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Appell's Hypergeometric Function F_2 and Periods of Certain Elliptic K3 Surfaces

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Introduction

In 1880 Appell introduced four types of hypergeometric functions F_1, F_2, F_3 and F_4 of two variables. These are generalizations of the Gauss hypergeometric function $F(\alpha, \beta, \gamma, x)$. There are several generalizations of the elliptic modular function $\lambda(\tau)$ or H. A. Schwarz's theory [14] using Appell's F_1 (see E. Picard [8, 9], T. Terada [17], P. Deligne and G. D. Mostow [2], H. Shiga [12, 13]). But there are no remarkable generalizations using F_2, F_3 and F_4.

In this paper we shall investigate an automorphic function of two variables derived from $F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y)$ with $\alpha=\beta=\beta'=1/2$ and $\gamma=\gamma'=1$. To make the situation clear, let us recall what $\lambda(\tau)$ is. Consider the family $\mathscr{F}_0$ of the following elliptic curves $C(\lambda)$:

$$C(\lambda):\ w^2=u(u-1)(u-\lambda)\ ,\quad \lambda\in \boldsymbol{P}_1(\boldsymbol{C})-\{0, 1, \infty\}\ .$$

Let $\{\gamma_1, \gamma_2\}$ be a basis of $H_1(C(\lambda), \boldsymbol{Z})$ and assume that the intersection multiplicity $\gamma_1\cdot\gamma_2=-1$. And let ω be a holomorphic 1-form on $C(\lambda)$. Then the periods $\eta_i=\int_{\gamma_i}\omega$ $(i=1, 2)$ satisfy the following differential equation:

$$\lambda(1-\lambda)\frac{d^2z}{d\lambda^2}+(1-2\lambda)\frac{dz}{d\lambda}-\frac{1}{4}z=0\ .$$

This is the Gauss differential equation with $\alpha=\beta=1/2$ and $\gamma=1$. For the family $\mathscr{F}_0$, we define the period map τ on the parameter space $\boldsymbol{P}_1-\{0, 1, \infty\}$ by $\tau(\lambda)=\eta_1(\lambda)/\eta_2(\lambda)$. Then we have the following:

(1) *The image of τ is contained in upper half plane H.*

(2) *The inverse map $\lambda=\lambda(\tau)$ of τ is a single-valued holomorphic function on H mapped to $\boldsymbol{P}_1-\{0, 1, \infty\}$, and it is an automorphic function*

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relative to the modular group $\Gamma(2)$ *which is the principal congruence subgroup of level* 2.

(3) *The map* λ *induces a biholomorphic equivalence between* $(H/\Gamma(2))^*$ *and* $P_1(C)$, *where* $(H/\Gamma(2))^*$ *denotes the compactification of the space* $H/\Gamma(2)$ *which is obtained by attaching three cusp points* $\{0, 1, \infty\}$.

We shall show, using some properties of the period map for a family of certain elliptic K3 surfaces, the properties similar to the above (1), (2) and (3) for $F_2(1/2, 1/2, 1/2, 1, 1, x, y)$.

Now, we sketch our method. The function $F_2(1/2, 1/2, 1/2, 1, 1, x, y)$ is represented by the following double integral:

$$F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, x, y\right)=\frac{1}{\pi^2}\int_0^1\int_0^1 \frac{dudv}{\sqrt{uv(1-u)(1-v)(1-xu-yv)}} .$$

So we consider the following surface:

$$w^2=uv(1-u)(1-v)(1-xu-yv) , \tag{0.1}$$

and the 2-form:

$$\varphi=\frac{du\wedge dv}{\sqrt{uv(1-u)(1-v)(1-xu-yv)}} ; \tag{0.2}$$

where the parameters (x, y) move in the domain Λ:

$$\Lambda=\{(x, y)\in C^2: xy(1-x)(1-y)(1-x-y)\neq 0\} ,$$

(see §1, (1.5), (1.5′) and Figure 1.1).

We compactify the surface (0.1) in a certain fibre space and denote it by $S(x, y)$. The surface $S(x, y)$ has 11 normal two-dimensional singularities: one of them is of type A_3 and the others are of type A_1. Let $\tilde{S}(x, y)$ be the minimal nonsingular model of $S(x, y)$, let $\mu: \tilde{S}(x, y)\rightarrow S(x, y)$ be the resolution map and put $\psi=\mu^*\varphi$. The surface $\tilde{S}(\lambda)$ $(\lambda=(x, y)\in\Lambda)$ is an elliptic K3 surface with 5 singular fibres of type I_0^*, I_0^*, I_2, I_2, I_2^*; and the 2-form ψ is a non-vanishing holomorphic 2-form on $\tilde{S}(x, y)$ (see §2, Propositions 2.1, 2.2). Since $H_2(\tilde{S}(\lambda), Z)$ is a free Z-module of rank 22, we have a basis $\{\Gamma_1(\lambda), \cdots, \Gamma_{22}(\lambda)\}$ of $H_2(\tilde{S}(\lambda), Z)$. And we can always take eighteen of them as algebraic cycles, so let us say that they are $\Gamma_5(\lambda), \cdots, \Gamma_{22}(\lambda)$. Therefore if we put $\eta_i(\lambda)=\int_{\Gamma_i(\lambda)}\psi$ $(i=1, \cdots, 22)$, then we have $\eta_i(\lambda)\equiv 0$ $(i=5, \cdots, 22)$. Hence we define the period map Φ_1 for $\mathscr{F}=\{\tilde{S}(\lambda): \lambda\in\Lambda\}$ by

$$\Phi_1: \Lambda\ni\lambda\longmapsto(\eta_1(\lambda): \eta_2(\lambda): \eta_3(\lambda): \eta_4(\lambda))\in P_3(C) .$$

In order to describe the image of the period map Φ_1, we change the coordinates by the following formula:

$$(\eta_1, \cdots, \eta_4)=(\eta'_1, \cdots, \eta'_4)P ,$$

where P is the regular matrix given by (4.11). We consider the quotient space $\Lambda/\sim$ of the parameter space Λ, where the equivalent relation $\sim$ is defined by the condition $\tilde{S}(\lambda)\cong_e\tilde{S}(\lambda')$ which is an isomorphism as elliptic surfaces (see (5.5), (5.6)).

Then we investigate the following "exact" period map

$$\Phi: \Lambda/\sim \ni \lambda \longmapsto \left(\frac{\eta'_1(\lambda)}{\eta'_2(\lambda)}, \frac{\eta'_4(\lambda)}{\eta'_2(\lambda)}, \frac{\eta'_3(\lambda)}{\eta'_2(\lambda)}\right) \in \boldsymbol{C}^3 .$$

But, in order to study the inverse map of Φ we must extend the domain $\Lambda/\sim$ to $\Lambda_0/\sim$ (see § 6, (6.2)).

The following are our main results.

(1°) *The image of* Φ *is contained in the Cartesian product space* $H\times H$ *of the upper half plane* H (Theorem 4.1).

(2°) *The inverse map* Ψ *of* Φ *is a single-valued holomorphic map on* $H\times H$, *and it is automorphic relative to the semi-direct product group* $\Gamma=\langle\iota\rangle\ltimes\Gamma_{1,2}\times\Gamma_{1,2}$, *where* $\langle\iota\rangle$ *is the group generated by the involution* $\iota: (z_1, z_2)\mapsto(z_2, z_1)$ *and* $\Gamma_{1,2}$ *is the modular group generated by two modular transformations* $z\mapsto z+2$ *and* $z\mapsto -1/z$, *i.e.*,

$$\Gamma_{1,2}=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2, \boldsymbol{Z})\colon ab\equiv 0,\ cd\equiv 0 \pmod 2\right\}\Big/ \pm I \quad \text{(Theorem 5.1)} .$$

(3°) *The map* Ψ *induces a biholomorphic equivalence between* $(H\times H/\Gamma)^*$ *and* $(\Lambda_0/\sim)^*\cong \boldsymbol{P}_2(\boldsymbol{C})$ (Theorem 6.1), *where* $(\ \ )^*$ *is a certain compactification defined in* § 6 (see (6.4), (6.7)).

REMARK. On the boundary of $(\Lambda_0/\sim)^*$, $\tilde{S}(\lambda)$ is not a K3 surface but is in general a rational elliptic surface with singular fibres I_0^*, I_0^*. If we restrict the period map there, the image of Φ is isomorphic to the upper half plain H, and its inverse is given by the lambda function which is an elliptic modular function (see Table 6.1 and Appendix).

We wish to find out a useful modular function of several variables in some way.

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CONTENTS

§1. Appell's hypergeometric function F_2.

We quote from T. Kimura [3] some results about F_2. $F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y)$ is defined by the following hypergeometric series of two variables:

$$(1.1)\qquad F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y)=\sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(1, m)(1, n)(\gamma, m)(\gamma', n)} x^m y^n ,$$

where $(a, k):=a(a+1)\cdots(a+k-1)$ for $k=1, 2, \cdots$; $(a, 0):=1$ for $a\neq 0$.

We can see that if the parameters $\alpha, \beta, \beta', \gamma, \gamma'$ are neither 0 nor negative integers, then F_2 is not a polynomial in x, y and the domain of convergence is $\{(x, y)\in \boldsymbol{C}^2: |x|+|y|<1\}$. And if the parameters satisfy the conditions $\operatorname{Re}\beta>0$, $\operatorname{Re}\beta'>0$, $\operatorname{Re}(\gamma-\beta)>0$ and $\operatorname{Re}(\gamma'-\beta')>0$, F_2 has an Euler integral representation:

$$(1.2)\qquad \begin{aligned} F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y) &= \Pi(\beta, \beta', \gamma, \gamma') \int_0^1\int_0^1 u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1} \\ &\quad \times (1-v)^{\gamma'-\beta'-1}(1-xu-yv)^{-\alpha}du dv , \end{aligned}$$

where $\Pi(\beta, \beta', \gamma, \gamma')=\Gamma(\gamma)\Gamma(\gamma')/(\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta'))$ and Γ indicates the gamma function.

Hence $F_2(1/2, 1/2, 1/2, 1, 1, x, y)$ is represented by the following double integral:

$$(1.3)\qquad F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, x, y\right)=\frac{1}{\pi^2}\int_0^1\int_0^1 \frac{du dv}{\sqrt{uv(1-u)(1-v)(1-xu-yv)}} .$$

This satisfies the following Appell's hypergeometric differential equation:

$$
(1.4)\qquad \begin{cases} x(1-x)\dfrac{\partial^2 z}{\partial x^2} - xy\dfrac{\partial^2 z}{\partial x \partial y} + (1-2y)\dfrac{\partial z}{\partial x} - \dfrac{1}{2}y\dfrac{\partial z}{\partial y} - \dfrac{1}{4}z = 0 \\ y(1-y)\dfrac{\partial^2 z}{\partial y^2} - xy\dfrac{\partial^2 z}{\partial x \partial y} + (1-2x)\dfrac{\partial z}{\partial y} - \dfrac{1}{2}x\dfrac{\partial z}{\partial x} - \dfrac{1}{4}z = 0 . \end{cases}
$$

The dimension of the solution space of (1.4) is four and solutions are in general multi-valued analytic functions in the following domain Λ:

$$
(1.5)\qquad \Lambda = \{(x, y) \in \boldsymbol{C}^2 : xy(1-x)(1-y)(1-x-y) \neq 0\} .
$$

From here on we study the following surfaces:

$$
(1.6)\qquad w^2 = uv(1-u)(1-v)(1-xu-yv) ,
$$

and the following 2-form:

$$
(1.7)\qquad \varphi = \frac{du \wedge dv}{\sqrt{uv(1-u)(1-v)(1-xu-yv)}} ;
$$

where parameters (x, y) move in the domain Λ. But, we regard the space Λ as the following subset of $\boldsymbol{P}_2(\boldsymbol{C})$:

$$
(1.5')\qquad \Lambda = \{(\xi_0 : \xi_1 : \xi_2) : \xi_0\xi_1\xi_2(\xi_0-\xi_1)(\xi_0-\xi_2)(\xi_0-\xi_1-\xi_2) \neq 0\} ,
$$

and regard the surfaces (1.6) as follows:

$$
(1.6')\qquad w^2 = uv(1-u)(1-v)(\xi_0-\xi_1 u-\xi_2 v) ;
$$

where $(\xi_0 : \xi_1 : \xi_2)$ are homogeneous coordinates of $\boldsymbol{P}_2(\boldsymbol{C})$ and we set $(x, y) = (\xi_1/\xi_0, \xi_2/\xi_0)$. Moreover, note that Λ is denoted as follows

$$
(1.7')\qquad \Lambda = \boldsymbol{P}_2(\boldsymbol{C}) - \bigcup_{k=0}^{5} L_k ,
$$

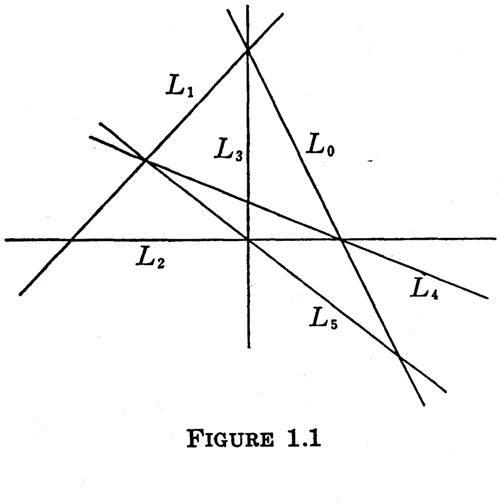


FIGURE 1.1

where $L_i=\{\xi_i=0\}$ $(i=0, 1, 2)$, $L_{2+j}=\{\xi_0-\xi_j=0\}$ $(j=1, 2)$, $L_5=\{\xi_0-\xi_1-\xi_2=0\}$ (see Figure 1.1).

