TOKYO J. MATH.
VOL. 12, NO. 1, 1989

The Γ-Antilocality of Stable Generators Whose Lévy Measures are Supported on a Cone

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(Communicated by N. Iwahori)

§1. Introduction.

The antilocality (to all directions) of an operator A states that, if $f=Af=0$ in a domain U then f is identically zero on the whole space. Goodman-Segal [2] and Murata [12] proved the antilocality in $\boldsymbol{R}^n$ of the operator $(m^2I-\Delta)^\lambda$, where Δ is the Laplacian and λ is a non-integral real number. The interest of the former physicists lied in the non-relativistic approximation of the relativistic quantum theory. Mathematically, both of them used classical methods connected with the Hardy space in the proof. Later, in his study on the complex powers of elliptic operators with analytic coefficients, Liess [11] showed the property using the hyperfunctions in connection with the theory of pseudodifferential operators with analytic symbols.

The notion of antilocality has been extended to that of the one-sided antilocality by Ishikawa [4], [5], and he proved it for the sums of one-dimensional stable generators. Recently, in connection with the potential theory corresponding to the operators of Lévy-Khintchin type, Kanda [8] has treated a similar problem.

Here we propose an extended notion of a biased antilocality (called Γ-antilocality). This states that if $f=Af=0$ in a domain U, then the zero's of f propagates from U to infinity only in some directions. This implies that if Γ is a (open or closed) convex cone in $\boldsymbol{R}^2$ and if $Af_1=Af_2$ and $f_1=f_2$ in a domain U for suitable functions f_1, f_2, then $f_1=f_2$ in $U+\Gamma$. Such an operator appears as an infinitesimal generator of a semi-group $(T_t)_{t>0}$ corresponding to a Markov process of jump type. In fact, the generator A of two-dimensional stable process whose Lévy measure is supported on a closed convex proper cone Γ has Γ-antilocality (Theorem

Received May 30, 1988
Revised November 9, 1988

2.4). Analytically, this phenomenon is closely connected with the propagation of (micro-) analyticity in the theory of micro-functions, and so Sato-Kashiwara-Kawai theory ([6], [7]) will prove to be a powerful tool.

We first prove the localized version of Γ-antilocality, that is, Lemma 3.1, and then apply it at each point in $U+\Gamma$ repeatedly to show the Γ-antilocality. In the procedure of the proof, we use the so-called "sweeping-out method".

The outline of the present paper is as follows. In § 2, we give some definitions and explanations for notation necessary to state our main result. In § 3 we prove our key lemma (Lemma 3.1), whose proof is divided into two cases according to the angle of the cone. § 4 is devoted to the proof of our sweeping-out method using the curve.

We restrict our study to a class of symbols of very simple form, that of stable generators. The result is valid for two-dimensional more general class of symbols. A sufficient condition for Γ-antilocality of a class satisfying assumptions (A1)-(A3) in § 2 under the restrictions on the angle of the corresponding convex cones is given in Appendix.

In conclusion, the author wishes to express his gratitude to Professor M. Kanda, who led him to this problem. Also, the author thanks Dr. N. Minami who read through the earlier manuscript and corrected the writing.

§ 2. Notation and main result.

In this section, to state our main result, we introduce a class of operators and a property connected with it which we shall study in this paper. For details on the general theory of pseudodifferential operators consult Hörmander [3] or Kumano-go [10].

Let $D\subset \boldsymbol{R}^2$ be a domain and Δ be an open convex cone in $\boldsymbol{R}^2-\{0\}$. Here and later we call Δ a cone if $t\Delta\subset\Delta$ for any $t>0$. For a compact $K\subset\subset D$, let us define the complex neighborhood $K_{\varepsilon,M,\Delta}$ by $K_{\varepsilon,M,\Delta}\equiv\{(z,\zeta)\in \boldsymbol{C}^2\times\boldsymbol{C}^2;\ \mathrm{dist}(z,K)<\varepsilon,\ |\zeta|>M,\ \mathrm{Re}\,\zeta\in\Delta,\ |\mathrm{Im}\,\zeta|<\varepsilon|\mathrm{Re}\,\zeta|\}$ where M and ε are positive numbers. Let $S^s_\Delta(D,\Delta)$ $(s\in\boldsymbol{R})$ be the set of all functions $a(x,\xi)\in C^\infty(D\times\Delta)$ such that for every compact $K\subset\subset D$ and for every open cone $\Delta'\subset\subset\Delta$ there are $\varepsilon>0$, $M>0$ and $C>0$ for which $a(x,\xi)$ extends to an analytic function on $K_{\varepsilon,M,\Delta'}$ which satisfies $|a(z,\zeta)|\leqq C(1+|\zeta|)^s$ on $K_{\varepsilon,M,\Delta'}$. Here we use $\Delta'\subset\subset\Delta$ for cones Δ' and Δ in the sense that the closure of $\Delta'\cap S^1$ is contained in $\Delta\cap S^1$.

