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$SO(n)$, $SU(n)$, $Sp(n)$-homology Spheres with Codimension Two Principal Orbits

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Introduction

In studying smooth actions of compact Lie groups on homology spheres (or on acyclic manifolds), a basic approach is to compare those smooth actions with linear actions on standard spheres (or on Euclidean spaces). Some basic relations between smooth and linear actions have been studied in [9] and [10].

Let G be a compact Lie group, and M a homology sphere (or an acyclic manifold) with a smooth action ψ of G. Let ϕ be a linear action of G on the standard sphere (or on Euclidean space) with the same dimension as of M. If ψ and ϕ have the same orbit types and the same slice representations of the corresponding orbits, then we say that ϕ is a *linear model* of ψ (see [6]). Denote by ρ_n, $(\mu_n)_R$ and $(\nu_n)_R$ the canonical inclusions of $SO(n)$, $SU(n)$ and Sp(n) into $O(n)$, $O(2n)$ and $O(4n)$, respectively. A smooth action of $SO(n)$, $SU(n)$ or $Sp(n)$ on M is called *regular* if its linear model is given by a representation $k\rho_n \oplus$trivial representation, $k(\mu_n)_R \oplus$trivial representation or $k(\nu_n)_R \oplus$trivial representation, respectively, where $k\phi$ is the direct sum of k copies of a representation ϕ. We shall also say that these representations are regular. In [5], M. Davis and W. C. Hsiang classified regular* $U(n)$ and $Sp(n)$-actions on homotopy spheres up to concordance. And in [7], these authors and J. W. Morgan classified regular* $O(n)$-actions on homotopy spheres up to concordance. In this paper, we treat smooth actions of compact, connected, simple classical Lie groups on homology spheres with linear models, and we shall prove that these actions are completely classified up to equivariant diffeomorphisms, if they have codimension two principal orbits. When a smooth

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* Precisely, "regularity" in our sence is obtained by restricting $U(n)$ (resp. $O(n)$) to its connected simple subgroup $SU(n)$ (resp. $SO(n)$).

manifold M with a smooth action ψ of G is equivariantly diffeomorphic to a smooth manifold M' with a smooth action ψ' of G, we shall say that ψ is equivariantly diffeomorphic to ψ'.

To state our theorem, we recall that the manifold W_k^{4m+1} given by

$$W_k^{4m+1}=\{(z_0, z_1, \cdots, z_{2m+1}) \in C^{2m+2} \mid z_0^k+z_1^2+\cdots+z_{2m+1}^2=0 \text{ and } |z_0|^2+|z_1|^2+\cdots+|z_{2m+1}|^2=1\}$$

is a homology sphere if k is odd (See Chapter I of [2]), and it is called the Brieskorn sphere. The group $SO(2m+1)$ acts naturally on C^{2m+1} with the coordinates $z_1, z_2, \cdots, z_{2m+1}$ as a subgroup of $U(2m+1)$, under which W_k^{4m+1} is invariant. Thus if k is odd, W_k^{4m+1} becomes an $SO(2m+1)$-homology sphere by this action. Let us denote it by $\psi_{SO(2m+1),k}$. In particular, W_k^9 also becomes an $Sp(2)$-homology sphere by the action $\psi_{SO(5),k}\circ\pi$ where π is the natural projection of $Sp(2)$ to $SO(5)$. Our main result is the following.

THEOREM. *Let M be a homology sphere, and let G be one of the three groups, $SO(n)$ $(n\neq 2, 4)$, $SU(n)$ or $Sp(n)$. Let ψ be a smooth action of G on M which has a codimension two principal orbit and a linear model. Then ψ is equivariantly diffeomorphic to the linear model unless it is equivariantly diffeomorphic to one of the actions of the following two types:*

(i) *$\psi_{SO(2m+1),k}$ on W_k^{4m+1} for $G=SO(n)$ $(n=2m+1, m\geqq 2)$ or* (ii) *$\psi_{SO(5),k}\circ\pi$ on W_k^9 for $G=Sp(2)$.*

REMARK. The linear model of the action $\psi_{SO(2m+1),k}$ is obtained from the linear action on the representation space of $2\rho_{2m+1}$ by restricting to the unit sphere. In particular, $\psi_{SO(3),k}$ is equivariantly diffeomorphic to the linear model (see Chapter I of [2]).

REMARK. Let G be a compact simple Lie group and M a homology sphere (or an acyclic manifold). Then it is stated in [6] that every smooth action of G on M with a non-trivial principal isotropy subgroup has a unique linear model. Thus the assumption that ψ has a linear model may be removed from the above theorem, except in the case of those actions with trivial principal isotropy subgroups. It will be seen that smooth actions in the above theorem which have trivial principal isotropy subgroups only appear as those modelled* on $2\rho_3$ or on $(\mu_2)_R\oplus$2-dimensional trivial representation (see Propositions 2.1 and 2.3).

In §1, we recall some basic notions and results. In §2, we shall list up all real representations which provide linear models in our thorem,

* For the definition, see §3.

and investigate all orbit types of the linear models. In §3, we define the *orbit datum* as a set of isotropy subgroups of G, and in terms of which we shall classify G-homology spheres in §4. Lastly, in §5 the proof of our theorem will be given.

§1. Preliminaries.

Let G be a compact connected Lie group and M a smooth G-manifold. For $x \in M$, the G-orbit through x and the isotropy subgroup at x are denoted by $G(x)$ and G_x, respectively. The orbit space is indicated by M^* or M/G. By $H<G$, we mean that H is a subgroup of G. Then we denote by (H) the conjugacy class of H, that is, $(H)=\{K<G|K$ is conjugate to $H\}$. And put $M_{(H)}=\{x \in M | G_x \in (H)\}$. If X is a G-invariant subspace of M, then we write $F(H, X)=\{x \in X|gx=x$ for all $g \in H\}$.

1.1. Recall a result of Montgomery-Samelson-Yang (see [12]). Order the conjugacy classes of isotropy subgroups by inclusions. They have a unique absolute minimum (H) for which $M_{(H)}$ is open dense in M. We call H a *principal isotropy subgroup* and the corresponding orbit G/H a *principal orbit*. Let P be a principal orbit. Then an orbit Q is called a *singular orbit* if $\dim P > \dim Q$. An orbit Q is called an *exceptional orbit*, if $\dim P = \dim Q$ and if the corresponding isotropy subgroup K is not conjugate to H.

1.2. Assume that M is equipped with a G-invariant Riemannian metric. Let $x \in M$. Then the induced action of G_x on the normal vector space V_x to $G(x)$ at x gives a representation $\psi_x\colon G_x \to O(l)$ ($l=\dim M - \dim G(x)$). ψ_x is called the *slice representation* of G_x at x. Then it is well-known that there is a small disk S_x in V_x such that a closed equivariant tubular neighbourhood of $G(x)$ is equivariantly diffeomorphic to $G\times_{G_x} S_x$, where G_x acts on S_x by ψ_x (see [2]). S_x is called a *slice* at x.

1.3. The results in this subsection are refered to Chapter VI of [2]. Let G/H be a principal orbit in a smooth G-manifold M and suppose that $Q=G/K$ is a non-principal orbit in M such that there are exactly two orbit types in a neighbourhood of Q. We may assume that $H<K$. Let $G\times_K \boldsymbol{R}^k$ be an open equivariant tubular neighbourhood of Q. We *assume* that $k=n+m$ and that K acts on $\boldsymbol{R}^k$ via a representation into $O(n) \hookrightarrow O(k)$ and is *transitive* on the unit sphere S^{n-1} in the orthogonal complemet $\boldsymbol{R}^n \times \{0\}$ to the fixed point set $F(K, \boldsymbol{R}^k)=\{0\}\times \boldsymbol{R}^m$. Let v_0 be a point in S^{n-1} with $K_{v_0}=H$. Then every point of $G\times_K \boldsymbol{R}^n$ is of the form $[g, v]$ where $v \in \boldsymbol{R}v_0$, and $G\times_K \boldsymbol{R}^n$ has the right action of $(N(H)\cap N(K))/H$ defined by

$[g, v] \mapsto [gs, v]$ for $s \in N(H) \cap N(K)$. We *assume* that this action is smooth.

M is called a *smooth special G-manifold* if there are at most two orbit types in the vicinity of each orbit and if the conditions above hold for each non-principal orbit.

Let $\mathbf{Homeo}_X^G(M)$ (resp. $\mathbf{Diffeo}_X^G(M)$) denote the set of G-equivariant homeomorphisms (resp. G-equivariant diffeomorphisms) of M inducing the identity on the orbit space $X = M/G$. And let $\pi_0 \mathbf{Homeo}_X^G(M)$ (resp. $\pi_0 \mathbf{Diffeo}_X^G(M)$) be the set of equivariant homotopy classes (resp. equivariant smooth isotopy classes) over X of elements in $\mathbf{Homeo}_X^G(M)$ (resp. $\mathbf{Diffeo}_X^G(M)$). Then the following theorem has been obtained in [2].

THEOREM (6.4, Chapter VI of [2]). *If M is a smooth special G-manifold over $X(=M/G)$, then the forgetful map $\pi_0 \mathbf{Diffeo}_X^G(M) \to \pi_0 \mathbf{Homeo}_X^G(M)$ is a one-one correspondence.*

We shall apply this theorem to Lemma 4.7 in §4.

1.4. Throughout this paper, we use the following notations: $G^0 =$ the identity component of a group G, $N(H) =$ the normalizer of a subgroup H of G, and $\mathrm{diag}(A_1, A_2, \cdots, A_r) =$ the matrix $\begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_r \end{bmatrix}$ where each A_i is a square matrix. $X \approx Y$ means that G-manifolds X and Y are equivariantly diffeomorphic, and $A \cong B$ means that the groups A and B are isomorphic. And we denote by ρ_n, μ_n and ν_n the canonical inclusions of $SO(n)$, $SU(n)$ and $Sp(n)$ into themselves, respectively.

§ 2. Linear models.

Let G be one of the three groups $SO(n)$ $(n \neq 2, 4)$, $SU(n)$ or $Sp(n)$. Our purpose in this section is to list up the candidates of the linear models in our theorem stated in the Introduction. This is equivalent to listing up all real representations of G with codimension three principal orbits. In Proposition 2.1, we shall give a list of these representations. And in Proposition 2.3, we shall observe which orbit types occur when the linear actions are restricted to the unit spheres. The results in Propositions 2.1 and 2.3 have been studied in [13]. In this section, we shall give the outlines of the proofs by treating only some typical cases.

From now on, we denote by ϕ_R the underlying real representation of a complex or symplectic representation ϕ, and by ϕ_C the underlying complex representation of a symplectic representation ϕ. ϕ^c denotes the complexification of a real representation ϕ, and θ denotes the real (or complex) one dimensional trivial representation. For the notational convenience, we denote by $k\phi$ the direct sum of k copies of a representation ϕ. We often write ϕ as $\phi_1-\phi_2$ if $\phi\oplus\phi_2$ is equivalent to ϕ_1. And, by the orbits (resp. the isotropy subgroups) of a representation of G, we mean the orbits (resp. the isotropy subgroups) of the linear action of G on the representation space.