§2. Minimal nonsingular model of $S(\lambda)$.

We shall construct a certain compactification of the surface (1.6). For two manifolds $W_0=\boldsymbol{P}_2(\boldsymbol{C})\times\boldsymbol{C}_0$, $W_1=\boldsymbol{P}_2(\boldsymbol{C})\times\boldsymbol{C}_1$, where $\boldsymbol{C}_0$, $\boldsymbol{C}_1$ are complex number planes $\boldsymbol{C}$, we form their union $W=W_0\cup W_1$ by identifying $(\zeta_0:\zeta_1:\zeta_2)\times u\in W_0$ with $(\zeta_0':\zeta_1':\zeta_2')\times u'\in W_1$ if and only if

$$\zeta_0=\zeta_0'\ ,\quad \zeta_1=\zeta_1'\ ,\quad \zeta_2=u^2\zeta_2'\ ,\quad uu'=1\ .$$

And we define

$$\Delta=\boldsymbol{C}_0\cup\boldsymbol{C}_1\ ,$$

where we identify $u\in\boldsymbol{C}_0$ with $u'\in\boldsymbol{C}_1$ if and only if $uu'=1$. By the projection from W onto Δ, W is a fibre bundle with the fibres $\boldsymbol{P}_2(\boldsymbol{C})$ over $\boldsymbol{P}_1(\boldsymbol{C})$. We define a compactification of the surface (1.6) as follows:

$$\text{(2.1)}\qquad \begin{cases}\zeta_0\zeta_2^2=u(1-u)\zeta_1(\zeta_0-\zeta_1)(\zeta_0-xu\zeta_0-y\zeta_1) & \text{in } W_0\ ,\\ \zeta_0'\zeta_2'^2=u'(u'-1)\zeta_1'(\zeta_0'-\zeta_1')(\zeta_0'u'-x\zeta_0'-y\zeta_1'u') & \text{in } W_1\ .\end{cases}$$

We denote the surface (2.1) by $S(\lambda)$ or $S(x, y)$, where we put $\lambda=(\xi_0\,\xi_1:\xi_2)$, $(x, y)=(\xi_1/\xi_0, \xi_2/\xi_0)$ and the parameters move in the domain Λ ((1.5), (1.5′)) as in §1.

Putting $v=\zeta_1/\zeta_0$, $w=\zeta_2/\zeta_0$, $v'=\zeta_1'/\zeta_0'$, $w'=\zeta_2'/\zeta_0'$ in (2.1), we have the following equations:

$$\text{(2.2)}\qquad \begin{cases}w^2=uv(1-u)(1-v)(1-xu-yv)\ ,\\ w'^2=u'v'(u'-1)(1-v')(u'-x-yu'v')\ .\end{cases}$$

We use the following notations in order to investigate the minimal nonsingular model $\tilde{S}=\tilde{S}(\lambda)$ of $S=S(\lambda)$:

$$\pi': S\longrightarrow\Delta \quad \text{projection}\ ,$$
$$\pi: \tilde{S}\longrightarrow\Delta \quad \text{projection}\ ,$$
$$u_1=0,\ u_2=1,\ u_3=\frac{1-y}{x},\ u_4=\frac{1}{x},\ u_5=\infty\ .$$

We can easily see that the fibre $\pi^{-1}(u)$ is a nonsingular elliptic curve for every u except u_i $(i=1, \cdots, 5)$. Hence the surface $\tilde{S}$ is an algebraic elliptic surface, and $\tilde{S}$ has the global holomorphic section $L=\{\zeta_1=\zeta_2=\zeta_1'=\zeta_2'=0\}$. That is, $\tilde{S}$ is a basic member. Following Kodaira [4], we describe types of singular fibres. The surface $\tilde{S}$ has 11 singular points P_{ij} ($\neq P_{14}, P_{24}$) shown in Figure 2.1 on the fibres $\pi'^{-1}(u_i)$ ($i=1, \cdots, 5$) in the hyperplane $\{w=w'=0\}$.

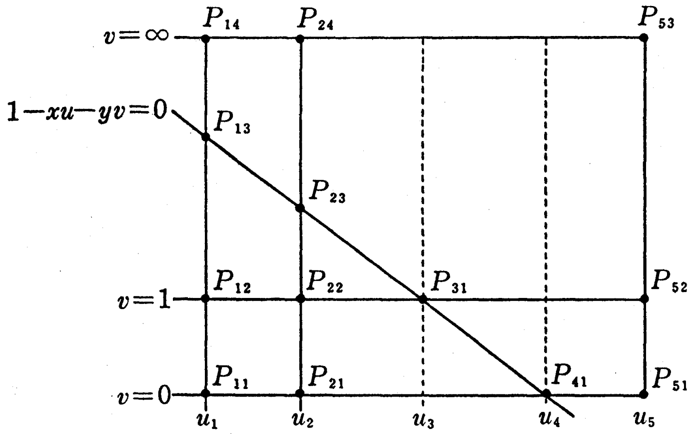


FIGURE 2.1

They are rational double points, and every point except P_{53} is of type A_1 and P_{53} is of type A_3. We carry out resolution of these singularities by blowing up along each curve $\pi'^{-1}(u_i)$ ($i=1, \cdots, 5$). Note that $P_{14}=(0:1:0)\times 0$ and $P_{24}(0:1:0)\times 1$ are not singular points, but if we put $u=0$, 1 in (2.1), rational curves $\Theta_{14}=\{\zeta_0=0, u=0\}$, $\Theta_{24}=\{\zeta_0=0, u=1\}$ occur and they meet $\pi'^{-1}(u_1)$, $\pi'^{-1}(u_2)$ transversely at P_{14}, P_{24} respectively. We obtain the following singular fibres $\pi^{-1}(u_i)$ ($i=1, \cdots, 5$):

$$\pi^{-1}(u_i)=2\Theta_{i0}+\Theta_{i1}+\Theta_{i2}+\Theta_{i3}+\Theta_{i4} \qquad (i=1, 2),$$

where Θ_{ij} ($i=1, 2$; $j=0, 1, \cdots, 4$) are nonsingular rational curves with $\Theta_{ij}^2=-2$ ($i=1, 2$; $j=0, 1, \cdots, 4$) and $\Theta_{i0}\cdot\Theta_{ik}=1$ ($i=1, 2$; $k=1, \cdots, 4$);

$$\pi^{-1}(u_i)=\Theta_{i0}+\Theta_{i1} \qquad (i=3, 4),$$

where Θ_{ij} ($i=3, 4$; $j=0, 1$) are nonsingular rational curves with $\Theta_{ij}^2=-2$ ($i=3, 4$; $j=0, 1$) and $\Theta_{i0}\cdot\Theta_{i1}=q_i+q_i'$ (q_i and q_i' indicate two different points) ($i=3, 4$);

$$\pi^{-1}(u_5)=2\Theta_{50}+\Theta_{51}+\Theta_{52}+2\Theta_{53}+2\Theta_{54}+\Theta_{55}+\Theta_{56},$$

where Θ_{5j} ($j=0, 1, \cdots, 6$) are nonsingular rational curves with $\Theta_{5j}^2=-2$ ($j=0, 1, \cdots, 6$) and $\Theta_{50}\cdot\Theta_{51}=\Theta_{50}\cdot\Theta_{52}=\Theta_{50}\cdot\Theta_{53}=\Theta_{53}\cdot\Theta_{54}=\Theta_{54}\cdot\Theta_{55}=\Theta_{54}\cdot\Theta_{56}=1$; where $\Theta\cdot\Theta'$ denotes the intersection number of two curves Θ and Θ', and Θ^2 denotes $\Theta\cdot\Theta$. Every component of each singular fibre does not have intersections excepting those aforementioned, and all those intersections are transverse.

Therefore $\pi^{-1}(u_1)$ and $\pi^{-1}(u_2)$ are singular fibres of type I_0^*, $\pi^{-1}(u_3)$ and $\pi^{-1}(u_4)$ are of type I_2 and $\pi^{-1}(u_5)$ is of type I_2^*. We note that each singular fibre has only one component, say Θ_{i1} $(i=1,\cdots,5)$, which intersects the section L.

Let $\tilde{S}=\tilde{S}(\lambda)$ be the elliptic surface obtained by the above resolution, then by the above argument, we obtain the following.

PROPOSITION 2.1. *The elliptic surface* $(\tilde{S}, \pi, \Delta)$ *is a basic member and it has five singular fibres of type* I_0^*, I_0^*, I_2, I_2 *and* I_2^*.

REMARK 2.1. From the equations (2.2), the functional invariant $\mathscr{J}$ of $\tilde{S}$ is represented by the following functions:

$$\begin{cases} \mathscr{J}(u)=\dfrac{4\{x^2u^2+(xy-2x)u+y^2-y+1\}^3}{27y^2(1-xu)^2(y-1+xu)^2}\,, \\ \mathscr{J}(u')=\dfrac{4\{(y^2-y+1)u'^2+(xy-2x)u'+x^2\}^3}{27y^2u'^2(u'-x)^2((y-1)u'+x)^2}\,. \end{cases}$$

Hence $\mathscr{J}$ is regular at points $u=0, 1$ and has poles of order 2 at $u=1/x$, $(1-y)/x$, ∞.

Next, let us show that $\tilde{S}$ is a K3 surface. By K3 surface, we mean a two-dimensional compact complex manifold with the canonical bundle $K=0$ and the first betti number $b_1=0$. Let $\mu\colon \tilde{S}\to S$ be the resolution map, and we define the 2-form ψ on $\tilde{S}$ by

$$\psi=\mu^*\varphi\,, \tag{2.3}$$

where $\varphi=(du\wedge dv)/w=-(du'\wedge dv')/w'$.

PROPOSITION 2.2. *The 2-form* ψ *is a non-vanishing holomorphic 2-form on* $\tilde{S}$ *and consequently* $\tilde{S}$ *is a* K3 *surface.*

PROOF. By elementary calculation, we can easily see that ψ is a non-vanishing holomorphic 2-form on $\tilde{S}$. Therefore the canonical bundle K of $\tilde{S}$ is trivial and we obtain $p_g=\dim H^0(\tilde{S}, \mathscr{O}(K))=1$. The Euler number $c_2=\chi(\tilde{S})$ of $\tilde{S}$ is

$$c_2=\chi(\tilde{S})=\sum_{i=1}^{5}\chi(\pi^{-1}(u_i))=6+6+2+2+8=24\,.$$

Moreover we have $c_1^2=0$ for elliptic surfaces. By the Noether formula:

$$c_1^2+c_2=12(p_g-q+1)$$

we obtain $q=0$. Hence we get $b_1=0$, consequently, $\tilde{S}$ is a K3 surface.

REMARK 2.2. We note that $\tilde{S}$ is the minimal nonsingular model of S from Proposition 2.2 and recall that twofold coverings of P_2 branched along a nonsingular curve of degree 6 are K3 surfaces.

§3. Monodromy of singular fibres and a basis of $H_2(\tilde{S}(\lambda), \boldsymbol{Z})$.

In this section we shall investigate the monodromy of the singular fibres of the elliptic surface $\tilde{S}(\lambda)$ and construct a basis of $H_2(\tilde{S}(\lambda), \boldsymbol{Z})$.

In §3 and §4, we use the following notation. Let $p, q_1, \cdots, q_r$ be fixed points on $\boldsymbol{P}_1(\boldsymbol{C})$. We denote by $\varepsilon(p, q_i)$ $(i=1, \cdots, r)$ the representative elements of $\pi_1(\boldsymbol{P}_1-\{q_1, \cdots, q_r\}, p)$ going around only q_i in the positive sense. And by the product $\gamma_1\gamma_2$ we mean the composite of two arcs γ_1 and γ_2 in this order.

(I) By Kodaira ([4] §9), the normal form of monodromy of singular fibres are given as the following table.