By $S^s_\Delta(D,\boldsymbol{R}^2,\Delta)$ we denote the space of $a(x,\xi)\in S^s_{1,0}(D,\boldsymbol{R}^2)$ such that the restriction $a|_{D\times\Delta}$ to $D\times\Delta$ belongs to $S^s_\Delta(D,\Delta)$. Here $S^s_{1,0}(D,\boldsymbol{R}^2)$ denotes the set of all functions $a(x,\xi)\in C^\infty(D\times\boldsymbol{R}^2)$ such that for every multi-index

k, l there exists $C_{k,l}>0$ for which $a(x, \xi)$ satisfies

$$|(\partial/\partial x)^l(\partial/\partial\xi)^k a(x, \xi)| \leqq C_{k,l}(1+|\xi|)^{s-|k|} \quad \text{on} \quad D\times \boldsymbol{R}^2 .$$

We denote by $SF_A^s(D, \Delta)$ the set of all formal sums $\sum_{j\geqq 0} a_j(x, \xi)$ with $a_j(x, \xi)\in S_A^{s-j}(D, \Delta)$, with the property below:

for every $K\subset\subset D$ and every open cone $\Delta'\subset\subset\Delta$ there are $\varepsilon>0$, $M>0$, $C>0$ and $A>0$ such that each $a_k(x, \xi)$ can be extended to an analytic function on $K_{\varepsilon,M,\Delta'}$ and satisfies

$$|a_k(z, \zeta)| \leqq CA^k k!(1+|\zeta|)^{s-k} \quad \text{on} \quad K_{\varepsilon,M,\Delta'} .$$

For $\sum a_j$, $\sum b_j \in SF_A^s(D, \Delta)$, we write $\sum a_j \sim \sum b_j$ in $SF_A^s(D, \Delta)$ if for every open cone $\Delta'\subset\subset\Delta$ there exist $\varepsilon>0$, $M>0$, $C>0$ and $A>0$ such that

$$|\sum_{j<t}(a_j(z, \zeta)-b_j(z, \zeta))| \leqq CA^t t!(1+|\zeta|)^{s-t} \quad \text{on} \quad K_{\varepsilon,M,\Delta'}$$

for every integer $t>0$.

In case that symbols are independent of the variable x, the classes $S_A^s(D, \Delta)$, $S_A^s(D, \boldsymbol{R}^2, \Delta)$, $SF_A^s(D, \Delta)$ are simply denoted by $S_A^s(\Delta)$, $S_A^s(\boldsymbol{R}^2, \Delta)$, $SF_A^s(\Delta)$ respectively.

REMARK 2.1. Let Δ_1 be an open cone in $\boldsymbol{R}^2-\{0\}$ such that $\Delta_1\subset\subset\Delta$. For $a\in S_A^s(D, \Delta)$, there exists an $a'\in S_A^s(D, \boldsymbol{R}^2, \Delta_1)$ such that $a'-a$ vanishes for (x, ξ) with large $|\xi|$, $\xi\in\Delta_1$.

Indeed, choose non-negative $\omega\in C^\infty(\boldsymbol{R}^2)$ such that $\omega=1$ in $\Delta_1-\{|\xi|\leqq 1\}$ and $\omega=0$ in $(\boldsymbol{R}^2-\Delta)\cup\{|\xi|<1/2\}$. Then $a'(x, \xi)=\omega(\xi)a(x, \xi)$ is in $S_A^s(D, \boldsymbol{R}^2, \Delta_1)$ and $a'-a$ has the desired property.

We define an operator A corresponding to $a(x, \xi)\in S_A^s(D, \Delta)$ by

$$Af(x)=\iint e^{i\langle x-y,\xi\rangle}a(y, \xi)f(y)dy\,đ\xi \tag{2.1}$$

where $đ\xi=(1/2\pi)^2 d\xi_1 d\xi_2$ and a is defined to be zero outside of $D\times\Delta$.

We call the operator A defined by (2.1) the operator corresponding to the symbol $a\in S_A^s(D, \Delta)$. (The operator A may be considered as the transposed ${}^t a(x, D)$ of the analytic pseudodifferential operator $a(x, D)$ associated with $a(x, \xi)$ in a natural way.)

In the following we fix a domain D in $\boldsymbol{R}^2$.

Let S be a simple (closed) analytic curve in D. A real function g on an open neighborhood of $x\in S$ is said to have an analytic extension at x with respect to the curve S if there is a real analytic function h near x such that $g-h=0$ inside of S near x.

DEFINITION 2.2. Let A be a linear operator: $C_0^\infty(D) \to C^\infty(D)$. Let Γ be a (open or closed) convex cone in $\boldsymbol{R}^2 - \{0\}$.

The operator A has Γ-antilocality if the following holds: if $f = Af = 0$ in an open non-empty set $U \subset D$ and if $f \in C_0^\infty(D)$ then $f = 0$ in $(U + \Gamma) \cap D$.

