\|g\|_{H^q}$$

where $1/\tilde r=1/r-\alpha/n$.

By Lemma 1 and Lemma 3,

$$\|M_{\alpha,1}h\cdot g^+\|_{L^p}\leqq\|M_{\alpha,1}h\|_{L^{\tilde r}}\|g^+\|_{L^q}$$
$$\leqq C\|h\|_{L^r}\|g\|_{H^q},$$
$$\|M_{\alpha,1}h\cdot g^{*(1-\alpha)}\|_{L^p}\leqq\|M_{\alpha,1}h\|_{L^{\tilde r}}\|g^{*(1-\alpha)}\|_{L^q}$$

$$
\begin{aligned}
&\leqq C\|h\|_{L^r}\|g\|_{H^q}\,, \\
\|M_{b,s_2}h\cdot M_{a,s_1}g^+\|_{L^p} &\leqq \|M_{b,s_2}h\|_{L^{\tilde{\tilde r}}}\|M_{a,s_1}g^+\|_{L^{\tilde{\tilde q}}} \\
&\leqq C\|h\|_{L^r}\|g^+\|_{L^q} \\
&\leqq C\|h\|_{L^r}\|g\|_{H^q}
\end{aligned}
$$

where $1/\tilde{\tilde q}=1/q-a/n$ and $1/\tilde{\tilde r}=1/r-b/n$.

By Lemma 2 and Lemma 3,

$$
\begin{aligned}
\|M_{0,1}h\cdot I_\alpha^* g\|_{L^p} &\leqq \|M_{0,1}h\|_{L^r}\|I_\alpha^* g\|_{L^{\tilde q}} \\
&\leqq C\|h\|_{L^r}\|g\|_{H^q}
\end{aligned}
$$

where $1/\tilde{q}=1/q-\alpha/n$.

So we have $\|h\cdot I_\alpha g-g\cdot I_\alpha h\|_{H^p}\leqq C\|h\|_{L^r}\|g\|_{H^q}$.

§4. Proof of Theorem 2.

We may assume that f is a p-atom such that

$$
\operatorname{supp} f\subset\{x;\ |x-x_0|<t\},\ \|f\|_{L^\infty}\leqq t^{-n/p} \quad\text{and}\quad \int f\,dx=0
$$

(see Latter [4]). As is the proof of Uchiyama [8], p. 465, we take $y_0\in R^n$ such that $|x_0-y_0|=Nt$ and set

$$
\begin{aligned}
h(x)&=N^{n-\alpha}\chi_{\{|x-y_0|<t\}}(x)\,, \\
g(x)&=-f(x)/I_\alpha h(x_0)\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|g\|_{H^q}\leqq Ct^{-\alpha-n/p+n/q}\,, \\
&\|h\|_{H^r}\leqq CN^{n-\alpha}t^{n/r}
\end{aligned}
$$

and

$$
\|g\|_{H^q}\|h\|_{H^r}\leqq CN^{n-\alpha}\,.
$$

By Lemma 5,

$$
\begin{aligned}
&\|f-(h\cdot I_\alpha g-g\cdot I_\alpha h)\|_{H^p} \\
&\quad\leqq C(\|f\cdot(I_\alpha h(x_0)-I_\alpha h)/I_\alpha h(x_0)\|_{H^p}+\|h\cdot I_\alpha g\|_{H^p}) \\
&\quad\leqq C\begin{cases} N^{-1-n+n/p} & (1>p>n/(n+1)) \\ N^{-1}\log N & (p=1)\,. \end{cases}
\end{aligned}
$$

So we have $\|f-(h\cdot I_\alpha g-g\cdot I_\alpha h)\|_{H^p}<1/2$ if N is sufficiently large. Repeating this process, we get the desired result.

§ 5. Applications.

DEFINITION. We define the commutator operators:

$$C^K(b, f)(x)=\int_{R^n}(b(x)-b(y))K(x-y)f(y)dy ,$$

where K is a Calderón-Zygmund kernel (see § 1);

$$C^\alpha(b, f)(x)=\int_{R^n}(b(x)-b(y))\frac{f(y)}{|x-y|^{n-\alpha}}dy ,$$

where $0<\alpha<n$.