PROPOSITION 2.1. *Let G be one of the three groups $SO(n)$ $(n\neq 2, 4)$,*

TABLE A[1]

G		representation ϕ	principal orbit	non-isolated singular orbit	isolated singular orbit
$SO(n)$	$n\neq 2,4$	$\rho_n\oplus 2\theta$	$SO(n-1)$	$SO(n)$	
		$2\rho_n$	$SO(n-2)$	$SO(n-1)$	
	$n=3$	$S^2\rho_3$	$(Z_2)^2$	$O(2)$	$SO(3)$
	$n=5$	$\mathrm{Ad}_{SO(5)}\oplus\theta$	T	$SO(2)\times SO(3)$ $U(2)$	$SO(5)$
	$n=6$	$\mathrm{Ad}_{SO(6)}$	T	$SO(2)\times U(2)$	$SO(2)\times SO(4)$ $U(3)$
	$n=7$	$\mathrm{Ad}_{SO(7)}$	T	$SO(2)^2\times SO(3)$ $U(2)\times SO(2)$	$U(2)\times SO(3)$ $SO(2)\times SO(5)$ $U(3)$
$SU(n)$	$n\geqq 2$	$(\mu_n)_R\oplus 2\theta$	$SU(n-1)$	$SU(n)$	
	$n=3$	$\mathrm{Ad}_{SU(3)}\oplus\theta$	T	$S(U(2)\times U(1))$	$SU(3)$
	$n=4$	$(\mu_4)_R\oplus\phi_2\oplus\theta$; $\phi_2^c=\Lambda^2\mu_4$	$SU(2)$	$Sp(2)$ $SU(3)$	$SU(4)$
	$n=5$	$(\Lambda^2\mu_5)_R\oplus\theta$	$SU(2)^2$	$SU(2)\times SU(3)$ $Sp(2)$	$SU(5)$
	$n=7$	$(\Lambda^2\mu_7)_R$	$SU(2)^3$	$Sp(2)\times Sp(1)$ $SU(2)^2\times SU(3)$	$Sp(3)$ $Sp(2)\times SU(3)$ $SU(2)\times SU(5)$
$Sp(n)$	$n\geqq 2$	$(\nu_n)_R\oplus 2\theta$	$Sp(n-1)$	$Sp(n)$	
	$n=3$	$\mathrm{Ad}_{Sp(3)}$	T	$U(1)^2\times Sp(1)$ $U(2)\times U(1)$	$U(2)\times Sp(1)$ $U(1)\times Sp(2)$ $U(3)$
		$\phi^c=\Lambda^2(\nu_3)_C$	$Sp(1)^3$	$Sp(1)\times Sp(2)$	$Sp(3)$
	$n=4$	$\phi^c=\Lambda^2(\nu_4)_C-\theta$	$Sp(1)^4$	$Sp(2)\times Sp(1)^2$	$Sp(2)^2$ $Sp(1)\times Sp(3)$

$SU(n)$ or $Sp(n)$. Then the Table A contains all possible real representations of G with codimension three principal orbits.

REMARK 2.2. Let $\tilde{G}$ be the universal covering group of G. In Table A, we omit representations of $\tilde{G}$ which become the liftings of representations of G. For example, the representation $(\Lambda^2\mu_4)_R$ of $SU(4)$ is excluded because it is the lifting of the representation $2\rho_6$ of $SO(6)$.

PROPOSITION 2.3. *Let ϕ be a representation in Proposition* 2.1, *and let V_ϕ be the representation space of ϕ. Then the restricted linear action on the unit sphere $S(V_\phi)$ has the orbit types in the Table A.*

REMARK 2.4. In Table A, we display the corresponding isotropy subgroups instead of orbits. And the symbol T denotes the specific maximal torus of G which is chosen according to Chapter 4 of [1]. The group $S(U(n_1)\times U(n_2))$ means the subgroup of $SU(n_1+n_2)$ consisting of matrices which are of the form $\mathrm{diag}(A_1, A_2)$ such that A_i is in $U(n_i)$ $(i=1, 2)$ and $\det A_1 \det A_2=1$. K^s is the direct product of s copies of K. The group $U(n)$ (resp. $Sp(n)$) is regarded as a subgroup of $SO(2n)$ (resp. $SU(2n)$) by the natural embedding. And $S^2\rho_3$ denotes the second symmetric power of ρ_3.

Before giving the proofs of Propositions 2.1 and 2.3, we recall some relations between the weight system of a representation of G and the root system. The results stated below are refered to [10].

Consider the linear action on V_ϕ induced by a real representation ϕ of G, where V_ϕ is the representation space of ϕ. For a maximal torus T of G, we denote by $\Omega(\phi)$ the system of non-zero weights of ϕ^c, and by $\Delta(G)$ the root system of G. Then, for $x\in V_\phi$, we may assume that the maximal torus T_x of G_x^0 is contained in T. And we may take the Lie algebra $L(T_x)$ of T_x as $\omega_{j_1}^\perp\cap\omega_{j_2}^\perp\cap\cdots\cap\omega_{j_t}^\perp$ for a suitable subcollection $\{\omega_{j_i}\}$ of weights in $\Omega(\phi)$, where $\omega_{j_i}^\perp$ means the set of vectors perpendicular to ω_{j_i}. On the other hand, for $x\in V_\phi$, we have the following equality:

$$\phi|G_x=(\mathrm{Ad}_G|G_x-\mathrm{Ad}_{G_x})\oplus\psi_x$$

where ψ_x is the slice representation at x. Thus we have

$$\Omega(\phi)|T_x=\Omega(\mathrm{Ad}_G|T_x-\mathrm{Ad}_{G_x}|T_x)+\Omega(\psi_x)\ ,$$

where $\Omega(\phi)|T_x$ is the restriction to $L(T_x)$ of $\Omega(\phi)$. And hence, the root system of G_x^0 must satisfy the following condition:

$$\Delta(G_x^0)\supset\Delta(G)|T_x-\Omega(\phi)|T_x \quad \text{(difference set)}\ . \tag{2.5}$$

Moreover, it is known that if $\Omega(\phi)|T_x - \Delta(G)|T_x = \Phi$, then T_x is a maximal torus of a suitable connected principal isotropy subgroup H^0, and that the root system $\Delta(H^0)$ of H^0 is given by the following equation:

$$\Delta(H^0) = \Delta(G)|T_x - \Omega(\phi)|T_x . \tag{2.6}$$

We remark that these results are valid for any compact, connected, simple Lie group.

Now let $G = Sp(r)$, and ϕ the representation such that $\phi^c = \Lambda^2(\nu_r)_C - \theta$. And let $\{x_1, \cdots, x_r\}$ be the basis of the Cartan subalgebra of $Sp(r)$ such that

$$\Delta(Sp(r)) = \{\pm 2x_i,\ 1 \leqq i \leqq r,\ \pm x_i \pm x_j,\ 1 \leqq i < j \leqq r\} .$$

Then we have

$$\Omega(\phi) = \{\pm x_i \pm x_j,\ 1 \leqq i < j \leqq r\} .$$

Since $\Omega(\phi) - \Delta(Sp(r)) = \Phi$, the principal isotropy subgroup H_ϕ of ϕ has the maximal rank. Thus, from (2.6), we have

$$\Delta(H_\phi^0) = \{\pm 2x_i,\ 1 \leqq i \leqq r\} .$$

This implies that $(H_\phi^0) = (Sp(1)^r)$. Also, for a singular isotropy subgroup G_x, (2.5) shows that G_x^0 is conjugate to $Sp(c_1) \times \cdots \times Sp(c_k)$, $c_1 + \cdots + c_k = r$.

Outline of the proof of Proposition 2.1. We only consider the case $G = Sp(r)$ $(r \geqq 2)$; because the proof for this case is typical and the proofs of the other cases are similar. See [13] for the details.

Let ϕ be a real representation of $Sp(r)$ with a codimension three principal orbit. Then the degree of ϕ does not exceed $\dim Sp(r) + 3$. Thus the complex degree of an irreducible direct summand of ϕ^c is not larger than $\dim Sp(r) + 3$.

(A) Denote by L_i $(1 \leqq i \leqq r)$ the highest weight of the i-th basic complex irreducible representation of $Sp(r)$. It is known that every complex irreducible representation of $Sp(r)$ is uniquely determined by the highest weight which can be written as $a_1L_1 + \cdots + a_rL_r$ with non-negative integers $\{a_i\}$. Denote by $d(a_1L_1 + \cdots + a_rL_r)$ the complex degree of a complex irreducible representation of $Sp(r)$ whose highest weight is $a_1L_1 + \cdots + a_rL_r$. The degree can be computed by Weyl's dimension formula (see Theorem 0.24, (0.148)-(0.155) of [8]). By this formula, we can see that, if a complex irreducible representation of $Sp(r)$ has the degree not larger than $\dim Sp(r) + 3$, then the highest weight is

$$L_1,\ L_2 \quad \text{or} \quad 2L_1 \quad \text{for} \quad r \neq 3 ,$$

$$L_1,\ L_2,\ L_3 \quad \text{or} \quad 2L_1 \quad \text{for} \quad r=3\,.$$

Notice that the complex irreducible representations corresponding to L_1, L_2 and $2L_1$ are $(\nu_r)_C$, $\Lambda^2(\nu_r)_C-\theta$ and $(\mathrm{Ad}_{Sp(r)})^c$, respectively, and that $d(L_3)=13$ for $r=3$.

(B) Let ψ be an irreducible direct summand of ϕ^c. Then, from (A), it follows that ψ is $(\nu_r)_C$, $\Lambda^2(\nu_r)_C-\theta$, $(\mathrm{Ad}_{Sp(r)})^c$ or ψ_1, where ψ_1 is the complex irreducible representation of $Sp(3)$ with the highest weight L_3. Suppose that ψ is ψ_1. Then ϕ^c also has the conjugate representation of ψ_1 as a direct summand, since there is no real representation whose complexification is ψ_1. Thus the degree of ϕ^c must be larger than $2d(L_3)=26$. This is a contradiction; because the degree of ϕ does not exceed $\dim Sp(3)+3=24$. Therefore ϕ^c is $k(\nu_r)_C\oplus l\theta$, $(\mathrm{Ad}_{Sp(r)})^c\oplus l\theta$ or $m(\Lambda^2(\nu_r)_C-\theta)\oplus l\theta$, where $m=1$ for $r\geqq 3$ and $m=1$ or 2 for $r=2$. Notice that $((\nu_r)_R)^c=2(\nu_r)_C$ and that $\Lambda^2(\nu_r)_C-\theta$ is the complexification of the isotropy representation of $SU(2r)/Sp(r)$ at the base point. Denote by η this isotropy representation. As is investigated above, the identity component of a principal isotropy subgroup of η is conjugate to $Sp(1)^r$. Also a principal isotropy subgroup of 2η for $r=2$ is conjugate to $Sp(1)$, since η is the lifting of ρ_5 Moreover, it is known that principal isotropy subgroups of $k(\nu_r)_R$ and $\mathrm{Ad}_{Sp(r)}$ are conjugate to $Sp(r-k)$ and a maximal torus of G, respectively. Thus we see that ϕ is $(\nu_r)_R\oplus 2\theta$ $(r\geqq 2)$, $\mathrm{Ad}_{Sp(r)}\oplus(3-r)\theta$ $(r=2, 3)$, $\eta\oplus(4-r)\theta$ $(r=2, 3, 4)$ or 2η $(r=2)$. Hence the required result for $G=Sp(r)$ is obtained.

Outline of the proof of Proposition 2.3. Here we only consider the case of $G=Sp(3)$ and $\phi^c=\Lambda^2(\nu_3)_C$, because the proof for this case is typical. The other cases are treated similarly, but the actual case-by-case proofs are somewhat long and tedious. See [13] for the other cases.