TABLE 3.1

type of singular fibres	I_0^*	I_2	I_2^*
normal form of monodromy matrix	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & -2\\ 0 & -1\end{pmatrix}$

But, in general, the monodromy representations are conjugate to the normal forms in $SL(2, \boldsymbol{Z})$. We fix parameters $(x, y)=(-1, -1)$ and consider the surface $\tilde{S}_0=\tilde{S}(-1, -1)$. The surfaces $\tilde{S}_0$ is represented, using the affine coordinates (u, v, w), as follows:

$$\tilde{S}_0:\ w^2=uv(1-u)(1-v)(1+u+v)\ .$$

We set

$$(3.1)\qquad \begin{cases} u_1=-2,\ u_2=-1,\ u_3=0,\ u_4=1,\ u_5=\infty\ , \\ \Delta'=\Delta-\{u_1, u_2, u_3, u_4, u_5\}\ . \end{cases}$$

The types of singular fibres of $\tilde{S}_0$ are given as follows:

$$(3.2)\qquad \begin{cases} \pi^{-1}(u_1),\ \pi^{-1}(u_2) \cdots\cdots I_2\ , \\ \pi^{-1}(u_3),\ \pi^{-1}(u_4) \cdots\cdots I_0^*\ , \\ \pi^{-1}(u_5) \cdots\cdots\cdots\cdots\cdots I_2^*\ . \end{cases}$$

We take a general point u_0 in Δ, say $u_0=-3/2$, and put $C=\pi^{-1}(u_0)$. Let us consider the projection from C onto v-sphere:

$$p: C \longrightarrow \boldsymbol{P}_1(\boldsymbol{C})\ ,$$

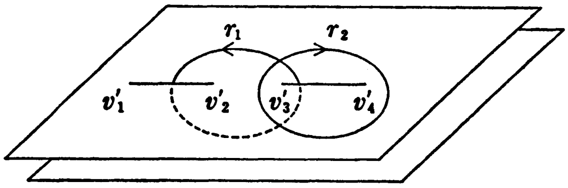
then C is a double covering over $P_1(C)$ branched at the four points $v_1=0$, $v_2=1/2$, $v_3=1$, $v_4=\infty$. Take a fixed point v_0 in v-sphere with $\mathrm{Im}\, v_0>0$. We choose a basis $\{\gamma_1, \gamma_2\}$ of $H_1(C, \boldsymbol{Z})$ such that

$$p(\gamma_1)=\varepsilon(v_0, v_2)\varepsilon(v_0, v_3) ,$$
$$p(\gamma_2)=\{\varepsilon(v_0, v_3)\varepsilon(v_0, v_4)\}^{-1} ,$$

and

$$\gamma_1\cdot\gamma_2=-1 ,$$

(see Figure 3.1).



(v_i' indicate the points on C with $p(v_i')=v_i$ $(i=1, 2, 3, 4)$)

FIGURE 3.1

Now, we put $\alpha_i=\varepsilon(u_0, u_i)$ $(i=1, \cdots, 5)$ and continue the above 1-cycles γ_1 and γ_2 analytically along the closed arcs α_i. Then α_i induces the monodromy transformation α_i^* of $H_1(C, \boldsymbol{Z})$. By elementary calculation (see Appendix), we obtain the following:

$$(3.3)\quad \alpha_1^*=\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix},\ \alpha_2^*=\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\ \alpha_3^*=\alpha_4^*=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ \alpha_5^*=\begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}.$$

Then it follows that

$$(3.4)\quad \alpha_1^*\alpha_2^*\alpha_3^*\alpha_4^*\alpha_5^*=1 .$$

The transformations $\{\alpha_i^*\}$ define the homological invariant of the elliptic surface $\tilde{S}_0$.

(II) In order to define a basis $H_2(\tilde{S}_0, \boldsymbol{Z})$, first we define a basis $\{G_1, \cdots, G_{22}\}$ over $\boldsymbol{Q}$. Since $\tilde{S}_0$ is a K3 surface, $H_2(\tilde{S}_0, \boldsymbol{Q})$ is a 22-dimensional vector space over $\boldsymbol{Q}$. We can choose 18 cycles of a basis of $H_2(\tilde{S}_0, \boldsymbol{Q})$ as algebraic cycles. Indeed, let $G_5, \cdots, G_{22}$ be such cycles, then it is sufficient to define them as follows:

$$(3.5)\quad \begin{aligned} &G_5=\Theta_{10},\ G_6=\Theta_{12},\ G_7=\Theta_{13},\ G_8=\Theta_{14},\ G_9=\Theta_{20},\ G_{10}=\Theta_{22} ,\\ &G_{11}=\Theta_{23},\ G_{12}=\Theta_{24},\ G_{13}=\Theta_{30},\ G_{14}=\Theta_{40},\ G_{15}=\Theta_{50},\ G_{16}=\Theta_{52} ,\\ &G_{17}=\Theta_{53},\ G_{18}=\Theta_{54},\ G_{19}=\Theta_{55},\ G_{20}=\Theta_{56},\ G_{21}=L ,\\ &G_{22}=C_u \ \text{(a general fibre)} . \end{aligned}$$

Let B be the intersection matrix defined by $G_5, \cdots, G_{22}$:

$$B=(G_i \cdot G_j)_{5\leq i,j\leq 22} .$$

Then it follows that $\det B \neq 0$.

Now, in order to define $G_1, \cdots, G_4$ we choose a point u^* in the lower half plane of Δ and take line segments l_i ($i=1, \cdots, 5$) connecting u_i and u^*. So far as the general point u_0 moves in $\Delta - \cup_{i=1}^5 l_i$, the basis $\{\gamma_1, \gamma_2\}$ is uniquely determined up to the homotopy equivalence. Hence if it is necessary we may take u_0 so that $\mathrm{Im}\, u_0 > 0$. We cotinue analytically the basis $\{\gamma_1, \gamma_2\}$ along an arc g in Δ', then we can consider the 1-cycles γ_1, γ_2 are transformed by α_i^* if their 1-cycles cross l_i along g in the positive sense. When we continue a 1-cycle γ on the general fibre $\pi^{-1}(u_0)$ analytically along an arc g on Δ' beginning at u_0, we get a 2-chain on $\tilde{S}_0$. If this 2-chain is a 2-cycle, we denote the 2-cycle by $\Gamma(\gamma, g)$.

Now, let us define closed arcs g_1, g_2, g_3 on Δ' as follows:

$$\text{(3.6)} \qquad \begin{cases} g_1 = \varepsilon(u_0, u_3)\varepsilon(u_0, u_4) , \\ g_2 = \varepsilon(u_0, u_2)\varepsilon(u_0, u_3) , \\ g_3 = \varepsilon(u_0, u_1)\varepsilon(u_0, u_4) . \end{cases}$$

The arcs g_1, g_2 and g_3 are homotopic to the arcs in Figure 3.2 respectively. We as well denote these arcs by g_1, g_2 and g_3 respectively.

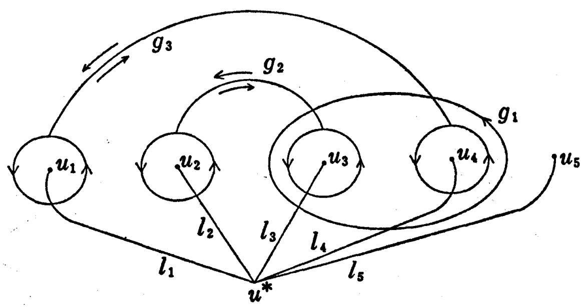


FIGURE 3.2

We first define 2-cycles G, G' as follows:

G: Continue the 1-cycle γ_1 along $\varepsilon(u_0, u_2)$ and continue the 1-cycle γ_2 along $\varepsilon(u_0, u_3)$,

G': Continue the 1-cycle $-\gamma_2$ along $\varepsilon(u_0, u_1)$ and continue the 1-cycle γ_1 along $\varepsilon(u_0, u_4)$.

REMARK 3.1. We can see that G and G' are well defined as 2-cycles by considering the local monodromy (3.3).

Now, we define 2-cycles G_1, G_2, G_3 and G_4 as follows:

$$(3.7)\qquad \begin{aligned} &G_1=\Gamma(\gamma_2, g_1^{-1}), && G_2=\Gamma(\gamma_1, g_1), \\ &G_3=G+G_2, && G_4=G'+G_1. \end{aligned}$$

Let A be the intersection matrix $(G_i\cdot G_j)_{1\le i,j\le 4}$. By elementary calculation, we get

$$(3.8)\qquad A=\begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}.$$

Let C be the intersection matrix $(G_i\cdot G_j)_{1\le i,j\le 22}$, then we have $C=A\oplus B$. Hence we have $\det C\neq 0$. This shows that $\{G_1, \cdots, G_{22}\}$ is a basis of $H_2(\tilde{S}_2, \boldsymbol{Q})$.

Next, in order to construct a basis of $H_2(\tilde{S}_0, \boldsymbol{Z})$, we take directed segments β_i $(i=1, 2, 3, 4)$ beginning at u_0 and ending at u_i (see Figure 3.3). We define the 2-cycles Γ_1, Γ_2, Γ_3, Γ_4 on $\tilde{S}_0$ as follows:

$$(3.9)\qquad \begin{aligned} &\Gamma_1:=\Gamma(\gamma_1, \beta_1^{-1}\beta_4), && \Gamma_2:=\Gamma(\gamma_2, \beta_2^{-1}\beta_3), \\ &\Gamma_3:=\Gamma(\gamma_1, \beta_4^{-1}\beta_3), && \Gamma_4:=\Gamma(\gamma_2, \beta_3^{-1}\beta_4). \end{aligned}$$

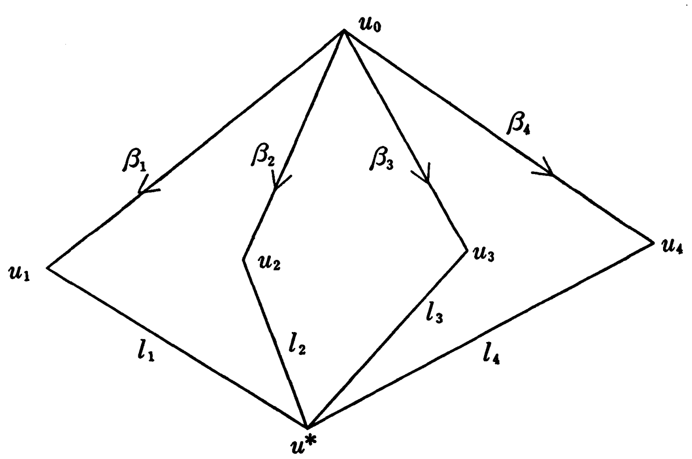


FIGURE 3.3

It is easily checked that Γ_i $(i=1, 2, 3, 4)$ are well-defined as 2-cycles.

The following holds for the 2-cycles G_i, Γ_j $(i, j=1, 2, 3, 4)$:

$$(3.10)\qquad G_i\cdot\Gamma_j=\delta_{ij} \qquad (i, j=1, 2, 3, 4),$$

where δ_{ij} indicates Kronecker's delta.

Now, let $\{\Gamma_5, \cdots, \Gamma_{22}\}$ be a $\boldsymbol{Z}$-basis of

$$\langle G_5, \cdots, G_{22}\rangle_{\boldsymbol{Q}} \cap H_2(\tilde{S}_0, \boldsymbol{Z}) ,$$

where the notation $\langle * \rangle_{\boldsymbol{Q}}$ indicates the subspace of $H_2(\tilde{S}_0, \boldsymbol{Q})$ generated by $*$. Then we obtain the following.

Proposition 3.1. *The system $\{\Gamma_1, \cdots, \Gamma_{22}\}$ defined in the above is a basis of $H_2(\tilde{S}_0, \boldsymbol{Z})$.*

Proof. Let Γ be any element of $H_2(\tilde{S}_0, \boldsymbol{Z})$, and we set

$$\Gamma' = \Gamma - \sum_{i=1}^{4} a_i \Gamma_i ,$$

where $a_i = \Gamma \cdot G_i$ $(i=1, 2, 3, 4)$.

From (3.10), we get

$$\Gamma' \cdot G_j = \Gamma \cdot G_j - \sum_{i=1}^{4} a_i \Gamma_i \cdot G_j = a_j - a_j = 0 \qquad (j=1, 2, 3, 4) .$$

Hence Γ' belongs to $\langle G_5, \cdots, G_{22}\rangle_{\boldsymbol{Q}} \cap H_2(\tilde{S}_0, \boldsymbol{Z})$, and this proves that Γ is represented by a $\boldsymbol{Z}$-linear combination of $\Gamma_1, \cdots, \Gamma_{22}$.