In this case we simply say that A has Γ-[AL]. If $\Gamma = \boldsymbol{R}^2 - \{0\}$ in the above definition, A is simply said to have the antilocality ([AL] for short).

Let $\Gamma \subset\subset \boldsymbol{R}^2 - \{0\}$ be a closed convex cone. Let Γ^* be another open convex cone given by $\Gamma^* = \mathrm{int}(\Gamma^\circ)$; the interior of the dual cone of Γ. Here the dual cone Γ° of Γ is given by $\Gamma^\circ = \{y \in \boldsymbol{R}^2 ;\ \langle y, x\rangle \geqq 0$ for every $x \in \Gamma\}$. The generator A of two-dimensional stable process (called the stable generator) with index $\alpha \in (0, 1) \cup (1, 2)$ whose *Lévy measure* is supported on Γ is the operator corresponding to the symbol $a(\xi)$ given by

$$a(\xi) = -Ce^{-i\alpha\pi/2}\int_{S^1 \cap \Gamma \cap \{\theta; \langle x, \xi\rangle > 0\}} |\langle x, \xi\rangle|^\alpha d\theta - Ce^{i\alpha\pi/2}\int_{S^1 \cap \Gamma \cap \{\theta; \langle x, \xi\rangle < 0\}} |\langle x, \xi\rangle|^\alpha d\theta \ , \tag{2.2}$$

where $x = (x_1, x_2) = (\cos\theta, \sin\theta)$ and $\xi = (\xi_1, \xi_2) = (\rho\cos\varphi, \rho\sin\varphi)$ (see Komatsu [9, sect. 11]).

We observe that the symbol $a(\xi)$ enjoys the following three properties:

(A1) $a(\xi) \in S_\Delta^\alpha(\Delta \cup -\Delta)$ for every open convex cone $\Delta \subset\subset \boldsymbol{R}^2 - \{0\}$,

(A2) $a(t\xi) = t^\alpha a(\xi)$ for $|\xi| > 0$, $t > 0$,

(A3) $a(\xi)$ has a hyperfunction expression

$$a(\xi) = A_+(\xi + i\Gamma^* 0) - A_-(\xi - i\Gamma^* 0) \ ,$$

where $A_\pm(\zeta)$ are holomorphic in wedges of type $\boldsymbol{R}^2 \pm i\Gamma^* 0$.

In fact, $A_\pm(\zeta)$ are given by

$$A_\pm(\zeta) = \mp C\int_{S^1 \cap \Gamma \cap \{\theta; \pm\langle x, \xi\rangle > 0\}} \exp \alpha \log\langle x, \zeta\rangle d\theta$$

with the branch of logarithm so chosen that $\log\langle x, \zeta\rangle|_{\zeta=\xi} = \log|\langle x, \xi\rangle| - i\pi/2$ if $\langle x, \xi\rangle > 0$. Note that $A_\pm(\xi \pm i\Gamma^* 0)$ does not vanish if and only if $\xi \in S^1 - (\mp\Gamma^\circ)$.

The properties (A1), (A2) imply that $a(\xi) \in C^\infty(\boldsymbol{R}^2 - \{0\})$ with the growth order α at infinity. By choosing $\omega \in C^\infty(\boldsymbol{R}^2)$ such that $\omega = 1$ in $\{|\xi| \geqq 1\}$ and $\omega = 0$ in $\{|\xi| < 1/2\}$, we can rewrite the operator A as

$$Af(x) = \iint e^{i\langle x-y, \xi\rangle}\omega(\xi)a(\xi)f(y)dy\bar{d}\xi + \iint e^{i\langle x-y, \xi\rangle}(1-\omega(\xi))a(\xi)f(y)dy\bar{d}\xi \ ,$$

where $\omega \cdot a \in S^{\alpha}_{1,0}(\boldsymbol{R}^2)$. Observe that, for $f \in C_0^\infty(D)$, the first term is C^∞ in D and that the second term is real analytic in D. Hence A defines an operator from $C_0^\infty(D)$ to $C^\infty(D)$.

REMARK 2.3. In case $\Gamma=\boldsymbol{R}^2-\{0\}$ in (2.2), the symbol $a(\xi)$ is reduced to $-C_1\rho^\alpha$, and so A is the constant multiple of $(-\Delta)^{\alpha/2}$. In this case the antilocality (to all directions) is well known, cf. Goodman-Segal [2], Murata [12].

Now we have our main theorem:

THEOREM 2.4. *Let Γ be a closed convex cone in $\boldsymbol{R}^2-\{0\}$ and let A be the stable generator with index $\alpha \in (0, 1) \cup (1, 2)$ whose Lévy measure is supported on Γ. If Γ is proper then A has Γ-[AL]. If $\Gamma \neq \boldsymbol{R}^2-\{0\}$ then A does not have [AL].*

An example which shows that A is not antilocal in case $\Gamma \neq \boldsymbol{R}^2-\{0\}$ is easily given: one has only to take $f \in C_0^\infty(D)$ such that $f \not\equiv 0$ and $\operatorname{supp} f \cap (U+\Gamma)=\varnothing$. And so in the following we assume that Γ is proper.