COROLLARY 2 (The refinement of the theorem in [1]). 1) *C^K is a bounded map of $L^p(R^n)$ to itself for $1<p<\infty$, if and only if b is in $BMO(R^n)$ (for the definition of BMO, see* [3]) *and operator norm $\|C^K\|_{L^p\to L^p}\approx\|b\|_{BMO}$.*

2) *C^α is a bounded map of $L^p(R^n)$ to $L^q(R^n)$ for $1/q=1/p-\alpha/n$, $1<p<n/\alpha$ and $0<\alpha<n$, if and only if b is in $BMO(R^n)$ and operator norm $\|C^\alpha\|_{L^p\to L^q}\approx\|b\|_{BMO}$.*

In [2] and [7], 1) is proved, and in [1], 2) is proved for some α.

PROOF. We shall prove 2). If C^α is a bounded map of L^p to L^q, then for any $h\in H^1$, we apply Theorem 2 and we have

$$\begin{aligned}\left|\int b(x)h(x)dx\right| &\leqq \sum\left|\int b(x)(f_j(x)I_\alpha g_j(x)-g_j(x)I_\alpha f_j(x))dx\right|\\ &\leqq\sum\left|\int g_j(x)(b(x)I_\alpha f_j(x)-I_\alpha(bf_j)(x))dx\right|\\ &\leqq\sum\|g_j\|_{L^{q'}}\|C^\alpha(b, f_j)\|_{L^q}\\ &\leqq C\sum\|g_j\|_{L^{q'}}\|f_j\|_{L^p}\|C^\alpha\|_{L^p\to L^q}\\ &\leqq C\|f\|_{H^1}\|C^\alpha\|_{L^p\to L^q}\end{aligned}$$

where $1/q+1/q'=1$. By the duality between H^1 and BMO (see [3]), we have $b\in BMO$. To prove the converse we use Theorem 1.

COROLLARY 3. *C^K is a bounded map of $H^p(R)^n$ to $L^q(R^n)$ for $1/q=1/p-\delta/n$, $n/(n+\delta)<p<n/\delta$ and $0<\delta<1$, if and only if b is in* $\mathrm{Lip}_\delta(R^n)$ *(we define $\|b\|_{\mathrm{Lip}_\delta}=\sup_{x\neq y}|b(x)-b(y)|/|x-y|^\delta$) and operator norm $\|C^K\|_{H^p\to L^q}\approx\|b\|_{\mathrm{Lip}_\delta}$.*

PROOF. Note that $q>1$, thus for any $g\in L^{q'}$ $(1/q+1/q'=1)$,

$$\int C^K(b, f)(x)g(x)dx=\int b(x)(f(x)Kg(x)-g(x)K'f(x))dx .$$

By the former half of the inequalities in Theorem A,

$$\|fKg-gK'f\|_{H^{n/(n+\delta)}}\leqq C\|f\|_{H^p}\|g\|_{L^{q'}}$$

and using the duality between H^p and $\mathrm{Lip}_{n(1/p-1)}$ (see [3]), we have

$$\|C^K(b, f)\|_{L^q}\leqq C\|b\|_{\mathrm{Lip}_\delta}\|f\|_{H^p} .$$

The converse is proved by using the latter half of the inequalities in Theorem A.

REMARK. Corollary 3 does not hold for $p\leqq n/(n+\delta)$. And a simple example shows that C^K is not bounded map of H^p to itself for $p\leqq 1$ even if b is in BMO.

We give an example for $n=1$. Let $b(x)=\chi_{(0,1)}$ and $f(x)=\chi_{(0,1)}-\chi_{(-1,0)}$, then $b\in BMO$ and $f\in H^p$ for $n/(n+1)<p\leqq 1$, but for $x>2$,

$$C^K(b, f)(x)=\int_0^1\frac{b(y)f(y)}{x-y}dy=\int_0^1\frac{1}{x-y}dy\geqq\frac{1}{x} ,$$

so, $C^K(b, f)\notin L^p$ for $p\leqq 1$.

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