(III) Finally, we construct a basis of $H_2(\tilde{S}(\lambda), \boldsymbol{Z})$ for all $\lambda \in \Lambda$. We set

$$\mathscr{F} = \{\tilde{S}(\lambda) : \lambda \in \Lambda\} . \tag{3.11}$$

Since $\mathscr{F}$ is locally trivial as the fibre space over Λ, we can easily define bases $\{\Gamma_1(\lambda), \cdots, \Gamma_{22}(\lambda)\}$ and $\{G_1(\lambda), \cdots, G_{22}(\lambda)\}$ of $H_2(\tilde{S}(\lambda), \boldsymbol{Z})$ and $H_2(\tilde{S}(\lambda), \boldsymbol{Q})$ for $\{\Gamma_1, \cdots, \Gamma_{22}\}$ and $\{G_1, \cdots, G_{22}\}$, respectively. Here we note that the 2-cycles $\Gamma_5(\lambda), \cdots, \Gamma_{22}(\lambda)$ are algebraic 2-cycles and

$$\Gamma_i(\lambda) \cdot G_j(\lambda) = \delta_{ij} \qquad \text{for all } \lambda \in \Lambda \quad (i, j=1, 2, 3, 4) . \tag{3.12}$$

Moreover, let $A(\lambda)$ be the intersection matrix $(G_i(\lambda) \cdot G_j(\lambda))_{1 \le i, j \le 4}$, then we have

$$A(\lambda) = A \qquad \text{for all } \lambda \in \Lambda , \tag{3.13}$$

where A is the matrix defined by (3.8).

§4. Period map Φ and its image.

In §3 we defined the second homology basis $\{\Gamma_1(\lambda), \cdots, \Gamma_{22}(\lambda)\}$ on the K3 surface $\tilde{S}(\lambda)$. We define periods $\eta_i = \eta_i(\lambda)$ along the 2-cycles $\Gamma_i(\lambda)$ $(i=1, \cdots, 22)$ as follows:

$$\eta_i(\lambda)=\int_{\Gamma_i(\lambda)}\psi(\lambda) \qquad \text{for all } \lambda\in\Lambda \quad (i=1,\cdots,22)\,, \tag{4.1}$$

where $\psi=\psi(\lambda)$ is the holomorphic 2-form on $\tilde{S}(\lambda)$ defined in (2.3). Since the cycles $\Gamma_5(\lambda),\cdots,\Gamma_{22}(\lambda)$ are algebraic, we have the following:

$$\eta_i(\lambda)\equiv 0 \qquad (i=5,\cdots,22)\,. \tag{4.2}$$

Hence we can define the period map Φ_1 for $\mathscr{F}$ as follows:

$$\Phi_1\colon \Lambda\ni\lambda\longmapsto(\eta_1(\lambda):\eta_2(\lambda):\eta_3(\lambda):\eta_4(\lambda))\in \boldsymbol{P}_3(\boldsymbol{C})\,. \tag{4.3}$$

Now, let us cosider the Riemann-Hodge relations. Let $\{e_1(\lambda),\cdots,e_{22}(\lambda)\}$ be the dual basis of $H^2(\tilde{S}(\lambda),\boldsymbol{Z})$ to the basis $\{\Gamma_1(\lambda),\cdots,\Gamma_{22}(\lambda)\}$: namely, denoting by $\omega_j=\omega_j(\lambda)$ the d-closed 2-form corresponding to $e_j=e_j(\lambda)$ under the de Rham theorem, we have the following:

$$e_j(\Gamma_i(\lambda)):=\int_{\Gamma_i(\lambda)}\omega_j(\lambda)=\delta_{ij} \qquad (i,j=1,\cdots,22)\,. \tag{4.4}$$

We set the integers a_{ij} as follows:

$$a_{ij}=e_i\cdot e_j \qquad (i,j=1,\cdots,22)\,, \tag{4.5}$$

where $e_i\cdot e_j$ indicates the cup product of e_i and e_j. Then it follows that

$$a_{ij}=\int_{\tilde{S}(\lambda)}\omega_i\wedge\omega_j \qquad (i,j=1,\cdots,22)\,. \tag{4.6}$$

When we set $M=(a_{ij})_{1\le i,j\le 22}$, the Riemann-Hodge relations are given by the following:

$$\eta M{}^t\eta=0\,, \tag{4.7}$$

$$\eta M{}^t\bar{\eta}>0\,, \tag{4.8}$$

where $\eta=(\eta_1,\cdots,\eta_{22})$ (see Kodaira [5, 6]).

From (3.12), (4.4) and (4.5), we obtain

$$a_{ij}=G_i\cdot G_j \qquad (i,j=1,2,3,4)\,.$$

Thus from (4.2), (4.7) and (4.8), we get the following:

$$(\eta_1,\eta_2,\eta_3,\eta_4)A\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{pmatrix}=0\,, \tag{4.9}$$

$$(4.10)\qquad (\eta_1, \eta_2, \eta_3, \eta_4)A\begin{pmatrix}\bar{\eta}_1\\ \bar{\eta}_2\\ \bar{\eta}_3\\ \bar{\eta}_4\end{pmatrix}>0\ ,$$

where A is the matrix in (3.8).

Let Ω be the subset of $\boldsymbol{P}_3(\boldsymbol{C})$ defined by (4.9) and (4.10), then the image of Φ_1 is contained in Ω. Let us show that the image of the period map Φ_1 is contained in the space biholomorphic to the Cartesian product space $H\times H$ of the upper half plane H. We define the matrix P of $SL(4, \boldsymbol{C})$ as follows:

$$(4.11)\qquad P=\begin{pmatrix}-\frac{\rho}{2} & 0 & 0 & \frac{1}{\rho}\\ 0 & -\frac{\rho}{2} & \frac{1}{\rho} & 0\\ 0 & \rho & 0 & 0\\ \rho & 0 & 0 & 0\end{pmatrix},\qquad \rho\in\boldsymbol{C}^*\ .$$

We set anew $\eta={}^t(\eta_1, \eta_2, \eta_3, \eta_4)$ and define $\eta'={}^t(\eta'_1, \eta'_2, \eta'_3, \eta'_4)$ by the relation:

$$(4.12)\qquad \eta=P\eta'\ .$$

Then we have

$$(4.13)\qquad {}^tPAP=A'\ ,\qquad A'=\begin{pmatrix}0&0&0&2\\0&0&2&0\\0&2&0&0\\2&0&0&0\end{pmatrix}.$$

Thus from (4.9) and (4.10), we obtain the following:

$$(4.14)\qquad \eta'_1\eta'_4+\eta'_2\eta'_3=0\ ,$$

$$(4.15)\qquad \eta'_1\bar{\eta}'_4+\eta'_2\bar{\eta}'_3+\eta'_3\bar{\eta}'_2+\eta'_4\bar{\eta}'_1>0\ .$$

Since η'_i $(i=1, 2, 3, 4)$ are never zero, we can set

$$(4.16)\qquad (z_1, z_2, z_3)=\left(\frac{\eta'_1}{\eta'_2}, \frac{\eta'_4}{\eta'_2}, \frac{\eta'_3}{\eta'_2}\right).$$

Hence from (4.14), (4.15) and (4.16) we get

$$(4.17)\qquad z_3+z_1z_2=0\ ,$$

(4.18) $$(\operatorname{Im} z_1)(\operatorname{Im} z_2)>0 .$$

The subset of C^3 defined by (4.17) and (4.18) has two components. The image of the period map Φ_1 is connected, so it must be contained in only one component. Let us denote the component by Ω_0, then we may set Ω_0 as follows:

(4.19) $$\Omega_0=\{(z_1, z_2, z_3)\in C^3\colon \operatorname{Im} z_1>0,\ \operatorname{Im} z_2>0,\ z_3=-z_1z_2\} .$$

In fact, we can see that $\operatorname{Im} z_1>0$ and $\operatorname{Im} z_2>0$ (see Appendix). The space Ω_0 is clearly biholomorphic to $H\times H$.

In general, periods $\eta_i(\lambda)$ are multi-valued holomorphic functions, and so are $\eta'_i(\lambda)$. Therefore setting anew the period map Φ for $\mathscr{F}$ as follows:

$$\Phi\colon \Lambda\ni\lambda\longmapsto\left(\frac{\eta'_1(\lambda)}{\eta'_2(\lambda)}, \frac{\eta'_4(\lambda)}{\eta'_2(\lambda)}, \frac{\eta'_3(\lambda)}{\eta'_2(\lambda)}\right)\in C^3 ,$$

we obtain the following theorem.

Theorem 4.1. *The period map Φ for $\mathscr{F}$ is a multi-valued holomorphic map from Λ into $H\times H$.*

Remark 4.1. The signature of A is (2.2), hence from (4.9) and (4.10), we can get the formulas:

$$\begin{cases}\tilde{\eta}_1^2+\tilde{\eta}_2^2-\tilde{\eta}_3^2-\tilde{\eta}_4^2=0 ,\\ |\tilde{\eta}_1|^2+|\tilde{\eta}_2|^2-|\tilde{\eta}_3|^2-|\tilde{\eta}_4|^2>0 ,\end{cases}$$

which show that Ω is isomorphic to a symmetric domain of type IV.

§ 5. Monodromy transformation group.

Let λ_0 be the point whose homogeneous coordinates is $(1:-1:-1)$ in Λ. The elements of $\pi_1(\Lambda, \lambda_0)$ induce monodromy transformations of $H_2(\tilde{S}(\lambda_0), Z)$. The algebraic cycles $\Gamma_5, \cdots, \Gamma_{22}$ are invariant under the transformations. Thus the transformations are regarded as that of the periods $\eta_i=\eta_i(\lambda)$ $(i=1, 2, 3, 4)$. In this section we shall study the representations into $GL(4, Z)$ of their transformations and determine a transformation group on $H\times H$.

(I) In order to define the generators of $\pi_1(\Lambda, \lambda_0)$, we use the following notations:

H: a general hyperplane passing through λ_0 in $P_2(C)$, assume that H and L_i $(i=0, 1, 2, 3, 4)$ intersect at one point respectively, where L_i are the lines defined in (1.7).

$\varepsilon(\lambda_0; H\cap L_i)$: a loop on H starting from λ_0 and going around only $H\cap L_i$ in the positive sense.

We set

$$\delta_i=\varepsilon(\lambda_0; H\cap L_i) \qquad (i=0, 1, 2, 3, 4)\ . \tag{5.1}$$

We as well denote by δ_i the homotopy class of δ_i, then $\{\delta_0, \delta_1, \delta_2, \delta_3, \delta_4\}$ are the generators of $\pi_1(\Lambda, \lambda_0)$. Let the δ_i^* be the monodromy representation induced by δ_i. δ_i^* is obtained by the analytic continuation of 2-cycles Γ_1, Γ_2, Γ_3 and Γ_4 along the loop δ_i. Let us study δ_1^*. We define the loop δ_1 using affine coordinates (x, y) as follows:

$$\delta_1: \begin{cases} x=-r(\theta)e^{i\theta} & (0\leqq\theta\leqq 2\pi) \\ y=-1 \end{cases},$$

where $r(\theta)$ is a continuous function such that $1/2\leqq r(\theta)\leqq 1$, $r(0)=r(2\pi)=1$ and $r(\pi)=1/2$. Then the critical points $1/x$ and $(1-y)/x$ are denoted by $1/x=-(1/r(\theta))e^{-i\theta}$ and $(1-y)/x=-(2/r(\theta))e^{-i\theta}$ respectively. Thus the segments β_i $(i=1, 2, 3, 4)$ defined in Figure 3.3 are transformed to the arcs β_i' in Figure 5.1.

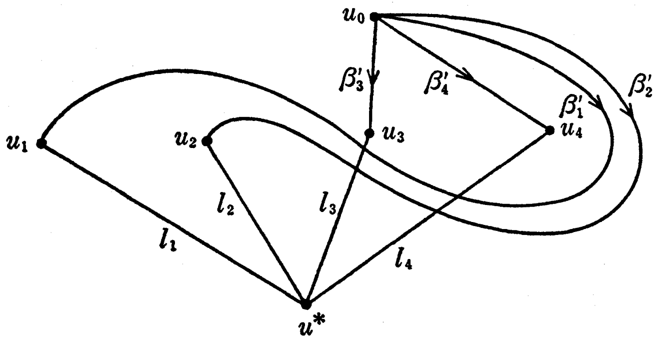


FIGURE 5.1

Suppose that Γ_i is transformed to Γ_i' by δ_1, then by using (3.10) (or (3.3)), we obtain

$$\Gamma_1'=\Gamma_1+2\Gamma_3\ ,\quad \Gamma_2'=\Gamma_2-2\Gamma_4\ ,\quad \Gamma_3'=\Gamma_3\ ,\quad \Gamma_4'=\Gamma_4\ .$$

Hence we get

$$\delta_1^*=\begin{pmatrix}1&0&2&0\\0&1&0&-2\\0&0&1&0\\0&0&0&1\end{pmatrix}. \tag{5.2}$$

By a similar way we obtain the following:

$$(5.3)\qquad \begin{aligned} &\delta_2^*=\begin{pmatrix}3&2&0&2\\-2&-1&-2&0\\0&0&3&-2\\0&0&2&-1\end{pmatrix}, \quad \delta_3^*=\begin{pmatrix}1&2&0&2\\0&-1&0&-2\\0&2&1&2\\0&0&0&1\end{pmatrix},\\ &\delta_4^*=\begin{pmatrix}-1&-4&-2&0\\-2&-3&-2&0\\4&8&5&0\\2&4&2&1\end{pmatrix}, \quad \delta_0^*=\begin{pmatrix}1&2&2&-2\\0&1&0&0\\0&0&1&-2\\0&0&0&1\end{pmatrix}. \end{aligned}$$

Here we have the following proposition.