§ 3. Proof of the main result (I).

In this section we shall show the following

LEMMA 3.1. *Let x be a point in a simple (closed) analytic curve $S=\{s(x)=0\}$ in D. Let $f \in C_0^\infty(D)$ be such that $V \cap \operatorname{supp} f \subset \{s(x) \geqq 0\}$ ($\equiv$ the outside of S) for some neighborhood V of x.*

Let A: $C_0^\infty(D) \to C^\infty(D)$ be the operator given in Theorem 2.4. Assume that the unit outer normal ν_x of S at x is in $S^1-(-\Gamma^\circ)$ and that Af has an analytic extension at x with respect to the curve S, then $x \notin \operatorname{supp} f$.

Before beginning the proof, we shall remark a special property of the operators with symbols in $S^{\alpha}_{\Delta}(D, \boldsymbol{R}^2, \Delta)$.

LEMMA 3.2 (cf. Liess [1, Theorems 2.6, 2.7]). *The operator A with symbol $a(x, \xi)$ in $S^{\alpha}_{\Delta}(D, \boldsymbol{R}^2, \Delta)$ has Δ-analytic pseudolocal property (Δ-[APL] for short), that is,*

$$\mathrm{SS}\, Af \cap (D \times \Delta) \subset \mathrm{SS}\, f\,.$$

In view of Remark 2.1, the statement just above also holds for operators with symbols in $S^{\alpha}_{\Delta}(D, \Delta)$.

The rest of this section is devoted to the proof of Lemma 3.1.

By performing a linear coordinate transformation, we may consider the case $\nu_x=e\equiv(0, 1)$ and $x=0$. Choose an open cone $\Delta_1 \subset\subset \boldsymbol{R}^2-\{0\}-(-\Gamma^\circ)$

such that $e\in\Delta_1$ and that angle$(\Delta_1)<\pi/2$.

Let V be an open connected neighborhood of 0 and let Φ: $x=\Phi(y)$ be an analytic coordinate transformation on V such that $\Phi(0)=0$, $(d\Phi/dy)_0=I$ and $\Phi^{-1}(S\cap V)\subset\{y_2=0\}$. Let $\varphi=_{\mathrm{def}}\Phi^*f$. Then $\Psi=_{\mathrm{def}}\Phi^*(Af)$ can be written

$$\Psi(y)=\iint e^{i\langle y-y',\eta\rangle}b(y',\eta)\varphi(y')dy'đ\eta \tag{3.1}$$

with $b(y',\eta)=a(((d\Phi/dy')^t_{y'})^{-1}\eta)$, where $((d\Phi/dy')^t_{y'})^{-1}$ denotes the inverse of transposed matrix $(d\Phi/dy')^t$ evaluated at y' (cf. Kumano-go [10, Chap. 2, Sect. 6]). Since $a(\xi)\in S^\alpha_A(\Delta_1\cup-\Delta_1)$, $\Phi(0)=0$, $(d\Phi/dy)_0=1$ and $y\mapsto(d\Phi/dy)_y$ is analytic, $b(y,\eta)\in S^\alpha_A(V_1,\Delta_2\cup-\Delta_2)$ for some $V_1\subset\subset V$ and some open cone Δ_2 such that $e\in\Delta_2\subset\subset\Delta_1$.

We recall the hyperfunction expression of the symbol $a(\xi)$:

$$a(\xi)=A_+(\xi+i\Gamma^*0)-A_-(\xi-i\Gamma^*0)\,.$$

Here $A_\pm(\zeta)$ are holomorphic in wedges of type $\boldsymbol{R}^2\pm i\Gamma^*0$ respectively. Hence, denoting $\xi=\Psi_y\eta$, $\Psi_y=((d\Phi/dy)^t_y)^{-1}$, $b(y,\eta)$ has the expression

$$b(y,\eta)=\Psi^*_yA_+(\eta+i\Psi^{-1}_y\Gamma^*0)-\Psi^*_yA_-(\eta-i\Psi^{-1}_y\Gamma^*0)\,.$$

Here $\Psi^*_yA_\pm(\zeta)$ are holomorphic in wedges of type $\boldsymbol{R}^2\pm i\Psi^{-1}_y\Gamma^*0$ respectively.

In the sequel we divide the discussion into two cases according to the angle of Γ.

Case I. angle $\Gamma\leqq\pi/2$.

In this case, we observe $\{x;\langle e,x\rangle>0\}\cap(\Gamma^*)\neq\emptyset$ if $e\in S^1-(-\Gamma^\circ)$. Hence, since $\Psi_0=I$, and $y\mapsto\Psi^{-1}_y\Gamma^*$ is continuous,

$$\{x;\langle e,x\rangle>0\}\cap(\Psi^{-1}_y\Gamma^*)\neq\emptyset \tag{3.2}$$

for y in a small neighborhood $V_2\subset\subset V_1$ of zero.