The orbit types of $\phi|S(V_\phi)$ equal those of ϕ because ϕ has just one dimensional trivial summand. In particular, $\phi|S(V_\phi)$ has exactly two fixed points as the isolated singular orbits. Since $\phi|S(V_\phi)$ has codimension two principal orbits and has singular orbits, $\phi|S(V_\phi)$ has no exceptional orbit and the orbit space $S(V_\phi)^*$ is a two dimensional disk whose boundary is B^*, where B^* is the set of all singular orbits in $S(V_\phi)$ (see Chapter IV of [2]). On the other hand, from the observation stated before the proof of Proposition 2.1, we see that the identity component of each isotropy subgroup is conjugate to $Sp(1)^3$, $Sp(1)\times Sp(2)$ or $Sp(3)$, and that $Sp(1)^3$ is the identity component of a principal isotropy subgroup. From these facts together with the fact that $N(Sp(1)\times Sp(2))=Sp(1)\times Sp(2)$, it follows that $G/(Sp(1)\times Sp(2))$ must occur in $S(V_\phi)$ as a non-isolated singular orbit. We show below which vector in $S(V_\phi)$ has $Sp(1)\times Sp(2)$ as its isotropy subgroup. At the same time, we show that the principal isotropy subgroup is connected.

As is mentioned in the proof of Proposition 2.1, the non-trivial irreducible direct summand η of ϕ is the isotropy representation of $SU(6)/Sp(3)$ at the base point. Let T and T' be maximal tori of $SU(6)$ and $Sp(3)$, respectively. Regarding the Lie algebra $L(T)$ of T as

$$\{\mathrm{diag}(d_1\sqrt{-1}, d_2\sqrt{-1}, \cdots, d_6\sqrt{-1}) | d_j \in \boldsymbol{R}, \textstyle\sum d_j = 0\} ,$$

each element in $L(T')$ can be expressed as

$$\mathrm{diag}(d_1\sqrt{-1}, -d_1\sqrt{-1}, d_2\sqrt{-1}, -d_2\sqrt{-1}, d_3\sqrt{-1}, -d_3\sqrt{-1})$$

(see Chapter 4 of [1]).

Here $\mathrm{diag}(d_1\sqrt{-1}, d_2\sqrt{-1}, \cdots, d_6\sqrt{-1})$ is a diagonal matrix of order 6. Let π be the projection of $L(SU(6))$ to $L(SU(6))/L(Sp(3))$. Then, for the linear action induced by η, we have

$$G_{\pi(v_1)} = Sp(1)^3,\ G_{\pi(v_2)} = Sp(1) \times Sp(2) ,$$

where $v_1 = \mathrm{diag}(l_1\sqrt{-1}, l_1\sqrt{-1}, l_2\sqrt{-1}, l_2\sqrt{-1}, l_3\sqrt{-1}, l_3\sqrt{-1})$ and $v_2 = \mathrm{diag}(l_1\sqrt{-1}, l_1\sqrt{-1}, l_2\sqrt{-1}, l_2\sqrt{-1}, l_2\sqrt{-1}, l_2\sqrt{-1})$ for each other different integers $\{l_j, 1 \leqq j \leqq 3\}$. This implies that the principal isotropy subgroup is connected and that the vector given by normalizing v_2 has $Sp(1) \times Sp(2)$ as its isotropy subgroup. Thus the required result for the case of $G = Sp(3)$ and $\phi^c = \Lambda^2(\nu_3)_C$ is obtained.

The smooth actions of G on homology spheres whose linear models are given by the representations $\rho_n \oplus 2\theta$, $2\rho_n$, $(\mu_n)_R \oplus 2\theta$ and $(\nu_n)_R \oplus 2\theta$ in Table A are regular. We shall also say that the smooth actions of G on homology spheres are regular if the linear models are given by the liftings of these representations. Then these actions of regular types have exactly two orbit types and the G-homology spheres become smooth special G-manifolds (See 1.3.) whose orbit spaces are two dimensional disks. It is known that these actions are equivariantly diffeomorphic to their linear models, the $SO(2m+1)$-actions on W_k^{4m+1} or the $Sp(2)$-actions on W_k^9 (k; odd) in the Introduction (see [3], [4] and Chapters V, VI of [2]). So, to prove our theorem, we have only to investigate smooth actions which are of different types from the above regular ones. In the later sections, we shall only treat such smooth actions.

§ 3. Orbit datum.

Throughout this section, let G be one of the three groups $SO(n)$

$(n\neq 2, 4)$, $SU(n)$ or $Sp(n)$, and let M be a homology sphere with a smooth action of G *modelled on* a representation in Table A which is not regular. By a G-homology sphere *modelled on* a representation ϕ of G, we mean a homology sphere with the smooth action of G whose linear model is obtained from the linear action on the representation space of ϕ by restricting to the unit sphere.

As is mentioned in Remark 2.2, we have excluded the representations of G from Table A which become the liftings of some representations in Table A. But, the results in this section are also valid for G-homology spheres modelled on such representations.

Let B be the set of all singular orbits in M. As in the proof of Propositions 2.3, the orbit space M^* is a two dimensional disk whose boundary is B^*. In particular, M always has finitely many isolated singular orbits because M is compact.

Now suppose that M has c isolated singular orbits. Denote by G/H and G/K_i $(i=1, \cdots, c)$ a principal orbit and isolated singular orbits, respectively. We may assume that $K_i>H$ $(i=1, \cdots, c)$. Let $\nu(G/K_i)$ be a closed equivariant tubular neighbourhood of G/K_i in M for each $i=1, \cdots, c$. For the natural projection p of M to M^*, put $A_i=p(\nu(G/K_i))$ and $z_i=p(G/K_i)$. By renumbering if necessary, we may assume that $\{z_i\}$ is cyclically ordered in B^*. The boundary $\partial p^{-1}(A_i)$ of $p^{-1}(A_i)$ becomes a G-manifold with codimension one principal orbits and with two singular orbits which are non-isolated singular orbits in M. Denote these two singular orbits by G/L_j $(j=i-1, i)$. The L_j $(j=i-1, i)$ can be taken to be $K_i>L_j>H$. Moreover, the L_i may be chosen up to conjugacy in K_i so that $\partial p^{-1}(A_i)$ is equivariantly diffeomorphic to $M_{\pi_{i-1}}\cup_{\mathrm{id}}M_{\pi_i}$, where M_{π_j} is the mapping cylinder of the natural projection π_j of G/H to G/L_j $(j=i-1, i)$, and id indicates the identity map of G/H (see p. 206 of [2]). For the union of the two mapping cylinders, we have the following lemma which we need in the proofs of some lemmas later.

LEMMA 3.1. *For $j=i-1, i$, the action of $(N(H)\cap N(L_j))/H$ on M_{π_j} which is induced by the commutative diagram*

$$\begin{array}{ccccccc} gH & \in & G/H & \xrightarrow{\pi_j} & G/L_j & \ni & gL_j \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ ga^{-1}H & \in & G/H & \xrightarrow{\pi_j} & G/L_j & \ni & ga^{-1}L_j \end{array}, \quad [a]\in (N(H)\cap N(L_j))/H$$

is smooth. In particular, the G-manifold $M_{\pi_{i-1}}\cup_{\mathrm{id}}M_{\pi_i}(\approx\partial p^{-1}(A_i))$ is a smooth special G-manifold.

PROOF. Let $G\times_{L_j}D^{n_j}$ $(j=i-1, i)$ be equivariant tubular neighbourhoods of G/L_j $(j=i-1, i)$ in $\partial p^{-1}(A_i)$, where D^{n_j} is an n_j-dimensional disk. Then M_{π_j} is equivariantly diffeomorphic to $G\times_{L_j}D^{n_j}$, and the action of $(N(H)\cap N(L_j))/H$ on M_{π_j} above induces the action of $(N(H)\cap N(L_j))/H$ on $G\times_{L_j}D^{n_j}$ defined by $[g, v]\mapsto[ga^{-1}, v]$ for $[g, v]\in G\times_{L_j}D^{n_j}$. Let v_o be a point in $S^{n_j-1}=\partial D^{n_j}$ with $(L_j)_{v_0}=H$. This action is smooth if the map h_a: $S^{n_j-1}\rightarrow S^{n_j-1}$ defined by $lv_0\mapsto(ala^{-1})v_0$ is orthogonal, where $l\in L_j$. So, to obtain the first statement of this lemma, it is sufficient to show that, for each $[a]\in(N(H)\cap N(L_j))/H$, the map h_a is orthogonal. To do this, we may assume that H is one of the principal isotropy subgroups in Table A and that L_j is one of the isotropy subgroups in Table A which correspond to non-isolated singular orbits in M. Moreover, from Table A and the slice representation of L_j, we see that it is sufficient to investigate G-homology spheres M modelled on $S^2\rho_3$, $\mathrm{Ad}_{SO(5)}\oplus\theta$, $(\mu_4)_R\oplus\phi_2\oplus\theta$, $(\Lambda^2\mu_5)_R\oplus\theta$ and ϕ such that $\phi^c=\Lambda^2(\nu_3)_C$. The other case is reduced to one of these five cases. And, for these five cases, by describing the map h_a explicitly, we can show that h_a is orthogonal. This is a straightforward verification for any case. So we only give here the actual proof for the last case, that is, for an $Sp(3)$-homology sphere M modelled on ϕ such that $\phi^c=\Lambda^2(\nu_3)_C$. The other cases are treated similarly.

From Table A, we may put $H=Sp(1)^3$ and $L_j=Sp(1)\times Sp(2)$. Then we have $(N(H)\cap N(L_j))/H=eH\cup bH$, where e is the identity element and $b=\mathrm{diag}\left(1, \begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$. And a closed equivariant tubular neighbourhood of G/L_j in $\partial p^{-1}(A_i)$ is equivariantly diffeomorphic to $(Sp(3)/Sp(1))\times_{Sp(2)}D^5$, where $Sp(2)$ acts on D^5 by the isotropy representation of the homogeneous space $SU(4)/Sp(2)$ at the base point. Thus, as in the proof of Proposition 2.3, we may put $v_o=\pi(w_o)$, where $w_0=\mathrm{diag}\,(1/2)(\sqrt{-1}, \sqrt{-1}, -\sqrt{-1}, -\sqrt{-1})$ and π is the natural projection of $L(SU(4))$ to $L(SU(4))/L(Sp(2))$. Since b is in $L_j=Sp(1)\times Sp(2)$, it is easily verified that, for $l\in L_j$, $(blb^{-1})v_o=-b(lv_0)$ holds. This implies that h_b is orthogonal. Thus, for an $Sp(3)$-manifold M modelled on the above representation ϕ, the first statement of this lemma is obtained.

The second statement immediately follows from the first statement, because the other conditions to be a smooth special G-manifold (see 1.3) are clearly satisfied. Q.E.D.

Next let $\{B_i\}$ $(i=1, \cdots, c)$ be a set of subsets of M^* which satisfies the following conditions: $\{B_i\}$ $(i=1, \cdots, c)$ do not mutually intersect, each B_i is adjacent to A_i and $A_{i+1}(A_{c+1}=A_1)$ and the space $(\cup_{i=1}^c A_i)\cup(\cup_{i=1}^c B_i)$ becomes a neighbourhood of ∂M^*. See Fig. 1 below. Then $p^{-1}(B_i)$ is a trivial M_{π_i}-bundle over $B_i \cap \partial M^*$ which is diffeomorphic to the unit interval $I=[0, 1]$. We write by X the complement of $\mathrm{Int}((\cup_{i=1}^c A_i) \cup (\cup_{i=1}^c B_i))$ in M^*. The orbit space M^* is illustrated as follows:

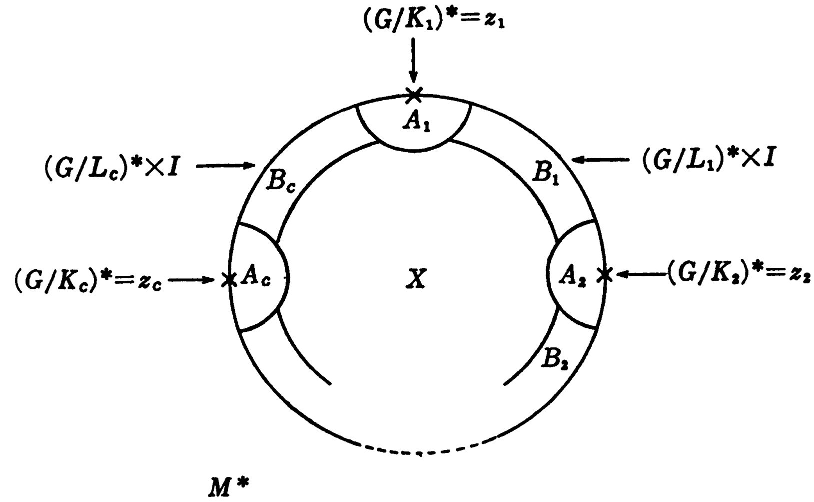

FIGURE. 1.