PROPOSITION 5.1. *The following properties hold for the transformations* δ_i^* $(i=0, 1, 2, 3, 4)$:

$$\det \delta_i^*=1 \quad (i=0, 1, 2)\ , \qquad \det \delta_i^*=-1 \quad (i=3, 4)\ ,$$
$${}^t\delta_i^* A \delta_i^*=A\ , \quad \delta_i^*\equiv 1 \pmod 2 \quad (i=0, 1, 2, 3, 4)\ ,$$

where A *is the matrix defined by* (3.8).

REMARK 5.1. The monodromy group of the system of hypergeometric differential equation for $F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y)$ is known in general case (see Sasaki and Takano [11]); but, in our case, we must describe in the concrete.

Now, let us study our transformation group on $H\times H$. We get the transformations $\delta_i^{*\prime}=P^{-1}\delta_i^* P$ $(i=0, 1, 2, 3, 4)$ by the change of basis (4.12). By using (4.16) and (4.17), we can regard $\delta_i^{*\prime}$ as transformations on $H\times H$. Let us denote by $\tilde{\delta}_i$ the transformations on $H\times H$ corresponding to δ_i^* $(i=0, 1, 2, 3, 4)$, then we obtain the following:

$$(5.4)\qquad \begin{aligned} &\tilde{\delta}_0\colon (z_1, z_2)\longmapsto\left(\frac{z_1}{-2z_1+1}, z_2+\rho^2\right),\\ &\tilde{\delta}_1\colon (z_1, z_2)\longmapsto(z_1, z_2+2\rho^2)\ ,\\ &\tilde{\delta}_2\colon (z_1, z_2)\longmapsto\left(\frac{-z_1+2}{-2z_1+3}, z_2\right),\\ &\tilde{\delta}_3\colon (z_1, z_2)\longmapsto\left(\frac{1}{-\frac{2}{\rho^2}z_2+2}, -\frac{\rho^2}{2z_1}+\rho^2\right),\\ &\tilde{\delta}_4\colon (z_1, z_2)\longmapsto\left(\frac{z_2}{2z_2+\frac{\rho^2}{2}}, \frac{\frac{\rho^2}{2}z_1}{-2z_1+1}\right). \end{aligned}$$

(II) In order to describe more exactly the moduli space of the surfaces $\tilde{S}(\lambda)$ and complete the monodromy transformation group on $H\times H$, we induce the equivalent relation $\sim$ in the space Λ as follows:

$$(5.5)\qquad \begin{array}{l}(\xi_0:\xi_1:\xi_2)\sim(\xi_0':\xi_1':\xi_2') \text{ if and only if } \tilde{S}(\xi_0:\xi_1:\xi_2) \\ \text{is isomorphic to } \tilde{S}(\xi_0':\xi_1':\xi_2') \text{ as elliptic surfaces .}\end{array}$$

This isomorphism as elliptic surfaces is given by regarding the base curve as u-sphere, so we call it u-isomorphism and denote it by

$$(5.6)\qquad \tilde{S}(\lambda)\cong_u \tilde{S}(\lambda') ,$$

where $\lambda=(\xi_0:\xi_1:\xi_2)$ and $\lambda'=(\xi_0':\xi_1':\xi_2')$. The u-isomorphism $\sigma\colon \tilde{S}(\lambda)\xrightarrow{\sim}\tilde{S}(\lambda')$ makes the following diagram commutative (Figure 5.2), where T is an automorphism on u-sphere Δ.

$$\begin{array}{ccc}\tilde{S}(\lambda) & \xrightarrow{\ \sigma\ } & \tilde{S}(\lambda') \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ \Delta & \xrightarrow[T]{} & \Delta\end{array}$$

FIGURE 5.2

Thus, if a u-isomorphism $\sigma\colon \tilde{S}(\lambda)\to\tilde{S}(\lambda')$ exists, then the arrangement of the singular fibres of $\tilde{S}(\lambda)$ coincides with that of $\tilde{S}(\lambda')$. From Proposition 2.1, the singular fibres of $\tilde{S}(\lambda)$ are as follows:

$$\begin{array}{ll} u=0,\ 1 & \cdots\cdots\ \mathrm{I}_0^* , \\ u=\dfrac{\xi_0}{\xi_1},\ \dfrac{\xi_0-\xi_2}{\xi_1} & \cdots\cdots\ \mathrm{I}_2 , \\ u=\infty & \cdots\cdots\ \mathrm{I}_2^* . \end{array}$$

Hence the automorphism $T\colon \Delta\to\Delta$ has to satisfy the following:

$$(5.7)\qquad T\colon \{0,1\}\longrightarrow\{0,1\} ,\quad T\colon \infty\longmapsto\infty ,$$

$$(5.8)\qquad T\colon \left\{\frac{\xi_0}{\xi_1},\ \frac{\xi_0-\xi_2}{\xi_1}\right\}\longrightarrow\left\{\frac{\xi_0'}{\xi_1'},\ \frac{\xi_0'-\xi_2'}{\xi_1'}\right\} .$$

From (5.7), we get

$$T=\mathrm{id} \quad\text{or}\quad T\colon u\longmapsto u'=1-u .$$

(1) The case : $T=\mathrm{id}$. In this case, we have only to consider the following

$$\frac{\xi_0}{\xi_1}=\frac{\xi_0'-\xi_2'}{\xi_1'}, \quad \frac{\xi_0-\xi_2}{\xi_1}=\frac{\xi_0'}{\xi_1'}. \tag{5.9}$$

Setting $(x, y)=(\xi_1/\xi_0, \xi_2/\xi_0)$ and $(x', y')=(\xi_1'/\xi_0', \xi_2'/\xi_0')$, from (5.9) we have

$$x'=\frac{x}{1-y}, \quad y'=\frac{-y}{1-y}. \tag{5.10}$$

Then the u-isomorphism σ_2: $\tilde{S}(x, y)\rightarrow\tilde{S}(x', y')$ is given by

$$\sigma_2: (u, v, w)\longmapsto(u', v', w')=\left(u, 1-v, \frac{w}{\sqrt{1-y}}\right). \tag{5.11}$$

In paticular, putting $(x, y)=(-1, -1)$, we get

$$\tilde{S}(-1, -1)\cong_u\tilde{S}\left(-\frac{1}{2}, \frac{1}{2}\right). \tag{5.12}$$

(2) The case : T: $u\mapsto u'=1-u$. In this case we have two cases.
(2-1) The case:

$$T: \frac{\xi_0}{\xi_1}\longmapsto 1-\frac{\xi_0}{\xi_1}=\frac{\xi_0'}{\xi_1'}$$

$$T: \frac{\xi_0-\xi_2}{\xi_1}\longmapsto 1-\frac{\xi_0-\xi_2}{\xi_1}=\frac{\xi_0'-\xi_2'}{\xi_1'}.$$

We have

$$\frac{\xi_1-\xi_0}{\xi_1}=\frac{\xi_0'}{\xi_1'}, \quad \frac{\xi_1+\xi_2-\xi_0}{\xi_1}=\frac{\xi_0'-\xi_2'}{\xi_1'}, \tag{5.13}$$

$$x'=\frac{x}{x-1}, \quad y'=\frac{-y}{x-1}. \tag{5.14}$$

Thus, in this case the u-isomorphism σ_1: $\tilde{S}(x, y)\rightarrow\tilde{S}(x', y')$ is given by

$$\sigma_1: (u, v, w)\longmapsto(u', v', w')=\left(1-u, v, \frac{w}{\sqrt{1-x}}\right). \tag{5.15}$$

And we get

$$\tilde{S}(-1, -1)\cong_u\tilde{S}\left(\frac{1}{2}, -\frac{1}{2}\right). \tag{5.16}$$

(2-2) The case:

$$T: \frac{\xi_0}{\xi_1}\longmapsto 1-\frac{\xi_0}{\xi_1}=\frac{\xi_0'-\xi_2'}{\xi_1'}$$

$$T: \frac{\xi_0-\xi_2}{\xi_1}\longmapsto 1-\frac{\xi_0-\xi_2}{\xi_1}=\frac{\xi_0'}{\xi_1'}.$$

We have

$$1-\frac{\xi_0}{\xi_1}=\frac{\xi_0'-\xi_2'}{\xi_1'}\,,\qquad \frac{\xi_1+\xi_2-\xi_0}{\xi_1}=\frac{\xi_0'}{\xi_1'}\,, \tag{5.17}$$

$$x'=\frac{x}{x+y-1}\,,\qquad y'=\frac{y}{x+y-1}\,. \tag{5.18}$$

Thus, in this case the u-isomorphism σ_3: $\tilde{S}(x, y)\to\tilde{S}(x', y')$ is given by

$$\sigma_3\colon (u, v, w)\longmapsto(u', v', w')=\left(1-u,\ 1-v,\ \frac{w}{\sqrt{1-x-y}}\right). \tag{5.19}$$

And we get

$$\tilde{S}(-1, -1)\cong_u\tilde{S}\left(\frac{1}{3}, \frac{1}{3}\right). \tag{5.20}$$

REMARK 5.2. The elliptic surface $\tilde{S}(\lambda)$ is also considered as elliptic surface on v-sphere, then the types of the singular fibres of two elliptic surfaces coincide with each other. By a similar way, we can concider v-isomorphisms, but v-isomorphisms are equivalent to u-isomorphisms: namely

$$\tilde{S}(\lambda)\cong_v\tilde{S}(\lambda)\quad\text{if and only if}\quad\tilde{S}(\lambda)\cong_u\tilde{S}(\lambda')\,.$$

Now, we consider the quotient space $\Lambda/\sim$ of Λ by the relation $\sim$. In the space $\Lambda/\sim$, $\lambda_0=(-1, -1)$ is identified with $\lambda_1=(1/2, -1/2)$, $\lambda_2=(-1/2, 1/2)$ and $\lambda_3=(1/3, 1/3)$. Let us denote the equivalent class of λ_0 by $[\lambda_0]$, then the monodromy transformations induced by $\pi_1(\Lambda/\sim, [\lambda_0])$ are obtained by adding two transformations to that induced by $\pi_1(\Lambda, \lambda_0)$.

If we take adequately three arcs τ_1, τ_2, τ_3 starting from λ_0 and ending at λ_1, λ_2, λ_3 respectively in Λ, then we can regard the arcs τ_1, τ_2, τ_3 as loops starting from $[\lambda_0]$ in $\Lambda/\sim$. We denote as well these loops by τ_i ($i=1, 2, 3$) and denote the representation of τ_i into $GL(4, \boldsymbol{Z})$ by τ_i^*. This monodromy τ_i^* means the following:

Let σ_{i*}: $H_2(\tilde{S}(\lambda_0), \boldsymbol{Z})\simeq H(S(\lambda_i), \boldsymbol{Z})$ be the isomorphism induced by the u-isomorphism σ_i and let $\tau_{i*}(\Gamma_1), \cdots, \tau_{i*}(\Gamma_4)$ be the 2-cycles on $\tilde{S}(\lambda_i)$ induced by τ_i. Then the monodromy τ_i^* is defined by the formula:

$$\begin{pmatrix}\tau_{i*}(\Gamma_1)\\ \vdots\\ \tau_{i*}(\Gamma_4)\end{pmatrix}=\tau_i^*\begin{pmatrix}\sigma_{i*}(\Gamma_1)\\ \vdots\\ \sigma_{i*}(\Gamma_4)\end{pmatrix}.$$

By carrying out calculation, we obtain

$$(5.21)\qquad \tau_1^* = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tau_2^* = \begin{pmatrix} 2 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\tau_3^* = \begin{pmatrix} 2 & 1 & 2 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

τ_i^* $(i=1, 2, 3)$ satisfy the following:

$$(5.22)\qquad \det \tau_i^* = 1, \quad {}^t\tau_i^* A \tau_i^* = A \quad (i=1, 2, 3),$$

$$(5.23)\qquad \tau_1^{*2} = \delta_1^*, \quad \tau_2^{*2} = \delta_2^*, \quad \tau_3^* = \tau_1^* \tau_2^*.$$

And by the same way which we got $\tilde{\delta}_i$ from δ_i^*, we get $\tilde{\tau}_i$ from τ_i^*:

$$(5.24)\qquad \begin{aligned} &\tilde{\tau}_1 : (z_1, z_2) \longmapsto (z_1, z_2+\rho^2), \\ &\tilde{\tau}_2 : (z_1, z_2) \longmapsto \left(\frac{1}{-z_1+2}, z_2\right), \\ &\tilde{\tau}_3 : (z_1, z_2) \longmapsto \left(\frac{1}{-z_1+2}, z_2+\rho^2\right). \end{aligned}$$

We denote by $G(\rho)$ the transformation group on $H \times H$ generated by $\tilde{\delta}_i$ $(i=0, 1, 2, 3, 4)$ and $\tilde{\tau}_j$ $(j=1, 2, 3)$.