Fix $y\in V_2$. Since $b(y,\eta)\in S^\alpha_A(V_2,\Delta_2\cup-\Delta_2)$, $b(y,\eta)$ can be extended to a holomorphic function $d(y,\zeta)$ in a complex neighborhood U^0 of Δ_2: $d(y,\zeta)=\Psi^*_yA_+(\zeta)-\Psi^*_yA_-(\zeta)$, $\zeta\in U^0$. Here we can use the analytic continuation of $d(y,\cdot)$ along curves in wedges of type $W\equiv(\Delta_2\cup-\Delta_2)-i(\{x;\langle e,x\rangle>0\}\cap\Psi^{-1}_y\Gamma^*)0$ from U^0 to extend the domain of $d(y,\cdot)$ to $-\Delta_2$. This is possible since $d(y,\zeta)$ does not vanish on W and W is simply connected. Then, on $-\Delta_2$, we have

$$\begin{aligned} d(y,\eta)&=e^{-i\alpha\pi}\Psi^*_yA_+(-\eta+i\Psi^{-1}_y\Gamma^*0)-\Psi^*_yA_-(\eta-i\Psi^{-1}_y\Gamma^*0)\\ &=e^{-i\alpha\pi}(\Psi^*_yA_+(-\eta+i\Psi^{-1}_y\Gamma^*0)-\Psi^*_yA_-(-\eta-i\Psi^{-1}_y\Gamma^*0))\\ &=e^{-i\alpha\pi}b(y,-\eta)\,. \end{aligned} \tag{3.3}$$

Since $b(y, \eta) \in S_A^\alpha(V_2, \Delta_2 \cup -\Delta_2)$, we see that $d(y, \eta)$ is also in $S_A^\alpha(V_2, \Delta_2 \cup -\Delta_2)$, and hence

$$\begin{aligned} b(y, \eta)-d(y, \eta) &= \Psi_y^* A_+(\eta+i\Psi_y^{-1}\Gamma^*0)-e^{-i\alpha\pi}\Psi_y^* A_+(-\eta+i\Psi_y^{-1}\Gamma^*0) \\ &= (e^{i\alpha\pi}-e^{-i\alpha\pi})\Psi_y^* A_+(-\eta+i\Psi_y^{-1}\Gamma^*0) \end{aligned}$$

defines a non-zero symbol in $S_A^\alpha(V_2, -\Delta_2)$, since $e \in S^1-(-\Gamma^\circ)$.

Let $\omega=\omega_{\Delta_2}$ be the cut-off function in Remark 2.1. Put $d'(y, \eta)=\omega(\eta)d(y, \eta)$. Then we see that $d' \in S_A^\alpha(V_2, \boldsymbol{R}^2, \Delta_2 \cup -\Delta_2)$ and that $\omega(b-d')$ defines a non-zero symbol in $S_A^\alpha(V_2, \boldsymbol{R}^2, -\Delta_2)$.

Here we prepare the following lemma. Let Δ be an open conic neighborhood of $e=(0, 1)$. Let $K \subset\subset V$ be a compact set such that $0 \in \operatorname{int} K$.

LEMMA 3.3 (Liess [11, Proposition 4.1]). *Let $b(y, \xi) \in S_A^s(V, \boldsymbol{R}^2, \Delta \cup -\Delta)$. Suppose there are constants ε, c, A, M such that b is analytic on $K_{\varepsilon,M,\Delta\cup-\Delta}$ and that for every $(y, \xi_1) \in K\times\boldsymbol{R}$ the function $\xi_2 \mapsto b(y, (\xi_1, \xi_2))$ can be extended analytically for $|\zeta_2| \geqq M(1+|\xi_1|)$, $\operatorname{Im}\zeta_2 \leqq 0$ and satisfy the estimate*

$$|b(y, (\xi_1, \zeta_2))| \leqq cA(1+|(\xi_1, \zeta_2)|)^s$$

on that set.

Let $f \in C_0^\infty(V)$, $\operatorname{supp} f \subset \{x; x_2 \geqq 0\} \cap K$. Then there are constants γ and C such that for every $g \in \mathscr{S}(\boldsymbol{R})$ which satisfies $|\hat{g}(\xi_1)| \leqq \exp(-\gamma|\xi_1|)$, it follows that

$$\left| D_x^\alpha \iint e^{i\langle x-y, \xi\rangle} \hat{g}(\xi_1) b(y, \xi) f(y) dy d\!\!\!{}^-\xi \right| \leqq C(C|\alpha|)^{|\alpha|}$$

if $x \in K$ and $x_2<0$.