By using the above facts, we shall next define the *orbit datum* of M which characterizes the action of G on M. To do this we first consider the following condition for a set of isotropy subgroups. The notations in the Condition P below are the same as those above.

CONDITION P. Fix a principal isotropy subgroup H. We say that a set of subgroups H, K_i, L_i $(i=1, \cdots, c)$ of G satisfies the Condition P if, for the given H, the subgroups $\{K_i\}, \{L_i\}$ satisfy the following conditions:

(i) $K_i, L_i > H$, $K_i > L_i, L_{i-1}$ for $i=1, \cdots, c$, where $L_0 = L_c$,

(ii) $\partial p^{-1}(A_i) \approx M_{\pi_{i-1}} \cup_{\mathrm{id}} M_{\pi_i}$ for $i=1, \cdots, c$, and $\pi_0 = \pi_c$,

(iii) $p^{-1}(B_i) \approx M_{\pi_i} \times I$ for $i=1, \cdots, c$, where $I=[0, 1]$,

(iv) $p^{-1}(X \cup (\cup_{i=1}^c B_i)) \approx (G/H \times X) \cup_{\mathrm{id}} (\cup_{i=1}^c M_{\pi_i} \times I)$.

We recall here that every equivariant map of a coset space G/A to itself is of the form ϕ_a which is defined by $\phi_a(gA) = ga^{-1}A$ for some $a \in N(A)$ (see Chapter I of [2]).

LEMMA 3.2. *Let M be a G-homology sphere modelled on a representation in Table* A *in* §2 *which is not regular. Then, for a principal isotropy subgroup H, we can always choose a set of isotropy subgroups K_i, L_i $(i=1, \cdots, c)$ satisfying the Condition* P, *where c is the number of the isolated singular orbits.*

PROOF. Fix a principal isotropy subgroup H in the conjugacy class.

Case 1. Suppose that $N(H)/H$ is finite. It is clear that $p^{-1}(X)$ is equivariantly diffeomorphic to $G/H\times X$. Thus we have

$$p^{-1}\Big(X\cup\Big(\bigcup_{i=1}^{c}B_i\Big)\Big)\approx(G/H\times X)\underset{(\psi_1,\cdots\psi_c)}{\cup}\Big(\bigcup_{i=1}^{c}p^{-1}(B_i)\Big)$$

where ψ_i is an equivariant diffeomorphism of $G/H\times(X\cap B_i)$ to $p^{-1}(X\cap B_i)$. First we choose arbitrarily subgroups L_0 and K_1 so that $p(G/K_1)=z_1$, $p(G/L_0)=\partial M^*\cap B_c\cap A_1$, $L_0>H$, $K_1>H$ and $K_1>L_0$. As is mentioned above, for given L_0 and K_1, we can choose L_1 so that $K_1>L_1>H$ and that $\partial p^{-1}(A_1)$ is equivariantly diffeomorphic to $M_{\pi_0}\cup_{\mathrm{id}}M_{\pi_1}$. This implies that we may put $\psi_1=$the identity map when $p^{-1}(B_1)$ is regarded as $M_{\pi_1}\times I$. Next choose K_2 so that $p(G/K_2)=z_2$, $K_2>H$ and $K_2>L_1$. Then we can also choose L_2 so as to satisfy i), ii) and iii) of the Condition P. And hence, we may put $\psi_2=$the identity map. In this way, we can choose a set $\{L_0, K_1, \cdots, K_c, L_c\}$ of subgroups of G which satisfies (i)-(iii) except the condition of $L_0=L_c$. In general, $L_0\neq L_c$, but, in our case, any equivariant map between M_{π_0} and M_{π_c} must be the identity map; because $p^{-1}(\partial X)$ is a trivial G/H- bundle and $N(H)/H$ is finite. This implies that $L_0=L_c$ and $\pi_0=\pi_c$. And hence, we may put $\psi_i=$the identity map for all $i\in\{1, \cdots, c\}$, namely (iv) of the Condition P holds. Thus the subgroups H, K_i, L_i $(i=1, \cdots, c)$ which are obtained by the above way, satisfy the Condition P.

Case 2. Suppose that $N(H)/H$ is not finite. In this case, M is modelled on one of the representations, $(\Lambda^2\mu_5)_R\oplus\theta$, $(\Lambda^2\mu_7)_R$ or $(\mu_4)_R\oplus\phi_2\oplus\theta$ in Table A. By the same way as in Case 1, we obtain a set $\{L_0, K_1, L_1, \cdots, L_c\}$ which satisfies (i)-(iii) of the Condition P except the condition of $L_0=L_c$. Regard $\partial X\cap B_i$ as I and ψ_i in Case 1 as an equivariant map of $G/H\times I$ to itself. If $L_0=L_c$ holds, then, for all i, $\psi_i|G/H\times\{0\}$ and $\psi_i|G/H\times\{1\}$ can be taken as the identity maps of G/H. Then, we may put $\psi_i=$the identity map, since $\psi_i(eH\times I)\subset(N(H)/H)^0\times I$. Thus (iv) of the Condition P holds. So we have only to show that we can choose a set of subgroups $\{L_0, K_1, \cdots, L_c\}$ so that $L_0=L_c$.

Any equivariant map f of M_{π_0} to M_{π_c} is given by

$$f(gH, t)=(ga_t^{-1}H, t),\ [a_t]\in(N(H)/H)^0 \quad\text{for}\quad t\neq 1\ ,$$
$$f(gL_0, 1)=(ga_1^{-1}L_0, 1),\ [a_1]\in(N(H)/H)^0\ ,$$

since $p^{-1}(\partial X)$ is a trivial G/H-bundle over ∂X. Thus we have $a_1L_0a_1^{-1}=L_c$. For each smooth action modelled on one of the representations, $(\Lambda^2\mu_5)_R\oplus\theta$, $(\Lambda^2\mu_7)_R$ or $(\mu_4)_R\oplus\phi_2\oplus\theta$, we may put $(H, L_0)=(SU(2)^2, SU(2)\times SU(3))$, $(SU(2)^3, SU(2)^2\times SU(3))$ or $(SU(2), Sp(2))$, respectively (see Table A). If (H, L_0) takes one of the first two types, then we have $N(H)^0\subset N(L_0)$.

Therefore, for these cases, $L_0=L_c$ holds. Next put $(H, L_0)=(SU(2), Sp(2))$ (in this case, $N(H)^0$ is not contained in $N(L_0)$). From Table A, it follows that $K_1=G=SU(4)$. Since the slice representation at x with $G_x=K_1$ is $(\mu_4)_R\oplus\phi_2$, we have $L_1\in(SU(3))$. Put $L_1'=SU(3)$. And let $M_{\pi_1'}$ be the mapping cylinder of the projection $\pi_1'\colon G/H\to G/L_1'$. Then $\partial p^{-1}(A_1)$ is equivariantly diffeomorphic to $M_{\pi_0}\cup\phi_a M_{\pi_1'}$, where ϕ_a is the equivariant map of G/H to itself defined by $\phi_a(gH)=ga^{-1}H$ for some $a\in N(H)$. Notice here that every element in $N(H)$ is written as bd for $b\in S(U(1)^4)$ and $d\in SU(2)^2$, and that $S(U(1)^4)\subset N(SU(3))$ and $SU(2)^2\subset N(Sp(2))$. So, taking a as bd, we have the following commutative diagram,

$$\begin{array}{ccccccccc}
gL_0\in G/L_0 & \xleftarrow{\pi_0} & G/H & \xrightarrow{\phi_a} & G/H & \xrightarrow{\pi_1'} & G/L_1'\ni gL_1' \\
\downarrow \quad \downarrow{\scriptstyle\phi_d} & & \downarrow{\scriptstyle\phi_d} & & \downarrow{\scriptstyle\phi_{b^{-1}}} & & \downarrow{\scriptstyle\phi_{b^{-1}}} \quad \downarrow \\
gd^{-1}L_0\in G/L_0 & \xleftarrow{\pi_0} & G/H & \xrightarrow{\mathrm{id}} & G/H & \xrightarrow{\pi_1'} & G/L_1'\ni gbL_1'\,.
\end{array}$$

Thus, from Lemma 3.1, we see that $\partial p^{-1}(A_i)$ is equivariantly diffeomorphic to $M_{\pi_0}\cup_{\mathrm{id}}M_{\pi_1'}$. Hence we may assume that $L_1=SU(3)$. For these H, L_0, K_1, L_1, we choose a set of subgroups $\{L_0, K_1, L_1, \cdots, L_c\}$ by the same way as in Case 1. Now suppose that $L_0\neq L_c$. Since L_c equals $b'L_0b'^{-1}$ for some $b'\in S(U(1)^4)(\subset N(SU(3)))$, we also have the following commutative diagram,

$$\begin{array}{ccccccc}
G/L_0 & \xleftarrow{\pi_0} & G/H & \xrightarrow{\mathrm{id}} & G/H & \xrightarrow{\pi_1} & G/L_1 \\
\downarrow{\scriptstyle\phi_{b'}} & & \downarrow{\scriptstyle\phi_{b'}} & & \downarrow{\scriptstyle\phi_{b'}} & & \downarrow{\scriptstyle\phi_{b'}} \\
G/L_c & \xleftarrow{\pi_c} & G/H & \xrightarrow{\mathrm{id}} & G/H & \xrightarrow{\pi_1} & G/L_1\,.
\end{array}$$

Thus $\partial p^{-1}(A_1)$ is equivariantly diffeomorphic to $M_{\pi_c}\cup_{\mathrm{id}}M_{\pi_1}$. And hence, by replacing L_0 by L_c, we obtain a set of subgroups $\{H, K_1, L_1, \cdots, K_c, L_c\}$ which satisfies the Condition P. Q.E.D.

DEFINITION. By an *orbit datum*, we shall mean the sequence of isotropy subgroups $(H, K_1, L_1, \cdots, K_c, L_c)$ satisfying the Condition P.

The Condition P implies that the classification of G-homology spheres with the same linear model and with the same orbit datum depends only on the choice of attaching maps of $\partial p^{-1}(A_i)(i=1, \cdots, c)$.

§ 4. The relation between orbit data and G-homology spheres.

Let G be a compact, connected, simple, classical Lie group as in §§2, 3. And we also assume that every G-homology sphere in this section is modelled on a representation in Table A which is not regular. Our purpose in this section is to show that all G-homology spheres with the same linear model and with the same orbit datum are equivariantly diffeomorphic. By the reason mentioned in the last paragraph in §3, we shall first investigate attaching maps of $\partial p^{-1}(A_i)$ (the notation $p^{-1}(A_i)$ is the same as in §3). Note that an attaching map of $\partial p^{-1}(A_i)$ is an equivariant diffeomorphism of $\partial p^{-1}(A_i)$ inducing the identity map on the orbit space $\partial p^{-1}(A_i)/G$. That is, the set of attaching maps of $\partial p^{-1}(A_i)$ is equal to the set $\mathbf{Diffeo}_I^G(\partial p^{-1}(A_i))$ in 1.3. I indicates the unit interval $[0, 1]$ which is diffeomorphic to $\partial p^{-1}(A_i)/G$.