In particular, putting $\rho=\sqrt{2}$, from (5.4) and (5.24), we get the following:

$$(5.25)\qquad \begin{aligned} &\tilde{\delta}_0 : (z_1, z_2) \longmapsto \left(\frac{z_1}{-2z_1+1}, z_2+2\right), \\ &\tilde{\delta}_1 : (z_1, z_2) \longmapsto (z_1, z_2+4), \\ &\tilde{\delta}_2 : (z_1, z_2) \longmapsto \left(\frac{-z_1+2}{-2z_1+3}, z_2\right), \\ &\tilde{\delta}_3 : (z_1, z_2) \longmapsto \left(\frac{1}{-z_2+2}, -\frac{1}{z_1}+2\right), \\ &\tilde{\delta}_4 : (z_1, z_2) \longmapsto \left(\frac{z_2}{2z_2+1}, \frac{z_1}{-2z_1+1}\right), \\ &\tilde{\tau}_1 : (z_1, z_2) \longmapsto (z_1, z_2+2), \end{aligned}$$

$$\tilde{\tau}_2\colon (z_1, z_2) \longmapsto \left(\frac{1}{-z_1+2}, z_2\right),$$

$$\tilde{\tau}_3\colon (z_1, z_2) \longmapsto \left(\frac{1}{-z_1+2}, z_2+2\right).$$

We denote by $\langle\iota\rangle$ the group generated by the involution $\iota\colon (z_1, z_2)\mapsto(z_2, z_1)$ and denote by $\Gamma_{1,2}$ the group generated by the modular transformations $T\colon z\mapsto z+2$ and $S\colon z\mapsto -1/z$, i.e.,

$$\Gamma_{1,2}=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2, \boldsymbol{Z})\colon\ ab\equiv 0,\ cd\equiv 0 \pmod 2\right\}\Big/\pm I.$$

We shall show that the transformation group $\Gamma=G(\sqrt{2})$ on $H\times H$ is the semi-direct product group $\langle\iota\rangle\ltimes\Gamma_{1,2}\times\Gamma_{1,2}$, where its operation is given as follows: Let $(\iota_1, (S_1, T_1))$ and $(\iota_2, (S_2, T_2))$ be elements of $\langle\iota\rangle\ltimes\Gamma_{1,2}\times\Gamma_{1,2}$, then

$$(\iota_1, (S_1, T_1))(z_1, z_2)=\begin{cases}(S_1(z_1), T_1(z_2)) & \text{if } \iota_1=\mathrm{id}\\ (T_1(z_2), S_1(z_1)) & \text{if } \iota_1=\iota,\end{cases}$$

$$\begin{aligned}(\iota_1, (S_1, T_1))(\iota_2, (S_2, T_2))&=(\iota_1\iota_2, (S_1, T_1)^{\iota_2}(S_2, T_2))\\ &=\begin{cases}(\iota_1\iota_2, (S_1S_2, T_1T_2)) & \text{if } \iota_2=1\\ (\iota_1\iota_2, (T_1S_2, S_1T_2)) & \text{if } \iota_2=\iota.\end{cases}\end{aligned}$$

THEOREM 5.1. *The transformation group Γ generated by $\tilde{\delta}_i$, $\tilde{\tau}_j$ $(i=0, 1, 2, 3, 4;\ j=1, 2, 3)$ in (5.25) is the semi-direct product group $\langle\iota\rangle\ltimes\Gamma_{1,2}\times\Gamma_{1,2}$:*

$$\Gamma=\langle\iota\rangle\ltimes\Gamma_{1,2}\times\Gamma_{1,2}.$$

PROOF. It is immediate that $\Gamma\subset\langle\iota\rangle\ltimes\Gamma_{1,2}\times\Gamma_{1,2}$. Thus we prove the converse. The group $\langle\iota\rangle\ltimes\Gamma_{1,2}\times\Gamma_{1,2}$ is generated by $(\iota, (I, I))$, $\left(1, \left(I, \begin{pmatrix}1&2\\0&1\end{pmatrix}\right)\right)$ and $\left(1, \left(I, \begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)\right)$, where $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. By the way, from (5.25), we have

$$\tilde{\delta}_0=\left(1, \left(\begin{pmatrix}1&0\\-2&1\end{pmatrix}, \begin{pmatrix}1&2\\0&1\end{pmatrix}\right)\right),$$

$$\tilde{\delta}_2=\left(1, \left(\begin{pmatrix}-1&2\\-2&3\end{pmatrix}, I\right)\right),$$

$$\tilde{\delta}_4=\left(\iota, \left(\begin{pmatrix}1&0\\-2&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix}\right)\right).$$

Hence we get

$$\tilde{\tilde{\delta}}_0 \cdot \tilde{\tilde{\delta}}_4 \cdot \tilde{\tilde{\delta}}_2 = (\iota, (I, I)) .$$

And we have

$$\tilde{\tau}_1 = \left(1, \left(I, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right)\right),$$

$$\iota \cdot \tilde{\tau}_2 \cdot \iota \cdot \tilde{\tau}_1 = \left(1, \left(I, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\right),$$

where we identified ι with $(\iota, (I, I))$. These show that $\Gamma \supset \langle \iota \rangle \ltimes \Gamma_{1,2} \times \Gamma_{1,2}$. Therefore we obtain

$$\Gamma = \langle \iota \rangle \ltimes \Gamma_{1,2} \times \Gamma_{1,2} .$$

REMARK 5.3. Let Γ' be the monodromy group generated by $\tilde{\tilde{\delta}}_i$ ($i=0, 1, 2, 3, 4$), then $\Gamma' \subsetneqq \langle \iota \rangle \ltimes \Gamma_{1,2} \times \Gamma_{1,2}$.

§6. Modular function Ψ.

In this final section, we shall investigate the inverse map Ψ of the period map

$$\Phi: \Lambda/\sim \longrightarrow H \times H/\Gamma ,$$

i.e., an automorphic map relative to $\Gamma = G(\sqrt{2})$. We call Ψ the "modular function" for the family $\mathscr{F}$. In order to make sure that Ψ is well-defined on $H \times H$, we must verify bijectivity of Φ by extending the domain $\Lambda/\sim$ if necessary. For this purpose, we set $\Lambda = P_2(C) - \bigcup_{k=0}^{5} L_k$ as in §1 and we study the behavior of the period map Φ on L_k ($k=0, 1, 2, 3, 4$).

(I) We set

$$P_0 = (0:1:0) , \quad P_1 = (0:0:1) , \quad P_2 = (1:0:0) , \quad P_3 = (1:1:0) ,$$
$$P_4 = (1:0:1) , \quad P_5 = (1:1:1) , \quad P_6 = (0:1:-1) \quad \text{(see Figure 6.1)}.$$

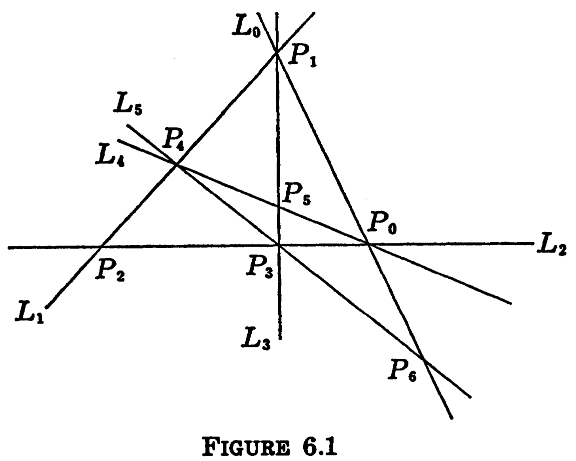


FIGURE 6.1

By elementary but careful calculation (see Appendix), we obtain the following table.

TABLE 6.1

boundary of Λ	image of Φ	$\tilde{S}(\lambda)$
P_5 P_6	$\mathfrak{p}_5=\left(\frac{2+i}{5}, i\right)$ $\mathfrak{p}_6=(i, i)$	elliptic K3 surface with singular fibres I_2^*, I_2^*, I_2^*
$L_0-\{P_0, P_1, P_6\}$ $L_3-\{P_1, P_3, P_5\}$ $L_4-\{P_0, P_4, P_5\}$ $L_5-\{P_3, P_4, P_6\}$	$H_0=\{z_1z_2+1=0\}-\mathfrak{p}_6$ $H_3=\{2z_1-z_1z_2-1=0\}-\mathfrak{p}_5$ $H_4=\{z_1-z_2-2=0\}-\mathfrak{p}_5$ $H_5=\{z_1=z_2\}-\mathfrak{p}_6$	elliptic K3 surface with singular fibres I_0^*, I_2, I_2^*, I_2^*
$L_1-\{P_1, P_2, P_4\}$ $L_2-\{P_0, P_2, P_3\}$	$\{(z_1, \infty)\colon z_1\in H\}$ $\{(-1, z_2)\colon z_2\in H\}$	elliptic rational surface with singular fibres I_0^*, I_0^*
P_0 P_1 P_3 P_4	$(-1, -1)$ (∞, ∞) $(-1, -1)$ (∞, ∞)	rational surface
P_2	$(-1, \infty)$	rational surface

REMARK 6.1. As the "image of Φ" we write representatives for equivalent classes relative to modulus Γ.

REMARK 6.2. $S(P_5)$ and $S(P_6)$ are denoted by

$$S(P_5)\colon w^2=uv(1-u)(1-v)(1-u-v)\ ,$$
$$S(P_6)\colon w^2=uv(1-u)(1-v)(-u+v),\ \text{respectively.}$$

And the Picard number of the surfaces $\tilde{S}(P_5)$ and $\tilde{S}(P_6)$ is 19.

We can regard the equivalent relation $\sim$ of the parameter space Λ as that obtained by a projective transformation group of $\boldsymbol{P}_2(\boldsymbol{C})$. Let us denote this group by G. By (5.9), (5.13) and (5.17) G is generated by the following transformations g_1, g_2 and g_3:

$$(6.1)\qquad \begin{cases} g_1\colon (\xi_0\colon \xi_1\colon \xi_2)\longmapsto(\xi_0'\colon \xi_1'\colon \xi_2')=(\xi_0-\xi_1\colon -\xi_1\colon \xi_2)\ , \\ g_2\colon (\xi_0\colon \xi_1\colon \xi_2)\longmapsto(\xi_0'\colon \xi_1'\colon \xi_2')=(\xi_0-\xi_2\colon \xi_1\colon -\xi_2)\ , \\ g_3\colon (\xi_0\colon \xi_1\colon \xi_2)\longmapsto(\xi_0'\colon \xi_1'\colon \xi_2')=(\xi_1+\xi_2-\xi_0\colon \xi_1\colon \xi_2)\ . \end{cases}$$

We immediately find that $g_i=g_jg_k=g_kg_j$ $(i, j, k=1, 2, 3)$ and $g_i^2=1$ $(i=1, 2, 3)$, thus G is isomorphic to the Klein four-group. G acts discontinuously on

$P_2(C)$. We note that g_1, g_2 and g_3 fix lines $\{\xi_1=0\}$, $\{\xi_2=0\}$ and $\{\xi_1+\xi_2-2\xi_0=0\}$ respectively and that the lines L_0, L_3, L_4 and L_5 are transformed one another by G. And the hypersurfaces H_0, H_3, H_4 and H_5 of $H\times H$ corresponding to these lines L_0, L_3, L_4 and L_5 belong to the same orbit of Γ. Moreover, by the above table, putting

$$\Lambda_0=P_2(C)-L_1\cup L_2\,, \tag{6.2}$$

we see that $\tilde{S}(\lambda)$ are elliptic K3 surfaces for all $\lambda\in\Lambda_0$. Therefore we can consider the period map Φ as the map from $\Lambda_0/\sim$ to $H\times H/\Gamma$, where the equivalent relation $\sim$ is obtained by restricting the projective transformation group G to Λ_0.