Set

$$G(x) \underset{\text{def}}{=} \iint e^{i\langle x-y, \xi\rangle} \hat{g}(\xi_1) b(y, \xi) f(y) dy d\!\!\!{}^-\xi$$

for the function g in the lemma. Since g is real analytic and

$$G(x)=g *' \iint e^{i\langle x-y, \xi\rangle} b(y, \xi) f(y) dy d\!\!\!{}^-\xi ,$$

where $*'$ denotes the convolution in x_1, the lemma implies that G is real analytic on $\{x_2<0\}$ and has an analytic extension at 0 with respect to $\{x_2=0\}$.

Now we begin the proof in case I. In the following we assume that f is a function in $C_0^\infty(D)$. We write $f=f_1+f_2$ with $f_1, f_2 \in C_0^\infty(D)$ such that

$\operatorname{supp} f_1 \subset \{|x|<\varepsilon\}$ and $\operatorname{supp} f_2 \subset \{|x| \geqq \varepsilon/2\}$. Here $\varepsilon>0$ is such that $V_2 \supset \{|x|<\varepsilon\}$. Put $\varphi_1=\Phi^* f_1$, $\varphi_2=\Phi^* f_2$.

Now we shall decompose

$$\begin{aligned}
&\iint e^{i\langle y-y', \eta\rangle}(b(y', \eta)-d'(y', \eta))\varphi(y')dy'd\!\!{}^-\eta \\
&\quad=\iint e^{i\langle y-y', \eta\rangle}(b(y', \eta)-d'(y', \eta))\varphi_1(y')dy'd\!\!{}^-\eta \\
&\qquad+\iint e^{i\langle y-y', \eta\rangle}b(y', \eta)\varphi_2(y')dy'd\!\!{}^-\eta \\
&\qquad-\iint e^{i\langle y-y', \eta\rangle}d'(y', \eta)\varphi_2(y')dy'd\!\!{}^-\eta \\
&\quad:=u_1+u_2-u_3 \quad \text{(say)}.
\end{aligned}$$

Since $d'(y, \eta) \in S^{\alpha}_{\Delta}(V_2, \boldsymbol{R}^2, \Delta_2 \cup -\Delta_2)$, in view of (3.2) the assumption of Lemma 3.3 is satisfied with $V=V_2$, $K=\{|x| \leqq \varepsilon\}$ and $\Delta=\Delta_2$. Hence $g*'\left(\iint e^{i\langle y-y', \eta\rangle}d'(y', \eta)\varphi_1(y')dy'd\!\!{}^-\eta\right)$ is real analytic in $\{|y|<\varepsilon', y_2<0\}$ for some $\varepsilon'>0$ and has an analytic extension at 0. Here $g \in \mathscr{S}(\boldsymbol{R})$ is the function in the lemma.

By the assumption that Af has an analytic extension h at 0 with respect to S, $\psi=\Phi^*(Af)$ has an analytic extension $h_1=\Phi^* h$ at 0 with respect to $\{y_2=0\}$ which coincides with ψ in $\{|y|<\varepsilon', y_2<0\}$. Hence

$$\begin{aligned}
g*'\Big(&\iint e^{i\langle y-y', \eta\rangle}(b(y', \eta)-d'(y', \eta))\varphi_1(y')dy'd\!\!{}^-\eta \\
&+\iint e^{i\langle y-y', \eta\rangle}b(y', \eta)\varphi_2(y')dy'd\!\!{}^-\eta-h_1\Big)
\end{aligned}$$

has an analytic extension w in $\{|y|<\varepsilon'\}$.

On the other hand, it follows that

$$(0, iedx\infty) \notin \mathrm{SS}\, u_1 \text{ and hence } (0, iedx\infty) \notin \mathrm{SS}(g*'u_1)\,. \tag{3.4}$$

Indeed, take any closed conic neighborhood $\Delta^\circ \subset\subset \Delta_2$ of e. Then, since $b(y, \eta)-d'(y, \eta)=0$ on $V_2 \times \Delta^\circ$, the result follows from the Paley-Wiener-Schwartz theorem ([6, Corollary 8.5.6]).

By Lemma 3.2 we see

$$(0, iedx\infty) \notin \mathrm{SS}\, u_2, \text{ and hence } (0, iedx\infty) \notin \mathrm{SS}(g*'u_2)\,. \tag{3.5}$$

From (3.4), (3.5) we observe that $v=_{\mathrm{def}} w-g*'(u_1+u_2-h_1)$ vanishes for $y_2 \leqq 0$ near zero and satisfies $(0, iedx\infty) \notin \mathrm{SS}\, v$. We apply the Kashiwara-Kawai theorem (K-K theorem for short, [6, Theorem 4.4.1], [3, Chap.