Let an *attaching diffeomorphism* of $\partial p^{-1}(A_i)$ mean an equivariant diffeomorphism in $\mathbf{Diffeo}_I^G(\partial p^{-1}(A_i))$. In Lemmas 4.4, 4.5, 4.7 and 4.8, we shall show that every attaching diffeomorphism of $\partial p^{-1}(A_i)$ can be extended to $p^{-1}(A_i)$. To do this, we need the following Lemmas 4.1, 4.2 and 4.3. The notations H, K_i, L_i, etc. are the same as those in §3.

The results in this section are also valid for G-homology spheres modelled on liftings of representations in Table A.

LEMMA 4.1. *If $N(H)/H$ is finite, then there is a one-one correspondence between $\mathbf{Diffeo}_I^G(\partial p^{-1}(A_i))$ and $(N(H)\cap N(L_i)\cap N(L_{i-1}))/H$.*

PROOF. The space $\partial p^{-1}(A_i)$ is equivariantly diffeomorphic to the space $G/H\times I$ with $G/H\times\{0\}$ callapsed to G/L_{i-1}, and with $G/H\times\{1\}$ to G/L_i. We denote this space by $(G/H\times I)/\sim$. Since $N(H)/H$ is finite, every attaching diffeomorphism of this space (that is, of $\partial p^{-1}(A_i)$) is naturally induced by the equivariant map $\bar{f}_a$ of $G/H\times I$ to itself which is defined by

$$\bar{f}_a(gH, t)=(ga^{-1}H, t) \quad \text{for} \quad [a]\in(N(H)\cap N(L_i)\cap N(L_{i-1}))/H\,.$$

Thus the correspondence of $\bar{f}_a$ to $[a]$ gives an injective correspondence from $\mathbf{Diffeo}_I^G(\partial p^{-1}(A_i))$ to $(N(H)\cap N(L_i)\cap N(L_{i-1}))/H$. The surjectivity of this correspondence follows from Lemma 3.1. Q.E.D.

Let f_a be the attaching diffeomorphism of $\partial p^{-1}(A_i)$ given by the map $\bar{f}_a$ in the proof of the above lemma. From 1.2, $p^{-1}(A_i)$ is equivariantly diffeomorphic to $G\times_{K_i}S$, where S is a slice at x with $G_x=K_i$. Thus the map f_a naturally induces the equivariant diffeomorphism of $G\times_{K_i}\partial S$. We also denote it by the same notation f_a.

LEMMA 4.2. *Suppose that $N(H)/H$ is finite. Let $a\in N(H)\cap N(L_i)\cap N(L_{i-1})$. If $N(H)\cap N(L_i)\cap N(L_{i-1})\subset N(K_i)$, then f_a maps ∂S to $\partial S'$, where S and S' are slices at x and $a^{-1}x$ with $G_x=G_{a^{-1}x}=K_i$, respectively.*

PROOF. Let Φ be an equivariant diffeomorphism of $(G/H\times I)/\sim$ to $G\times_{K_i}\partial S$. Then we have

$$\begin{aligned}&\Phi((K_i/H\times I)/\sim)=\partial S \quad\text{and}\\&\Phi(f_a((K_i/H\times I)/\sim))=\Phi((K_ia^{-1}/H\times I)/\sim)=\Phi((a^{-1}K_i/H\times I)/\sim)\\&\qquad=a^{-1}\Phi((K_i/H\times I)/\sim)=a^{-1}\partial S=\partial S'\,.\end{aligned}$$ Q.E.D.

Let $(G_1, G_2, \cdots, G_k)$ and $(G'_1, G'_2, \cdots, G'_k)$ be sets of ordered subgroups of G. We say that $(G_1, G_2, \cdots, G_k)$ and $(G'_1, G'_2, \cdots, G'_k)$ are simultaneously

TABLE B

	G	representation ϕ	H	(K_i, L_i, L_{i-1})
1	$SO(5)$	$\mathrm{Ad}_{SO(5)}\oplus\theta$	T	$(SO(5), SO(2)\times SO(3), U(2))$
2	$SO(6)$	$\mathrm{Ad}_{SO(6)}$	T	a. $(SO(2)\times SO(4), SO(2)\times U(2), SO(2)\times L)$; L is conjugate to $U(2)$ in $SO(4)$
				b. $(U(3), U(2)\times SO(2), SO(2)\times U(2))$
3	$SO(7)$	$\mathrm{Ad}_{SO(7)}$	T	a. $(U(2)\times SO(3), SO(2)^2\times SO(3), U(2)\times SO(2))$
				b. $(SO(2)\times SO(5), SO(2)\times U(2), SO(2)^2\times SO(3))$
				c. $(U(3), U(2)\times SO(2), SO(2)\times U(2))$
4	$SU(3)$	$\mathrm{Ad}_{SU(3)}\oplus\theta$	T	$(SU(3), S(U(1)\times U(2)), S(U(2)\times U(1)))$
5	$Sp(3)$	$\mathrm{Ad}_{Sp(3)}$	T	a. $(U(2)\times Sp(1), U(1)^2\times Sp(1), U(2)\times U(1))$
				b. $(U(1)\times Sp(2), U(1)\times U(2), U(1)^2\times Sp(1))$
				c. $(U(3), U(2)\times U(1), U(1)\times U(2))$
6	$SO(3)$	$S^2\rho_3$	Z_2^2	$(SO(3), O(2), N)$; N is conjugate to $O(2)$ in $SO(3)$
7	$Sp(3)$	$\phi^c=\Lambda^2\nu_3$	$Sp(1)^3$	$(Sp(3), Sp(1)\times Sp(2), Sp(2)\times Sp(1))$
8	$Sp(4)$	$\phi^c=\Lambda^2\nu_4-\theta$	$Sp(1)^4$	a. $(Sp(2)^2, Sp(2)\times Sp(1)^2, Sp(1)^2\times Sp(2))$
				b. $(Sp(1)\times Sp(3), Sp(1)^2\times Sp(2), Sp(1)\times Sp(2)\times Sp(1))$
9	$SU(5)$	$(\Lambda^2\mu_5)_R\oplus\theta$	$SU(2)^2$	$(SU(5), SU(2)\times SU(3), Sp(2))$
10	$SU(7)$	$(\Lambda^2\mu_7)_R$	$SU(2)^3$	a. $(Sp(3), Sp(2)\times Sp(1), Sp(1)\times Sp(2))$
				b. $(Sp(2)\times SU(3), SU(2)^2\times SU(3), Sp(2)\times Sp(1))$
				c. $(SU(2)\times SU(5), Sp(1)\times Sp(2), SU(2)^2\times SU(3))$
11	$SU(4)$	$(\mu_4)_R\oplus\phi_2\oplus\theta$	$SU(2)$	$(SU(4), Sp(2), SU(3))$

conjugate, if there exists an element g in G such that $g^{-1}G_i g = G'_i$ for all i.

LEMMA 4.3. *Let M be a G-homology sphere modelled on a representation in Table* A *which is not regular. Then, for a principal isotropy subgroup H in the following Table* B, *all triples (K_i, L_i, L_{i-1}) in the Table* B *satisfy* i) *and* ii) *of the Condition* P *in* §3. *Conversely, every triple (K_i, L_i, L_{i-1}) which satisfies* (i) *and* (ii) *of the Condition* P *for H above, is simultaneously conjugate to one in the Table* B.

We only prove Lemma 4.3 for M modelled on $\mathrm{Ad}_{SO(6)}$; because the proof for this case is typical and the others are similar.

PROOF. Suppose that (K_i, L_i, L_{i-1}) satisfies (i) of the Condition P. Let S be a slice at x with $G_x = K_i$. And let N_{π_j} $(j = i-1, i)$ be the mapping cylinders of the projections π_j: $K_i/H \to K_i/L_j (j = i-1, i)$. If ∂S is K_i-equivariantly diffeomorphic to $N_{\pi_{i-1}} \cup_{\mathrm{id}} N_{\pi_i}$, then $G \times_{K_i} \partial S$ is equivariantly diffeomorphic to $M_{\pi_{i-1}} \cup_{\mathrm{id}} M_{\pi_i}$, namely, (K_i, L_i, L_{i-1}) satisfies (i) and (ii) of the Condition P.

Let M be modelled on the representation $\mathrm{Ad}_{SO(6)}$ of $SO(6)$. From Proposition 2.3, we have $K_i \in (SO(2) \times SO(4))$ or $K_i \in (U(3))$. Put $K_i = U(3)$. Since the slice representation at x with $G_x = K_i$ is $\mathrm{Ad}_{U(3)} - \theta$, we can see that L_i and L_{i-1} are conjugate to $U(2) \times U(1)$. Put $L_i = U(2) \times U(1)$ and $L_{i-1} = U(1) \times U(2)$. Then ∂S is K_i-equivariantly diffeomorphic to $N_{\pi_{i-1}} \cup_{\phi_a} N_{\pi_i}$, where ϕ_a is a K_i-equivariant map of K_i/H to itself which is given by $\phi_a(kH) = ka^{-1}H$ for some $[a] \in (N(H) \cap K_i)/H$. It is easily verified that

$$(N(H) \cap K_i)/H = eH \cup b_1H \cup b_2H \cup b_3H \cup b_1b_2H \cup b_1b_3H ,$$

where $b_1 = \mathrm{diag}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 1\right)$, $b_2 = \mathrm{diag}\left(1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$ and $b_3 = \begin{bmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{bmatrix}$ as complex matrices of order 3. Since $b_1 \in N(L_i)$ and $b_2 \in N(L_{i-1})$, we can see that $N_{\pi_{i-1}} \cup_{\phi_{b_1}} N_{\pi_i}$, $N_{\pi_{i-1}} \cup_{\phi_{b_2}} N_{\pi_i}$ and $N_{\pi_{i-1}} \cup_{\phi_{b_1b_2}} N_{\pi_i}$ are K_i-equivariantly diffeomorphic to $N_{\pi_{i-1}} \cup_{\mathrm{id}} N_{\pi_i}$ (see Lemma 3.1 and the diagrams in the proof of Lemma 3.2). Similarly both $N_{\pi_{i-1}} \cup_{\phi_{b_3}} N_{\pi_i}$ and $N_{\pi_{i-1}} \cup_{\phi_{b_1b_3}} N_{\pi_i}$ are K_i-equivariantly diffeomorphic to $N_{\pi_i} \cup_{\mathrm{id}} N_{\pi_i}$; because $b_3 L_{i-1} b_3^{-1} = L_i$ holds. But, by the Mayer-Vietoris exact sequence for the triad $(N_{\pi_i} \cup_{\mathrm{id}} N_{\pi_i}, N_{\pi_i}, N_{\pi_i})$, we see that $N_{\pi_i} \cup_{\mathrm{id}} N_{\pi_i}$ is not diffeomorphic to ∂S. Thus ∂S is K_i-equivariantly diffeomorphic to $N_{\pi_{i-1}} \cup_{\mathrm{id}} N_{\pi_i}$. And hence, $(K_i, L_i, L_{i-1}) = (U(3), U(2) \times U(1), U(1) \times U(2))$ satisfies (i) and (ii) of the Condition P. For $K_i = U(3)$, suppose that another triple (K_i, L'_i, L'_{i-1}) satisfies (i) and (ii) of the Condition P. Then $N_{\pi_{i-1}} \cup_{\mathrm{id}} N_{\pi_i}$ and $N_{\pi'_{i-1}} \cup_{\mathrm{id}} N_{\pi'_i}$ are equivariantly diffeomoprhic. Thus we have $L'_j = aL_ja^{-1}(j=i-1, i)$ for some $[a] \in (N(H) \cap K_i)/H$. Also, taking another K''_i in $(U(3))$, it follows from the above results that a triple $(K''_i, L''_i, L''_{i-1})$ satisfying (i) and (ii) of the Condition P must be of the form $(gK_ig^{-1}, gL_ig^{-1}, gL_{i-1}g^{-1})$ for some $g \in G$. When $K_i \in (SO(2) \times SO(4))$, we also have the required result by the same way as in the case of $K_i \in (U(3))$.