REMARK 6.3. In general, the elements of $\Lambda_0/\sim$ consist of four points of Λ_0 except the equivalent classes of points on the line $L=\{\xi_1+\xi_2-2\xi_0=0\}$ fixed by g_3. On the line L, u-isomorphism σ_3: $\tilde{S}(\lambda)\to\tilde{S}(\lambda')$ corresponding to g_3 (see (5.19)) becomes the automorphism of order 4 of K3 surface $\tilde{S}(\lambda)$ $(\lambda\in L)$: namely

$$\begin{array}{ccl}\sigma_3:\tilde{S}(\lambda) & \xrightarrow{\sim} & \tilde{S}(\lambda)\\ \cup\!\!\!| & & \cup\!\!\!|\\ (u,\,v,\,w) & \longmapsto & (u',\,v',\,w')=(1-u,\,1-v,\,-iw)\,.\end{array}$$

(II) Next let us show that the period map Φ is an injection from $\Lambda_0/\sim$ to $H\times H/\Gamma$. For this purpose we define a "marked K3 surface". Here we employ the following notations:

S: an algebraic K3 surface,

$\mathscr{L}$: a free Z-module of rank 22 with an even integer valued unimodular symmetric bilinear form of signature (3.19),

l: a fixed element of $\mathscr{L}$.

A marked K3 surface is defined as a triple (S,φ,D) satisfying the following conditions:

(1) φ is an isomorphism from $\mathscr{L}$ to $H_2(S,Z)$,

(2) D is an effective divisor on S such that $D^2>0$, $D\cdot D'\geqq 0$ for any effective divisor D' and $\varphi(l)=D$.

Two marked K3 surfaces (S,φ,D) and (S',φ',D') are identified if there exists an isomorphism f from S to S' such that $\varphi'=f_*\cdot\varphi$ (modulo effective divisors) and $f_*(D)=D'$, where f_* is the map from $H_2(S,Z)$ to $H_2(S',Z)$ induced by f.

We denote by $M(l)$ a family of all marked K3 surfaces (S,φ,D) with fixed l. Let (S,φ,D) be a marked K3 surface and let $(l_1,\cdots,l_{22})$ be a basis of $\mathscr{L}$. Setting $\Gamma_i=\varphi(l_i)$ $(i=1,\cdots,22)$, we see that $\{\Gamma_1,\cdots,\Gamma_{22}\}$ is a basis of $H_2(S, \boldsymbol{Z})$. So we put $\eta_i=\int_{\Gamma_i}\psi$ $(i=1, \cdots, 22)$ and define a map $\tau: M(l)\to \boldsymbol{P}_{21}(\boldsymbol{C})$ by

$$\tau: M(l)\ni(S, \varphi, D)\longmapsto(\eta_1, \cdots, \eta_{22})\in \boldsymbol{P}_{21}(\boldsymbol{C}) ,$$

where ψ is a holomorphic 2-form on S. Then following Pjateckiĭ-Šapiro and Šafarevič [10], we obtain the Torelli theorem for algebraic K3 surfaces.

THEOREM T. *The period map τ is injective.*

Now in order to show injectivity of Φ we define a marking on $\tilde{S}(\lambda)$ $(\lambda\in \Lambda_0)$. We put

$$\lambda_0=(1:-1:-1) , \quad S_0=\tilde{S}(\lambda_0) , \quad \mathscr{L}=H_2(S_0, \boldsymbol{Z}) ,$$

and define $l\in\mathscr{L}$ by

$$l=L+2G , \tag{6.3}$$

where L is the global section on $\tilde{S}(\lambda)$ and G is a fibre $\pi^{-1}(u)$. It is trivial to verify that l is an effective divisor. We define an isomorphism $\varphi: \mathscr{L}\to H_2(S(\lambda), \boldsymbol{Z})$ by the canonical isomorphism from $H_2(S_0, \boldsymbol{Z})$ to $H_2(\tilde{S}(\lambda), \boldsymbol{Z})$ and an effective divisor D on $\tilde{S}(\lambda)$ by $D=L+2G$. Note that $D\cdot D'\geqq 0$ for any effective divisor D' on $\tilde{S}(\lambda)$. Hence $(\tilde{S}(\lambda), \varphi, D)$ is a marked K3 surface. The injectivity of Φ follows immediately from the following lemma.

LEMMA 6.1. *Let $(\tilde{S}(\lambda), \varphi, D)$ and $(\tilde{S}(\lambda'), \varphi', D)$ be two marked* K3 *surfaces, where $\lambda, \lambda'\in\Lambda_0$. If $(\tilde{S}(\lambda), \varphi, D)=(\tilde{S}(\lambda'), \varphi', D)$, then there exists a u-isomorphism from $\tilde{S}(\lambda)$ onto $\tilde{S}(\lambda')$.*

PROOF. By applying the fact that $H^0(\tilde{S}(\lambda), \mathscr{O}([D]))=0$, $H^1(\tilde{S}(\lambda), \mathscr{O}([D]))=0$ and Serre's duality theorem to the Riemann-Roch theorem, we obtain $\dim H^0(\tilde{S}(\lambda), \mathscr{O}([D]))=3$. Hence we infer that a coordinate t of based curve $\Delta=P_1$ is written by a ratio of two holomorphic sections of $\mathscr{O}([D])$. By the condition $(\tilde{S}(\lambda), \varphi, D)=(\tilde{S}(\lambda'), \varphi', D)$, there exists a biholomorphic map $f: \tilde{S}(\lambda)\to\tilde{S}(\lambda')$. Let $(\tilde{S}(\lambda), \pi, \Delta)$ and $(\tilde{S}(\lambda'), \pi', \Delta)$ be two elliptic surfaces, then $t'=\pi'\cdot f$ is also written by a ratio of two holomorphic sections of $\mathscr{O}([D])$. Thus the transformation $T: \Delta\ni t\to t'\in\Delta$ is an isomorphism on Δ and the following diagram (Figure 6.2) is commutative. Therefore we obtain $\tilde{S}(\lambda)\cong_u\tilde{S}(\lambda')$.

$$\begin{array}{ccc} \tilde{S}(\lambda) & \xrightarrow{\ f\ } & \tilde{S}(\lambda') \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \Delta & \xrightarrow[T]{} & \Delta \end{array}$$

FIGURE 6.2

By virtue of Theorem T and Lemma 6.1, we obtain the following proposition.

PROPOSITION 6.1. *The period map*

$$\Phi: \Lambda_0/\sim \longrightarrow H\times H/\Gamma$$

is injective.

(III) Finally, instead of showing the surjectivity of Φ we show that the period map Φ is extended as biholomorphic map from $(\Lambda_0/\sim)^*$ onto $(H\times H/\Gamma)^*$, where X^* indicates a compactification of X. Then, we first mention the compactification of $\Lambda_0/\sim$ and $H\times H/\Gamma$.

The equivalent relation $\sim$ in Λ_0 was defined as the restriction to Λ_0 of the projective transformation group G on $\boldsymbol{P}_2(\boldsymbol{C})$, hence we define the compactification $(\Lambda_0/\sim)^*$ by

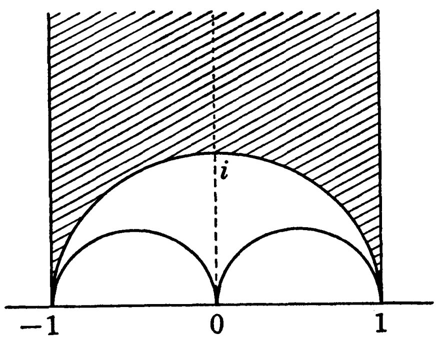
$$(\Lambda_0/\sim)^* := \boldsymbol{P}_2(\boldsymbol{C})/G = \boldsymbol{P}_2(\boldsymbol{C})\ . \tag{6.4}$$

In this definition, we can easily verify that the sign of equality holds. On the other hand, in view of $\Gamma = \langle\iota\rangle \ltimes \Gamma_{1,2}\times\Gamma_{1,2}$ we can consider as follows:

$$H\times H/\Gamma = (H/\Gamma_{1,2})\times(H/\Gamma_{1,2})/\iota\ . \tag{6.5}$$

Here $H/\Gamma_{1,2}$ is compactified by attaching two cusp points $\{1, \infty\}$ and the compactification $(H/\Gamma_{1,2})^*$ of $H/\Gamma_{1,2}$ is isomorphic to $\boldsymbol{P}_1(\boldsymbol{C})$: namely,

$$(H/\Gamma_{1,2})^* = \boldsymbol{P}_1(\boldsymbol{C})\ . \tag{6.6}$$



(fundamental domain of $H/\Gamma_{1,2}$)

Thus we define our compactification of $H\times H/\Gamma$ by the following:

$$(H\times H/\Gamma)^* := (H/\Gamma_{1,2})^* \times (H/\Gamma_{1,2})^*/\iota = \boldsymbol{P}_1(\boldsymbol{C})\times \boldsymbol{P}_1(\boldsymbol{C})/\iota \,. \tag{6.7}$$

Here we have

$$\boldsymbol{P}_1\times\boldsymbol{P}_1/\iota=\boldsymbol{P}_2 \,. \tag{6.8}$$

In fact, the map

$$\boldsymbol{P}_1\times\boldsymbol{P}_1/\iota \ni (\zeta_0\colon \zeta_1)\times(\nu_0\colon \nu_1)\longmapsto(\zeta_0\nu_0\colon \zeta_0\nu_1+\zeta_1\nu_0\colon \zeta_1\nu_1)\in\boldsymbol{P}_2$$

is an isomorphism. Hence we obtain

$$(H\times H/\Gamma)^*=\boldsymbol{P}_2(\boldsymbol{C}) \,. \tag{6.9}$$

Next, let us show that the map Φ is extended to a biholomorphic map from $(\Lambda_0/\sim)^*$ onto $(H\times H/\Gamma)^*$. For this purpose we use two lemmas.

Lemma 6.2. *Let Ω be an open set in $\boldsymbol{C}^n$ and $f\colon \Omega\to\boldsymbol{C}^n$ an injective holomorphic map. Then f is a biholomorphic map from Ω onto $f(\Omega)$.*

Proof. See Theorem 5 in p. 86, Narasimhan [7].

The following lemma follows immediately from the above lemma.

Lemma 6.3. *Let M and N be connected compact complex manifolds such that* $\dim M=\dim N$ *and let $f\colon M\to N$ be an injective holomorphic map. Then f is a biholomorphic map from M onto N.*

Proof. It is obvious.

Now, we can make sure that Φ is extended as an injective map onto $(\Lambda_0/\sim)^*=\boldsymbol{P}_2(\boldsymbol{C})$. In fact, we can see that the inverse map of the period map Φ restricted to the boundary of $(\Lambda_0/\sim)^*$ is given by the lambda function which is an elliptic modular function (see Appendix). Therefore, by the above argument we obtain the following theorem:

Theorem 6.1. *The period map $\Phi\colon \Lambda_0/\sim\to H\times H$ is extended to a biholomorphic map from $(\Lambda_0/\sim)^*$ onto $(H\times H/\Gamma)^*$. Consequently, the inverse map Ψ of Φ is defined as a single-valued holomorphic map on $H\times H$, and it is automorphic relative to the monodromy group Γ. And it follows that the modular function Ψ for $\mathscr{F}$ induces the biholomorphic map:*

$$(H\times H/\Gamma)^*\xrightarrow{\sim}\boldsymbol{P}_2(\boldsymbol{C})=(\Lambda_0/\sim)^* \,.$$

Appendix

Here we shall give calculation of the monodromy representation α_i^* in (3.3) and that of Table 6.1.

(I) We study α_1^*. In order to make our calculation easy, we rewrite Figure 3.1 as follows:

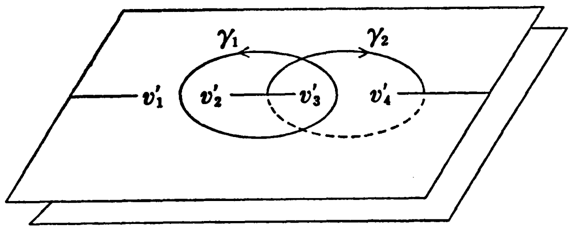


FIGURE A.1

The 1-cycles γ_1, γ_2 in Figure A.1 are clearly homotopic to the 1-cycles γ_1, γ_2 in Figure 3.1 respectively. General fibres $C(u)$ of $\tilde{S}_0$ have four branch points $v=0,\ 1,\ -1-u,\ \infty$. Putting the arc α_1 as follows:

$$\alpha_1:\ u+2=\frac{1}{2}e^{i\theta} \quad (0\leqq\theta\leqq 2\pi)\ ,$$

the branch point $v=-1-u$ encircles the point $v=1$ from $v=1/2$ along the arc $v-1=-(1/2)e^{i\theta}$ $(0\leqq\theta\leqq 2\pi)$. Thus the 1-cycles γ_1, γ_2 are transformed to 1-cycles γ_1', γ_2' in Figure A.2 by α_1. It is clear that $\gamma_1'=\gamma_1$. And we can see that the intersection numbers $\gamma_2'\cdot\gamma_1=1$, $\gamma_2'\cdot\gamma_2=2$, hence we get $\gamma_2'=-2\gamma_1+\gamma_2$. Therefore we obtain $\alpha_1^*=\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

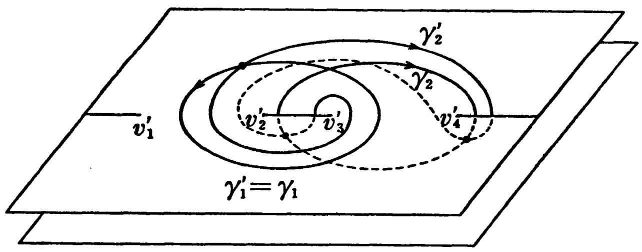


FIGURE A.2

α_2^* is obtained by using Figure 3.1. And we can get the others in a similar way.