VIII]) to see that v vanishes near zero. Hence

$$(3.6)\qquad (0, -iedx\infty)\notin \mathrm{SS}(g*'(u_1+u_2)) .$$

It also follows from Lemma 3.2 that

$$(3.7)\qquad (0, -iedx\infty)\notin \mathrm{SS}\,u_3, \text{ and hence } (0, -iedx\infty)\notin \mathrm{SS}(g*'u_3) .$$

Combining these we obtain

$$(3.8)\qquad \begin{aligned}(0, -iedx\infty)&\notin \mathrm{SS}(g*'(u_1+u_2-u_3))\\ &=\mathrm{SS}\Big(\iint e^{i\langle y-y',\eta\rangle}\hat{g}(\eta_1)(b(y',\eta)-d'(y',\eta))\varphi(y')dy'd\!\!^-\eta\Big) .\end{aligned}$$

We see that $\omega(\eta)(b(y,\eta)-d'(y,\eta))$ defines a symbol in $S^\alpha_A(V_2, \boldsymbol{R}^2, -\Delta_2)$. Putting $R(y,\eta)\equiv(b(y,\eta)-d'(y,\eta))^{-1}\in S^{-\alpha}_A(V_2, -\Delta_2)$, this satisfies $\omega(\eta)(b(y,\eta)-d'(y,\eta))\cdot R(y,\eta)\sim 1$ in $SF^0_A(V_2, \Delta_2)$. So applying the regularity theorem ([11, Theorem 2.8], [7, Cor. 4.1.6]), we have

$$(3.9)\qquad \begin{aligned}(0, -iedx\infty)&\notin \mathrm{SS}\Big(\int e^{i\langle y,\eta\rangle}\hat{g}(\eta_1)\hat{\varphi}(\eta)d\!\!^-\eta\Big)\\ &=\mathrm{SS}(g*'\varphi) ,\end{aligned}$$

which in turn implies $(0, -iedx\infty)\notin \mathrm{SS}\,\varphi$. It follows $0\notin \operatorname{supp}\varphi$ by the K-K theorem, and hence $0\notin \operatorname{supp} f$. This establishes the proof in case I.

Case II. $\pi/2<\text{angle}\,\Gamma<\pi$.

In this case, we divide Γ into the direct sum of two closed cones Γ_1, Γ_2 with each angle $<\pi/2$: $\Gamma=\Gamma_1\oplus\Gamma_2$.

Let $\Gamma^*_j=_{\mathrm{def}}\operatorname{int}(\Gamma^\circ_j)$ $(j=1, 2)$. We then see $\Gamma^*=\Gamma^*_1\cap\Gamma^*_2$ and $\text{angle}(\Gamma^*_1+\Gamma^*_2)=\pi$. Put the half space $H=_{\mathrm{def}}\overline{\Gamma^*_1+\Gamma^*_2}$. We divide $\boldsymbol{R}^2-\{0\}-(-\Gamma^\circ)$ into disjoint three parts:

$$\begin{aligned}\boldsymbol{R}^2-\{0\}-(-\Gamma^\circ)&=H+CH-(-\Gamma^\circ_2)+CH-(-\Gamma^\circ_1)\\ &:=H+J+K , \qquad \text{here } CH=\boldsymbol{R}^2-H .\end{aligned}$$

As in case I, we observe that $\{x; \langle e, x\rangle>0\}\cap(\Psi_y^{-1}\Gamma^*_j)\neq\emptyset$ for $e\in S^1-(-\Gamma^\circ_j)$ $(j=1, 2)$ and for y in a small neighborhood $V_2\subset\subset V_1$ of zero. Choose the conic neighborhood Δ_2 so small that $\Delta_2\subset\boldsymbol{R}^2-\{0\}-(-\Gamma^\circ)$ (resp. $\Delta_2\subset K, J$) in case $e\in H$ (resp. $e\in K, J$).

Let $a_j(\xi)$ be the symbol of the stable generator whose Lévy measure is supported on Γ_j given by the expressions (2.2), (2.3) with Γ replaced by Γ_j:

$$a_j(\xi)=A_{j+}(\xi+i\Gamma^*_j0)-A_{j-}(\xi-i\Gamma^*_j0)\qquad (j=1, 2) .$$

Obviously $a(\xi)=a_1(\xi)+a_2(\xi)$.

Let $b_j(y, \eta)$ denote the modification obtained from $a_j(\xi)$ in an obvious way. Fix $y \in V_2$. Then as in case I, $b_j(y, \eta)$ permits a holomorphic extension to a complex neighborhood U^0 of Δ_2: $d_j(y, \zeta)=\Psi_y^* A_{j+}(\zeta)-\Psi_y^* A_{j-}(\zeta)$, $\zeta \in U^0$. Put $b=b_1+b_2$ and $d=d_1+d_2$.