Similarly we can prove this lemma for the other cases, omitting the details.

Now suppose that $N(H)/H$ is finite. In this case, from Lemma 4.1, every attaching diffeomorphism of $\partial p^{-1}(A_i)$ is of the form f_a for $[a] \in (N(H) \cap N(L_i) \cap N(L_{i-1}))/H$. Also, from Lemma 4.3, to prove the existence of an extension of f_a to $p^{-1}(A_i)$, it is sufficient to prove it for the triples in Table B. For the case of 2-b, 4, 6, 7 or 8-b in Table B, we can take the identity map only as an attaching diffeomorphism of $\partial p^{-1}(A_i)$, since $(N(H) \cap N(L_i) \cap N(L_{i-1})) = H$. So we shall show that f_a has an extension to $p^{-1}(A_i)$ for the other triples in Table B. Here the extension of f_a means an equivariant diffeomorphism of $p^{-1}(A_i)$.

In the proofs of the following lemmas, the symbols i and j mean the imaginary unit and the quarternionic unit such that $i^2 = j^2 = -1$, respectively, excepting those appeared as suffices.

LEMMA 4.4. *Suppose that a triple (K_i, L_i, L_{i-1}) is one of those in* 1, 2-a, 3-b, 5-b *and* 8-a. *Then, for each $[a] \in (N(H) \cap N(L_i) \cap N(L_{i-1}))/H$, the map $f_a|\partial S$ is a linear map, where S is a slice at x with $G_x = K_i$. In particular, f_a can be extended to $p^{-1}(A_i)$.*

PROOF. For these cases, it is easily verified that $N(H) \cap N(L_i) \cap N(L_{i-1}) \subset K_i$. Therefore, from Lemma 4.2, every f_a maps ∂S onto ∂S.

Case of 1. In this case, we have $(N(H) \cap N(L_i) \cap N(L_{i-1}))/H = eH \cup bH$, where e = the identity matrix and $b = \mathrm{diag}(C, C, 1)$ for $C = \mathrm{diag}(1, -1)$. So an attaching diffeomorphism of $\partial p^{-1}(A_i)$ is either the identity map or f_b. Let Φ be the equivariant diffeomorphism in the proof of Lemma 4.2. Then, for $x = \Phi([eH, t]) \in \partial S$, we have $f_b(x) = b^{-1}x$. Since the K_i-action on S is $\mathrm{Ad}_{SO(5)}$ and x is in $F(H, \partial S) = F(T, \partial S)$, x can be written as the matrix X; $X = \mathrm{diag}(D(d_1), D(d_2), 0)$, $D(d_i) = \begin{bmatrix} 0 & -d_i \\ d_i & 0 \end{bmatrix}$, $d_i \in \boldsymbol{R}\, (i=1, 2)$, where $d_1 = d_2$ if $t=0$, and $d_2 = 0$ if $t=1$ (see Chapter 4 of [1]). Thus we have $f_b(x) = b^{-1}x = b^{-1}Xb = -X = -x$ and $f_b(gx) = gf_b(x) = g(-X)g^{-1} = -gx$ for $g \in K_i (= SO(5))$. This shows that $f_b|\partial S$ is linear, since every element in ∂S is of the form gx for some $g \in K_i$.

Case of 2-a. Instead of $\mathrm{Ad}_{SO(6)}$, we consider the lifting $\mathrm{Ad}_{SU(4)}$ to $SU(4)$ to simplify the computation. Then, $H(=T)$ corresponds to a maximal torus $\tilde{H}(=T)$ of $SU(4)$. And the triple (K_i, L_i, L_{i-1}) in 2-a corresponds to the triple $(\tilde{K}_i, \tilde{L}_i, \tilde{L}_{i-1})=(S(U(2)^2), S(U(1)^2\times U(2)), S(U(2)\times U(1)^2))$. The required result for the triple (K_i, L_i, L_{i-1}) are deduced from the following results for the triple $(\tilde{K}_i, \tilde{L}_i, \tilde{L}_{i-1})$. We have $(N(\tilde{H})\cap N(\tilde{L}_i)\cap N(\tilde{L}_{i-1}))/H=eH\cup b_1H\cup b_2H\cup b_3H$, where $b_1=\mathrm{diag}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\-1&0\end{bmatrix}\right)$, $b_2=\mathrm{diag}\left(\begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$ and $b_3=b_1b_2$. Since the K_i-action on S is $((\mathrm{Ad}_{U(2)}-\theta)\otimes\theta\oplus\theta\otimes(\mathrm{Ad}_{U(2)}-\theta))|S(U(2)\times U(2))$, $x=\Phi([eH, t])$ corresponds to the matrix X; $X=\mathrm{diag}(X_1, X_2)$, $X_j=\mathrm{diag}(d_ji, -d_ji)$, $d_j\in\boldsymbol{R}$ $(j=1, 2)$. And we have the following relation for $g=\mathrm{diag}(g_1, g_2)\in S(U(2)\times U(2))$: $f_{b_1}(gXg^{-1})=(gb_1^{-1}Xb_1g^{-1})=\mathrm{diag}(g_1X_1g_1^{-1}, -g_2X_2g_2^{-1})$, $f_{b_2}(gXg^{-1})=-gXg^{-1}$ and $f_{b_3}(gXg^{-1})=\mathrm{diag}(-g_1X_1g_1^{-1}, g_2X_2g_2^{-1})$. Thus $f_{b_j}|\partial S$ is linear for $j=1, 2$ and 3.

Case of 3-b. This case is reduced to the case of 1, since we have $G\times_{K_i}\partial S=(SO(7)/SO(2))\times_{SO(5)}\partial S$, where $SO(5)$ acts on S by $\mathrm{Ad}_{SO(5)}$.

Case of 5-b. This case is also reduced to the case of 1 by the same reason as in the case of 3-b.

Case of 8-a. In this case, the action ψ of $K_i=Sp(2)^2$ on a slice S (diffeomorphic to a 10-dimensional disk) is given by $\psi=\psi_1\oplus\psi_2$, where $\psi_j^c=\Lambda^2(\nu_2)_C-\theta (j=1, 2)$. And we have $(N(H)\cap N(L_i)\cap N(L_{i-1}))/H=eH\cup b_1H\cup b_2H\cup b_3H$, where $b_1=\mathrm{diag}\left(1, 1, \begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$, $b_2=\mathrm{diag}\left(\begin{bmatrix}0&1\\1&0\end{bmatrix}, 1, 1\right)$ and $b_3=b_1b_2$. Now consider the map $s\colon I=[0, 1]\to\partial S$ defined by $s(t)=\Phi([eH, t])$. Then we have $(K_i)_{s(0)}=L_{i-1}=Sp(1)^2\times Sp(2)$ and $(K_i)_{s(1)}=L_i=Sp(2)\times Sp(1)^2$. Regarding $s(t)$ as a vector in S, $s(t)$ is described as $s(0)\alpha+s(1)\beta$ for $\alpha, \beta\in\boldsymbol{R}$. And each vector w in ∂S is of the form $gs(t)=g_1(s(0))\alpha+g_2(s(1))\beta$ for some $g=(g_1, g_2)\in Sp(2)\times Sp(2)=K_i$ On the other hand, if we put $b_j=(b_{j,1}, b_{j,2})\in Sp(2)\times Sp(2)$, then we have $f_{b_j}(s(t))=b_j^{-1}s(t)=b_{j,1}^{-1}(s(0))\alpha+b_{j,2}^{-1}(s(1))\beta$, $b_{j,1}^{-1}(s(0))=\pm s(0)$ and $b_{j,2}^{-1}(s(1))=\pm s(1)$. Because both actions ψ_1 and ψ_2 on the unit spheres are transitive, $N(Sp(1)^2)/Sp(1)^2$ is isomorphic to Z_2 and $b_{j,1}, b_{j,2}\in N(Sp(1)^2)$, where $N(Sp(1)^2)$ is the normalizer of $Sp(1)^2$ in $Sp(2)$. Thus for $g=(g_1, g_2)\in K_i$, we have $f_{b_j}(gs(t))=gf_{b_j}(s(t))=\pm g_1(s(0))\alpha\pm g_2(s(1))\beta$. This shows that $f_{b_j}|\partial S$ is a linear map.

The second statement follows immediately from the first statement; because a linear map $f_a|\partial S$ can be extended to S and $p^{-1}(A_i)$ is equivariantly diffeomorphic to $G\times_{K_i}S$. Q.E.D.

LEMMA 4.5. *Suppose that a triple (K_i, L_i, L_{i-1}) is one of those in* 3-a, 3-c, 5-a *and* 5-c. *Then, for each $[a]\in(N(H)\cap N(L_i)\cap N(L_{i-1}))/H$, the map f_a can be extended to $p^{-1}(A_i)$.*

PROOF. For these cases, it is verified that $N(H)\cap N(L_i)\cap N(L_{i-1})\subset$

$N(K_i)$ but $N(H)\cap N(L_i)\cap N(L_{i-1})\not\subset K_i$. Put $x=\Phi([eH, t])$ (see the proof of Lemma 4.4). And let S and S' be slices at x and $a^{-1}x$, respectively. From Lemma 4.2, f_a maps ∂S to $\partial S'$. Such $f_a|\partial S$ is given by the composition of the following two maps:

$$h\colon \partial S \longrightarrow \partial S \quad \text{defined by} \quad h(gx)=aga^{-1}x \quad \text{for} \quad g\in K_i\,,$$
$$L_a\colon \partial S \longrightarrow \partial S' \quad \text{defined by} \quad L_a(v)=a^{-1}v \quad \text{for} \quad v\in\partial S\,.$$

L_a clearly has the extension $\tilde{L}_a$ such that $\tilde{L}_a(w)=a^{-1}w$ for $w\in S$. So, if h is a linear map, then $f_a|\partial S$ can be extended to S, and hence f_a is extended to $p^{-1}(A_i)$. When $a\in K_i$, the proof of this lemma is given by the same way as in Lemma 4.4.

Case of 5-c. In this case, we have $(N(H)\cap N(L_i)\cap N(L_{i-1}))/H=eH\cup bH$, where $b=\mathrm{diag}(j, j, j)(\notin U(3)=K_i)$. And $bgb^{-1}=\bar{g}$ holds for each $g\in K_i$, where $\bar{g}$ is the conjugate matrix of g. Let X be a matrix $\mathrm{diag}(d_1 i, d_2 i, d_3 i)$, where $d_j\in \boldsymbol{R}$ and $d_1+d_2+d_3=0$. Then x is identified with X, since the action of K_i on S is given by $\mathrm{Ad}_{U(3)}-\theta$. And gx is identified with the matrix gXg^{-1}. It is clear that $\bar{g}x$ corresponds to $-(\overline{gXg^{-1}})$. Thus the above map h is a linear map.

Case of 3-c. In this case, we have $(N(H)\cap N(L_i)\cap N(L_{i-1}))/H=eH\cup bH$, where $b=\mathrm{diag}(C, C, C, -1)(\notin K_i=U(3)\subset SO(7))$, $C=\mathrm{diag}(1, -1)(\in O(2))$. And $bgb^{-1}=\bar{g}$ holds for each $g\in K_i$. So the rest of the proof is similar to that of the case of 5-c. We omit the detail.