(II) Calculation of Table 6.1. In (4.11), we put $\rho=\sqrt{2}$, then by (4.12) we get the following:

$$\text{(A.1)} \qquad \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}}\eta_1' + \frac{1}{\sqrt{2}}\eta_4' \\ -\frac{1}{\sqrt{2}}\eta_2' + \frac{1}{\sqrt{2}}\eta_3' \\ \sqrt{2}\,\eta_2' \\ \sqrt{2}\,\eta_1' \end{pmatrix} .$$

First, we calculate $\mathfrak{p}_5 = \Phi(P_5)$. We note that the 2-cycles Γ_1, Γ_2, Γ_3 and Γ_4 on $\tilde{S}(\lambda)$ $(\lambda = \xi_0 : \xi_1 : \xi_2) \in \Lambda)$ are defined by using the arcs β_1, β_2, β_3 in Figure A.3 as follows:

$$\Gamma_1 = \Gamma(\beta_1, \gamma_1) , \quad \Gamma_2 = \Gamma(\beta_2, \gamma_2) ,$$
$$\Gamma_3 = \Gamma(\beta_3^{-1}, \gamma_1) , \quad \Gamma_4 = \Gamma(\beta_3, \gamma_2) ,$$

where γ_1, γ_2 are 1-cycles on a general fibre C of $\tilde{S}(\lambda)$ defined as Figure A.4.

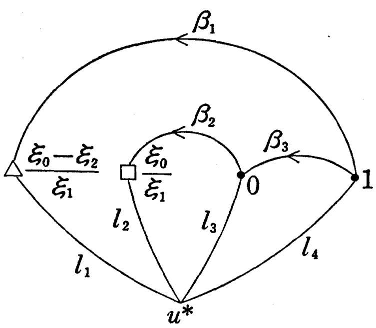


FIGURE A.3

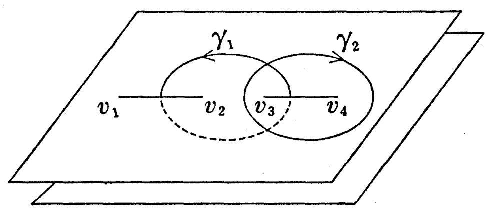


FIGURE A.4

Here $P(v_1)=0$, $P(v_2)=(\xi_0-\xi_1 u)/\xi_2$, $P(v_3)=1$ and $P(v_4)=\infty$, where P is a projection from C onto v-sphere.

When a point $\lambda=(\xi_0 : \xi_1 : \xi_2) \in \Lambda$ tends to $P_5=(1:1:1)$, the critical points $(\xi_0-\xi_2)/\xi_1$ and ξ_0/ξ_1 converge to 0 and 1 respectively. Thus the arcs β_1, β_2, β_3 in Figure A.3 are transformed as the following figure while λ tends to P_5:

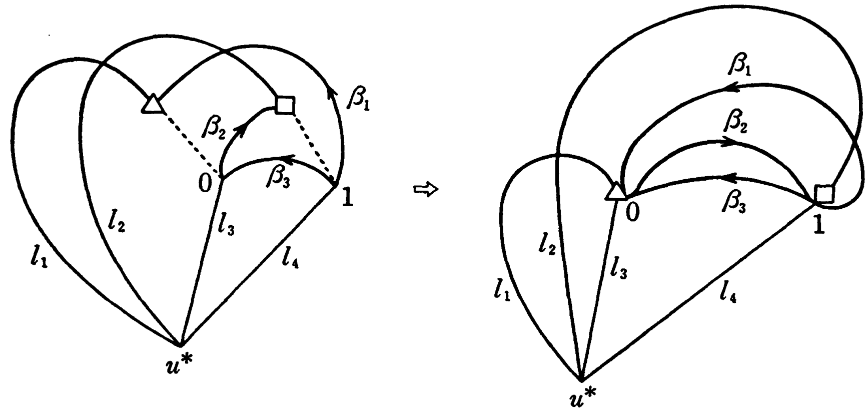
(Figure A.3) ⇨ 

FIGURE A.5

In Figure A.5 the arc β_1 crosses the arc l_2 in the positive sense, hence the 1-cycle γ_1 continued along the arc β_1 is transformed to $\gamma_1+2\gamma_2$ by the monodromy transformation $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ (see § 3). Therefore we get

$$\Gamma_1=-\Gamma_3+2\Gamma_4\,, \quad \Gamma_2=-\Gamma_4\,,$$

namely we get

$$\eta_1=-\eta_3+2\eta_4\,, \quad \eta_2=-\eta_4\,. \tag{A.2}$$

From (A.1) and (A.2), we obtain

$$\begin{cases} -\dfrac{1}{\sqrt{2}}\eta_1'+\dfrac{1}{\sqrt{2}}\eta_4'=-\sqrt{2}\,\eta_2'+2\sqrt{2}\,\eta_1'\,, \\[2mm] -\dfrac{1}{\sqrt{2}}\eta_2'+\dfrac{1}{\sqrt{2}}\eta_3'=-\sqrt{2}\,\eta_1'\,. \end{cases}$$

Thus by (4.16) and (4.17), $\Phi(P_5)$ is given as the intersection of the following two hypersurfaces:

$$\begin{cases} 5z_1-z_2-2=0\,, \\ 2z_1-z_1z_2-1=0\,. \end{cases}$$

Hence we obtain $\Phi(P_5)=((2+i)/5,\, i)$. Note that $\tau_2(i,\, i)=((2+i)/5,\, i)$.

Next, we calculate $\Phi(L_1-\{P_1,\, P_2,\, P_4\})$. When we put $\xi_1=0$, the critical points $(\xi_0-\xi_2)/\xi_1$ and ξ_0/ξ_1 go to the point at infinity. Putting $\xi_1=0$ in (1.6′), we have

$$w^2=uv(1-u)(1-v)(\xi_0-\xi_2 v)\,.$$

We set

(A.3) $$\omega_i=\int_{\gamma_i}\frac{dv}{\sqrt{v(1-v)(\xi_0-\xi_2 v)}}\qquad (i=1,\ 2)\ ,$$

where γ_1, γ_2 are 1-cycles on a general fibre of $\tilde{S}(\xi_0\colon 0\colon \xi_2)$ with $\gamma_1\cdot\gamma_2=-1$. Then we have the following:

$$\eta_1=\int_{\Gamma_1}\frac{du\wedge dv}{w}=\int_\infty^1 du\int_{\gamma_1}\frac{dv}{w}=\omega_1\int_\infty^1\frac{du}{\sqrt{u(1-u)}}\ ,$$
$$\eta_3=\int_{\Gamma_3}\frac{du\wedge dv}{w}=\omega_1\int_0^1\frac{du}{\sqrt{u(1-u)}}=\pi\omega_1\ ,$$
$$\eta_4=\int_{\Gamma_4}\frac{du\wedge dv}{w}=\omega_2\int_1^0\frac{du}{\sqrt{u(1-u)}}=-\pi\omega_2\ .$$

From (A.1) and (4.16), we get

(A.4) $$\begin{cases} z_1=\dfrac{\eta_1'}{\eta_2'}=\dfrac{\eta_4}{\eta_3}=-\dfrac{\omega_2}{\omega_1}\ ,\\ z_2=\dfrac{\eta_4'}{\eta_2'}=\dfrac{2\eta_1+\eta_4}{\eta_3}=\left(\omega_1\displaystyle\int_\infty^1\frac{du}{\sqrt{u(1-u)}}-\pi\omega_2\right)\Big/\pi\omega_1=\infty\ .\end{cases}$$

Since $\gamma_1\cdot\gamma_2=-1$, we have $\operatorname{Im} z_1=\operatorname{Im}(-\omega_2/\omega_1)>0$. Hence the points on $L_1-\{P_1, P_2, P_4\}$ are mapped into $H\times\{\infty\}$ by the period map Φ, where $H=\{z\in \boldsymbol{C}\colon \operatorname{Im} z>0\}$.

Now, let us study the behavior of the map Φ on L_1. Since $\xi_1\equiv 0$ on L_1, if we put $\lambda=\xi_0/\xi_2$, we have $P_1=0$, $P_2=\infty$, $P_4=1$. Thus $L_1-\{P_1, P_2, P_4\}$ coincides with $\boldsymbol{P}_1-\{0, 1, \infty\}$. And if we restrict the projective transformations g_1, g_2 and g_3 in (6.1) to L_1, we have that

$$\begin{cases} g_1\colon (\xi_0\colon 0\colon \xi_2)\longmapsto(\xi_0\colon 0\colon \xi_2)\ ,\\ g_2\colon (\xi_0\colon 0\colon \xi_2)\longmapsto(\xi_0-\xi_2\colon 0\colon -\xi_2)\ ,\\ g_3\colon (\xi_0\colon 0\colon \xi_2)\longmapsto(\xi_2-\xi_0\colon 0\colon \xi_2)\ .\end{cases}$$

Hence we get $g_1=\mathrm{id}$, $g_2=g_3\colon \lambda\mapsto 1-\lambda$. We can define the period map Φ on $L_1-\{P_1, P_2, P_4\}$ by $\Phi(\lambda)=\eta_1'(\lambda)/\eta_2'(\lambda)=\eta_4(\lambda)/\eta_3(\lambda)=\omega_2(\lambda)/\omega_1(\lambda)$. Then, from (A.3), the inverse map of Φ is essentially the lambda function. On the λ-function, it is well known that $z'\equiv z \pmod{SL(2, \boldsymbol{Z})}$ $(z, z'\in H)$ if and only if $\lambda(z')$ coincides with one of

$$\lambda(z),\ \ 1-\lambda(z),\ \ \frac{1}{\lambda(z)},\ \ \frac{1}{1-\lambda(z)},\ \ \frac{\lambda(z)}{\lambda(z)-1},\ \ \frac{\lambda(z)-1}{\lambda(z)}\ .$$

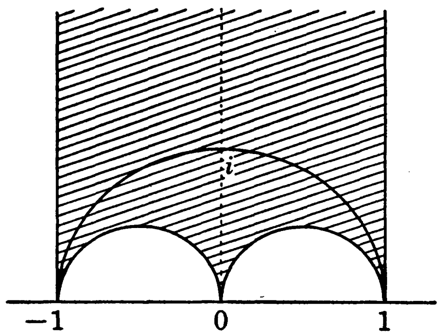
In particular, we have

$$z'=-\frac{1}{z} \quad \text{if and only if} \quad \lambda(z')=1-\lambda(z) .$$

By the way, λ-function is invariant under $\Gamma(2)$ the principal congruence subgroup of level 2. The subgroup of $SL(2, \boldsymbol{Z})$ generated by $\Gamma(2)$ and the transformation $S: z\mapsto -1/z$ is exactly the modular group $\Gamma_{1,2}$. Therefore we obtain the following:

$$\lambda: H/\Gamma_{1,2} \xrightarrow{\sim} \boldsymbol{P}_1-\{0, 1, \infty\}/\sim ,$$

where equivalent relation $\sim$ is defined by $\lambda\sim\lambda'$ if and only if $\lambda'=1-\lambda$.



(fundamental domain of $H/\Gamma(2)$)

Moreover, by Figure A.4 we can see that $\Phi(P_1)=\Phi(0)=0$, $\Phi(P_2)=\Phi(\infty)=-1$, $\Phi(P_4)=\Phi(1)=\infty$. (These facts do not contradict the results of Table 6.1.) This shows that the map Φ is well-defined as an injective holomorphic map on $L_1/\sim$. We can consider the period map Φ on L_2 in a similar way. Hence we obtain the following:

PROPOSITION A.1. *On the boundary $L_1\cup L_2/\sim$ of $(\Lambda_0/\sim)^*$, the period map Φ is an injective holomorphic map and its inverse map is given by the lambda function.*

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