Case of 5-a. In this case, we have $(N(H)\cap N(L_i)\cap N(L_{i-1}))/H=eH\cup b_1H\cup b_2H\cup b_3H\cup b_4H\cup b_1b_4H\cup b_2b_4H\cup b_3b_4H$, where $b_3=b_2b_1$, $b_1=\mathrm{diag}(1, 1, j)$, $b_2=\mathrm{diag}(C, 1)$, $C=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $b_4=\mathrm{diag}(j, j, 1)$. It is clear that $b_1, b_2, b_2b_1\in K_i$ and $b_4, b_1b_4, b_2b_4, b_3b_4\notin K_i$. Since the action of K_i on S is given by $(\mathrm{Ad}_{U(2)}-\theta)\otimes\theta\oplus\theta\otimes\mathrm{Ad}_{Sp(1)}$, x is identified with the matrix X; $X=\mathrm{diag}(d_1 i, -d_1 i, d_2 i)$ where $d_1, d_2\in\boldsymbol{R}$. When $a=b_1, b_2$ and b_3, we have the following relations for $g=(g_1, g_2)\in K_i=U(2)\times Sp(1)$: $f_{b_1}(gx)=gb_1^{-1}Xb_1g^{-1}=(g_1\,\mathrm{diag}\,(d_1 i, -d_1 i)g_1^{-1}, -g_2(d_2 i)g_2^{-1})$, $f_{b_2}(gx)=(-g_1\,\mathrm{diag}\,(d_1 i, -d_1 i)g_1^{-1}, g_2(d_2 i)g_2^{-1})$ and $f_{b_3}(gx)=-gx$. Thus the map $f_a|\partial S$ is linear, and hence, it has an extension to $p^{-1}(A_i)$. Next put $a=b_4$. Then we have $b_4gb_4^{-1}=(\bar{g}_1, g_2)$ for all $g=(g_1, g_2)\in K_i=U(2)\times Sp(1)$. Thus the above map h is linear (see the case of 5-c). Thirdly we put $a=b_1b_4$. Then it is clear that $f_{b_1b_4}(gx)=f_{b_1}\circ f_{b_4}(gx)$. Since f_{b_1} and f_{b_4} have extensions, $f_{b_1b_4}$ also has an extension to $p^{-1}(A_i)$. Similarly, it is seen that $f_{b_2b_4}$ and $f_{b_3b_4}$ have extensions to $p^{-1}(A_i)$.

Case of 3-a. In this case, we have $(N(H)\cap N(L_i)\cap N(L_{i-1}))/H=eH\cup b_1H\cup b_2H\cup b_3H\cup b_4H\cup b_1b_4H\cup b_2b_4H\cup b_3b_4H$, where $b_3=b_1b_2$, $b_1=\mathrm{diag}(1, 1, 1, 1,$

$1, -1, -1)$, $b_2=\mathrm{diag}(C, 1, 1, 1)$, $C=\left[\begin{array}{c|c} 0 & \begin{matrix}1&0\\0&1\end{matrix} \\ \hline \begin{matrix}1&0\\0&1\end{matrix} & 0 \end{array}\right]$ and $b_4=\mathrm{diag}(1, -1, 1, -1, 1, 1, 1)$. Also it is seen that the action of K_i on S is given by $(\mathrm{Ad}_{U(2)}-\theta)\otimes\theta\oplus\theta\otimes\rho_3$. The rest of the proof is similar to that of the case of 5-a. We omit the detail. Q.E.D.

From Lemmas 4.4 and 4.5, we get the following proposition.

PROPOSITION 4.6. *Suppose that $N(H)/H$ is finite. Then all G-homology spheres with the same orbit datum are equivariantly diffeomorphic.*

PROOF. Fix one orbit datum. And denote by $(a_1, a_2, \cdots, a_c)$ the G-homology sphere such that the attaching diffeomorphism of $\partial p^{-1}(A_i)$ is given by f_{a_i} for each i. Put $M_1=(a_1, a_2, \cdots, a_c)$ and $M_2=(e, e, \cdots, e)$ where e indicates the identity element. Since M_1 and M_2 have the same orbit datum, $M_1-(\cup_{i=1}^{c}\mathrm{Int}(p^{-1}(A_i)))$ is identified with $M_2-(\cup_{i=1}^{c}\mathrm{Int}(p^{-1}(A_i)))$. Then by an extension ψ_{a_i} of f_{a_i} to $p^{-1}(A_i)$, we can construct a map ψ of M_2 to M_1 as follows:

$$\psi|\Big(M_2-\Big(\bigcup_{i=1}^{c}\mathrm{Int}\big(p^{-1}(A_i)\big)\Big)\Big)=\text{the identity map}\,, \quad \psi|p^{-1}(A_i)=\psi_{a_i}\,.$$

Clearly ψ is an equivariant diffeomorphism of M_2 to M_1. Q.E.D.

Next we consider the case that $N(H)/H$ is not finite, namely G-homology spheres modelled on $(\Lambda^2\mu_5)_R\oplus\theta$, $(\Lambda^2\mu_7)_R$ or $(\mu_4)_R\oplus\phi_2\oplus\theta$ where $\phi_2^c=\Lambda^2\mu_4$.

Let Φ be the equivariant diffeomorphism of $(G/H\times I)/\sim$ to $\partial p^{-1}(A_i)$ in the proof of Lemma 4.2, where $I=[0, 1]$. Then every attaching diffeomorphism f of $\partial p^{-1}(A_i)$ uniquely determines an arc $s\colon I\to N(H)/H$ by $f\circ\Phi([eH, t])=\Phi([s(t), t])$. Such an arc s satisfies the conditions: $s(0)\in(N(H)\cap N(L_{i-1}))/H$ and $s(1)\in(N(H)\cap N(L_i))/H$. Moreover, by the connectedness of I, $s(t)$ is in N_k/H for some connected component N_k of $N(H)$ such that $N_k\cap N(L_{i-1})\neq\varnothing$ and $N_k\cap N(L_i)\neq\varnothing$. An arc s satisfying the above conditions will be called a *cross-sectioning arc*. Conversely, a cross-sectioning arc s uniquely determines an equivariant homeomorphism f of $\partial p^{-1}(A_i)$ by $f\circ\Phi([eH, t])=\Phi([s(t), t])$.

LEMMA 4.7. *Suppose that an $SU(7)$-homology sphere M is modelled on $(\Lambda^2\mu_7)_R$. Then every attaching diffeomorphism of $\partial p^{-1}(A_i)$ can be extended to $p^{-1}(A_i)$.*

PROOF. Put $H=SU(2)^3$ (see Proposition 2.3). Then it is easily verified that $N(SU(2)^3)$ has six connected components N_k $(0\leqq k\leqq 5)$ and that the identity component $N_0=(N(SU(2)^3))^0$ is $S(U(2)^3\times U(1))$. To prove this lemma, it is sufficient to study the following three cases (namely, the cases of 10-a, b and c in Table B), by virtue of Lemma 4.3.

Case 1. Let (K_i, L_i, L_{i-1}) be $(Sp(3), Sp(2)\times Sp(1), Sp(1)\times Sp(2))$. If $N_k\cap N(L_{i-1})\neq\emptyset$ and $N_k\cap N(L_i)\neq\emptyset$, then it is shown that $N_k=N_0$. And every element X in $N_k\cap N(L_i)$ is written as diag(A, B, C, D), where A, B and C lie in $U(2)$, D in $U(1)$ and $\det A=\det B$. Similarly, if X is an element in $N_0\cap N(L_{i-1})$, then $\det B=\det C$ holds. Therefore we have the following diffeomorphisms: $N_0/H\sim S^1\times S^1\times S^1$, $(N_0\cap N(L_i))/H\sim\Delta(S^1\times S^1)\times S^1$, $(N_0\cap N(L_{i-1}))/H\sim S^1\times\Delta(S^1\times S^1)$, where $\Delta(S^1\times S^1)=\{(x, x)|x\in S^1\}\subset S^1\times S^1$. And hence a cross-sectioning arc $s\colon I\to N_0/H$ which is given by each attaching diffeomorphism f of $\partial p^{-1}(A_i)$ can be regarded as the map s below;

$$s(t)=(s_1(t), s_2(t), s_3(t))\in S^1\times S^1\times S^1$$

such that

$$\left.\begin{array}{l} s_1(1)=s_2(1)\\ s_2(0)=s_3(0)\end{array}\right\} \quad (\mathrm{A})\,.$$

First consider the map $f_1\colon I\times I=p(\partial p^{-1}(A_i))\times I\to N_0/H$ defined by $f_1(t, u)=(s_1(t), s_2(t), s_3((1-u)t))$. Then we have $f_1(t, 0)=s(t)$, $f_1(t, 1)=(s_1(t), s_2(t), s_3(0))$. Put $s'(t)=f_1(t, 1)$. Next we define the map $f_2\colon I\times I\to N_0/H$ by $f_2(t, u)=(s_1'(t)\rho(t, u), s_2'(t), s_3'(t))$, where ρ is the map of $I\times I$ to S^1 given by $\arg\rho(t, u)=u(\arg(s_1'(t)^{-1}s_2'(t)))$. Then we have $f_2(t, 0)=s'(t)$ and $f_2(t, 1)=(s_2'(t), s_2'(t), s_3'(t))$. Put $s''(t)=f_2(t, 1)$. Finally we define the map $f_3\colon I\times I\to N_0/H$ by $f_3(t, u)=(s_1''((1-u)t), s_2''((1-u)t), s_3''(t))$. Then f_3 satisfies $f_3(t, 0)=s''(t)$ and $f_3(t, 1)=(s_2(0), s_2(0), s_2(0))$ $(\in(N_0\cap N(L_i)\cap N(L_{i-1})))$. Put $s'''(t)=f_3(t, 1)$. Since $(N_0\cap N(L_i)\cap N(L_{i-1}))/H$ is diffeomorphic to $\Delta(S^1\times S^1\times S^1)$, s''' is homotopic to s_0, where s_0 is given by $s_0(t)=eH$. And hence s is homotopic to s_0. We write this homotopy as $F(t, u)=F_u(t)$. The F_u satisfies the above condition (A) for each u. Thus F_u induces an equivariant homotopy between the given f and identity map of $\partial p^{-1}(A_i)$. That is, we have $\pi_0\mathbf{Homeo}_I^G(\partial p^{-1}(A_i))=1$. Hence, from Lemma 3.1 and the Theorem in 1.3, it follows that $\pi_0\mathbf{Diffeo}_I^G(\partial p^{-1}(A_i))=1$. Thus f can be extended to $p^{-1}(A_i)$.

Case 2. Let (K_i, L_i, L_{i-1}) be $(Sp(2)\times SU(3), SU(2)^2\times SU(3), Sp(2)\times Sp(1))$. If $N_k\cap N(L_i)\neq\emptyset$ and $N_k\cap N(L_{i-1})\neq\emptyset$, then it is verified that N_k

must be N_0 or $N_1=CN_0$ for $C=\mathrm{diag}\left(\left[\begin{array}{c|c} 0 & \begin{matrix}1&0\\0&1\end{matrix} \\ \hline \begin{matrix}1&0\\0&1\end{matrix} & 0\end{array}\right], 1, 1, 1\right)$. And we have $N_k\cap N(L_i)=N_k$, $(N_k\cap N(L_{i-1}))/H\sim\Delta(S^1\times S^1)\times S^1$ for $k=0, 1$. Thus, by the way similar to that in Case 1, we see that if $k=0$ (resp. $k=1$), then every attaching diffeomorphism of $\partial p^{-1}(A_i)$ is equivariantly isotopic to the identity map (resp. the map f_C). For the definition of the map f_C, see the first paragraph following Lemma 4.1. Since $C\in K_i$, it is verified that f_C can be extended to $p^{-1}(A_i)$ by the same way as in Lemma 4.5. Hence we also deduce the required result for this case.

Case 3. Let (K_i, L_i, L_{i-1}) be $(SU(2)\times SU(5), Sp(1)\times Sp(2), SU(2)^2\times SU(3))$. If $N_k\cap N(L_i)\neq\emptyset$ and $N_k\cap N(L_{i-1})\neq\emptyset$, then it is verified that $N_k=N_0$. And we see that every attaching diffeomorphism of $\partial p^{-1}(A_i)$ is equivariantly isotopic to the identity map. The proof is similar to that in Case 1. We omit the detail. Q.E.D.

Similarly we have the following lemma. The proof is omitted.

LEMMA 4.8. *Let G be $SU(5)$ or $SU(4)$. Suppose that a G-homology sphere M is modelled on $(\Lambda^2\mu_5)_R\oplus\theta$ or $(\mu_4)_R\oplus\phi_2\oplus\theta$, where $\phi_2^c=\Lambda^2\mu_4$. Then every attaching diffeomorphism of $\partial p^{-1}(A_i)$ can be extended to $p^{-1}(A_i)$.*

From Lemmas 4.7 and 4.8, we get the following proposition. The proof is omitted, since it is similar to that of Proposition 4.6.

PROPOSITION 4.9. *Suppose that $N(H)/H$ is not finite. Then all G-homology spheres with the same orbit datum are equivariantly diffeomorphic.*

§5. The proof of the Theorem.

In this section, we shall prove our theorem stated in the Introduction. Let G be one of the three groups, $SO(n)$ $(n\neq 2, 4)$, $SU(n)$ or $Sp(n)$. And let M be a G-homology sphere which satisfies the assumptions in our theorem. As is mentioned in §2, we may assume that M is a G-homology sphere with a smooth action of G which is not regular. By Propositions 4.6 and 4.9, we have only to prove that M has the same orbit datum as that of its linear model.

PROOF OF THEOREM. Let M and N be G-homology spheres with the same linear model. And let $(H, K_1, L_1, \cdots, K_c, L_c)$ and $(H', K_1', L_1', \cdots, K_c', L_c')$ be the orbit data of M and N, respectively. Now suppose that

$(H', K_1', L_1', \cdots, K_c', L_c')$ is equal to one of the followings:

(i) $(gHg^{-1}, gK_1g^{-1}, gL_1g^{-1}, \cdots, gK_cg^{-1}, gL_cg^{-1})$ for $g \in G$,

(ii) $(H, K_j, L_j, K_{j+1}, L_{j+1}, \cdots, K_c, L_c, K_1, L_1, \cdots, K_{j-1}, L_{j-1})$ for $1 \leqq j \leqq c$,

(iii) $(H, K_1, L_c, K_c, L_{c-1}, \cdots, K_2, L_1)$.

Then it is clear that N is equivariantly diffeomorphic to M or $-M$. The manifold $-M$ is equivariantly diffeomorphic to M. So we shall say that $(H, K_1, L_1, \cdots, L_c)$ and $(H', K_1', L_1', \cdots, L_c')$ are *equivalent* if the latter is equal to (i), (ii) or (iii) above. We shall prove that every G-homology sphere in our theorem has the same orbit datum as that of its linear model up to equivalence.

Case 1. Suppose that M is modelled on a representation other than the representations $(\Lambda^2\mu_7)_R$, $S^2\rho_3$ and $(S^2\rho_3)\circ\pi$ of $SU(7)$, $SO(3)$ and $SU(2)$, respectively, where π is the projection of $SU(2)$ to $SO(3)$. Then, from the following relation of Euler characteristics,

$$\chi(M) = \chi(G/H) + \sum_{i=1}^{c} \chi(G/K_i) - \sum_{i=1}^{c} \chi(G/L_i) \tag{5.1}$$

we can show that M has the same orbit detum as that of its linear model up to equivalence. We only give here the actual proof for M modelled on the representation ϕ of $Sp(4)$ where $\phi^c = \Lambda^2(\nu_4)_C - \theta$. We omit the proofs for the other cases, since they are given similarly.

Suppose that M is modelled on the representation ϕ above. By Proposition 2.3, we have $H \in (Sp(1)^4)$ and $L_i \in (Sp(1)^2 \times Sp(2))$. And we have $K_i \in (Sp(2)^2)$ or $K_i \in (Sp(1) \times Sp(3))$. Since $Sp(1)^4$ is a subgroup of $Sp(4)$ with the maximal rank, the Euler characteristic $\chi(Sp(4)/Sp(1)^4)$ equals $|W(Sp(4))|/|W(Sp(1)^4)| = 2$, where $|W(G)|$ is the order of the Weyl group of G. Similarly we have $\chi(Sp(4)/(Sp(1)^2 \times Sp(2))) = 12$, $\chi(Sp(4)/Sp(2)^2) = 6$ and $\chi(Sp(4)/(Sp(1) \times Sp(3))) = 4$. Let c_1 (resp. c_2) be the number of isolated singular orbits whose types are $Sp(4)/Sp(2)^2$ (resp. $Sp(4)/(Sp(1) \times Sp(3))$). Then, by (5.1), we have $c_1 = 1$ and $c_2 = 2$ (we remark that $\chi(M) = 2$, since the dimension of M is even). Thus, from Lemma 4.3, we can deduce that the orbit datum of M is uniquely determined as $(Sp(1)^4, Sp(1) \times Sp(3), Sp(1) \times Sp(2) \times Sp(1), Sp(3) \times Sp(1), Sp(2) \times Sp(1)^2, Sp(2)^2, Sp(1)^2 \times Sp(2))$ up to equivalence. This shows that M has the same orbit datum as that of the linear model up to equivalence.

Unfortunately, for M modelled on $(\Lambda^2\mu_7)_R$, $S^2\rho_3$ or $(S^2\rho_3)\circ\pi$, the equation (5.1) is useless. In fact, if M is modelled on $(\Lambda^2\mu_7)_R$, then both Euler characteristic of M and that of each orbit are zero. And if M is modelled on $S^2\rho_3$ or $(S^2\rho_3)\circ\pi$, then the Euler characteristics of M and the principal orbit are zero, and that of each singular orbit is one. So, for these cases, we shall prove our theorem by computing directly the homology of M.

Case 2. Suppose that M is modelled on $S^2\rho_3$. By Hudson [11], $H_2(M; \boldsymbol{Z}) \neq 0$ unless the number of the isolated singular orbits is two. Thus the orbit datum of M must be $(Z_2^2, SO(3), N_1, SO(3), N_2)$ where N_1, $N_2 \in (O(2))$ and $N_1 \neq N_2$. This orbit datum is the same as that of the linear model up to equivalence.

Case 3. Suppose that M is modelled on $(S^2\rho_3)\circ\pi$. The result in Case 2 naturally induces that M has the same orbit datum as that of the linear model up to equivalence.

Case 4. Suppose that M is modelled on $(\Lambda^2\mu_7)_R$. By Proposition 2.3, we have $H \in (SU(2)^3)$. And K_i is conjugate to $Sp(3)$, $Sp(2)\times SU(3)$ or $SU(2)\times SU(5)$, and L_i is conjugate to $Sp(2)\times Sp(1)$ or $SU(2)^2\times SU(3)$.

Now we consider the following three fibre bundles:

$$S^3 = SU(2) \longrightarrow SU(7)/SU(2)^2 \xrightarrow{p_1} SU(7)/SU(2)^3 ,$$

$$S^5 = SU(4)/Sp(2) \longrightarrow SU(7)/(Sp(2)\times Sp(1)) \xrightarrow{p_2} SU(7)/(SU(4)\times SU(2)) ,$$

$$S^5 = SU(3)/SU(2) \longrightarrow SU(7)/SU(2)^3 \xrightarrow{p_3} SU(7)/(SU(2)^2\times SU(3)) .$$

Then, by the Gysin sequences of the bundles p_1, p_2, p_3 we have

$$H_4(SU(7)/SU(2)^3; \boldsymbol{R}) = \boldsymbol{R}\oplus\boldsymbol{R},\ H_4(SU(7)/(Sp(2)\times Sp(1)); \boldsymbol{R}) = \boldsymbol{R} \quad \text{and}$$
$$H_4(SU(7)/(SU(2)^3\times SU(3)); \boldsymbol{R}) = \boldsymbol{R}\oplus\boldsymbol{R} .$$

Next let $\nu(G/K_i)$ be the normal vector bundle of an isolated singular orbit G/K_i in M, and let $S\nu(G/K_i)$ be the associated sphere bundle (we use the same notations for the total spaces of these bundles). Since the dimension of M is 41 and the codimension of G/K_i is larger than 10 for every K_i, by the Gysin sequence of the bundle $S\nu(G/K_i)\to G/K_i$, we have

$$H_q(S\nu(G/K_i); \boldsymbol{R}) \cong H_q(\nu(G/K_i); \boldsymbol{R}) \cong H_q(G/K_i; \boldsymbol{R}) \quad \text{for} \quad q=4, 5 .$$

Put $Y = M - \cup_{i=1}^{c}\mathrm{Int}(\nu(G/K_i)) = M - \cup_{i=1}^{c}\mathrm{Int}(p^{-1}(A_i))$. Then, by the Mayer-Vietoris exact sequence, we have

$$\bigoplus_{i=1}^{c} H_q(S\nu(G/K_i); \boldsymbol{R}) \cong \bigoplus_{i=1}^{c} H_q(\nu(G/K_i); \boldsymbol{R}) \oplus H_q(Y; \boldsymbol{R})$$

for $0<q<\dim M-1=40$. From the two isomorphisms above, it follows that $H_4(Y; \boldsymbol{R}) = H_5(Y; \boldsymbol{R}) = 0$. Moreover, regarding Y as $(\cup_{i=1}^{c} p^{-1}(B_i)) \cup (G/H\times X)$ (see Fig. 1 in § 3), we have the isomorphism

$$\underbrace{H_4(G/H\times I;\ \boldsymbol{R})\oplus\cdots\oplus H_4(G/H\times I;\ \boldsymbol{R})}_{c}$$
$$\cong\Big(\bigoplus_{i=1}^{c} H_4(p^{-1}(B_i);\ \boldsymbol{R})\Big)\oplus H_4(G/H\times X;\ \boldsymbol{R})\,.$$

And hence we have

$$\underbrace{H_4(G/H\times I;\ \boldsymbol{R})\oplus\cdots\oplus H_4(G/H\times I;\ \boldsymbol{R})}_{c}$$
$$\cong\Big(\bigoplus_{i=1}^{c} H_4(G/L_i;\ \boldsymbol{R})\Big)\oplus H_4(G/H;\ \boldsymbol{R})\,.$$

In the orbit datum of M, let m (resp. n) be the number of L_i's such that $L_i\in(Sp(2)\times Sp(1))$ (resp. $L_i\in(SU(2)^2\times SU(3))$). Then the above isomorphism of homologies shows that $m+2n+2=2(m+n)$, namely, $m=2$. On the other hand, from Lemma 4.3, each triple (L_{i-1}, K_i, L_i) in the orbit datum $(H, K_1, L_1, \cdots, K_c, L_c)$ must be simultaneously conjugate to $(Sp(1)\times Sp(2), Sp(3), Sp(2)\times Sp(1))$, $(SU(2)^2\times SU(3), SU(2)\times SU(5), Sp(1)\times Sp(2))$ or $(Sp(2)\times Sp(1), Sp(2)\times SU(3), SU(2)^2\times SU(3))$. Moreover, from the definition of a linear model, each of these three triples must appear at least one time in the orbit datum. Thus $m=2$ implies $n=1$. And hence, from Lemma 4.3, it is deduced that the orbit datum of M is uniquely determined as $(SU(2)^3, Sp(3), Sp(2)\times Sp(1), Sp(2)\times SU(3), SU(2)^2\times SU(3), SU(2)\times SU(5), Sp(1)\times Sp(2))$ up to equivalence. This shows that M has the same orbit datum as that of the linear model up to equivalence. Q.E.D.

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