(i) In case $e \in H$, we can perform the analytic continuation of d along curves in wedges of type $(\Delta_2 \cup -\Delta_2)-i(\{x;\ \langle e, x\rangle>0\} \cap \Psi_y^{-1}\Gamma^*)0$ from U^0 as in case I. On $-\Delta_2$ we have $d(y, \eta)=e^{-i\alpha\pi}b(y, -\eta)$, and hence

$$\begin{aligned} b(y, \eta)-d(y, \eta)&=(e^{i\alpha\pi}-e^{-i\alpha\pi})(\Psi_y^* A_{1+}+\Psi_y^* A_{2+})(-\eta+i\Psi_y^{-1}\Gamma^*0)\\ &=(e^{i\alpha\pi}-e^{-i\alpha\pi})\Psi_y^* A_+(-\eta+i\Psi_y^{-1}\Gamma^*0) \end{aligned}$$

defines a non-zero symbol in $S_A^\alpha(V_2, -\Delta_2)$, since $S^1 \cap \Delta_2 \subset S^1-(-\Gamma^\circ)$.

(ii) In case $e \in K$, we have $\{x;\ \langle e, x\rangle > 0\} \cap (\Psi_y^{-1}\Gamma_1^*) \neq \emptyset$ and $\{x;\ \langle e, x\rangle<0\} \cap (\Psi_y^{-1}\Gamma_2^*) \neq \emptyset$ for $y \in V_2$. And we perform the analytic continuation of $d_1(y, \cdot)$ (resp. $d_2(y, \cdot)$) along curves in wedges of type $(\Delta_2 \cup -\Delta_2)-i(\{x;\ \langle e, x\rangle>0\} \cap \Psi_y^{-1}\Gamma_1^*)0$ (resp. $(\Delta_2 \cup -\Delta_2)+i(\{x;\ \langle e, x\rangle<0\} \cap \Psi_y^{-1}\Gamma_2^*)0$) from U^0. On $-\Delta_2$ we have $d_1(y, \eta)=e^{-i\alpha\pi}b_1(y, -\eta)$ and $d_2(y, \eta)=e^{-i\alpha\pi}b_2(y, \eta)$. And so

$$\begin{aligned} b(y, \eta)-d(y, \eta)&=(b_1(y, \eta)-d_1(y, \eta))+(b_2(y, \eta)-d_2(y, \eta))\\ &=(\Psi_y^* A_{1+}(\eta+i\Psi_y^{-1}\Gamma_1^*0)-e^{-i\alpha\pi}\Psi_y^* A_{1+}(-\eta+i\Psi_y^{-1}\Gamma_1^*0)\\ &\quad -\Psi_y^* A_{1-}(\eta-i\Psi_y^{-1}\Gamma_1^*0)+e^{-i\alpha\pi}\Psi_y^* A_{1-}(-\eta-i\Psi_y^{-1}\Gamma_1^*0))\\ &\quad +(\Psi_y^* A_{2+}(\eta+i\Psi_y^{-1}\Gamma_2^*0)-e^{-i\alpha\pi}\Psi_y^* A_{2+}(-\eta+i\Psi_y^{-1}\Gamma_2^*0)\\ &\quad -\Psi_y^* A_{2-}(\eta-i\Psi_y^{-1}\Gamma_2^*0)+e^{-i\alpha\pi}\Psi_y^* A_{2-}(-\eta-i\Psi_y^{-1}\Gamma_2^*0))\\ &=\Psi_y^* A_{1+}(\eta+i\Psi_y^{-1}\Gamma_1^*0)-e^{-i\alpha\pi}\Psi_y^* A_{1+}(-\eta+i\Psi_y^{-1}\Gamma_1^*0)\\ &\quad -(\Psi_y^* A_{2-}(\eta-i\Psi_y^{-1}\Gamma_2^*0)-e^{-i\alpha\pi}\Psi_y^* A_{2-}(-\eta-i\Psi_y^{-1}\Gamma_2^*0))\\ &=(e^{i\alpha\pi}-e^{-i\alpha\pi})(\Psi_y^* A_{1+}(-\eta+i\Psi_y^{-1}\Gamma_1^*0)-\Psi_y^* A_{2-}(-\eta-i\Psi_y^{-1}\Gamma_2^*0))\ , \end{aligned}$$

which also defines a non-zero symbol in $S_A^\alpha(V_2, -\Delta_2)$, since $S^1 \cap \Delta_2 \subset (S^1-(-\Gamma_1^\circ)) \cap (S^1-(\Gamma_2^\circ))$.

(iii) In case $e \in J$, by the similar argument as in (ii), we see that $b(y, \eta)-d(y, \eta)=(e^{i\alpha\pi}-e^{-i\alpha\pi})(\Psi_y^* A_{2+}(-\eta+i\Psi_y^{-1}\Gamma_2^*0)-\Psi_y^* A_{1-}(-\eta-i\Psi_y^{-1}\Gamma_1^*0))$ defines a non-zero symbol in $S_A^\alpha(V_2, -\Delta_2)$, since $S^1 \cap \Delta_2 \subset (S^1-(-\Gamma_2^\circ)) \cap (S^1-(\Gamma_1^\circ))$.

Let $\omega=\omega_{\Delta_2}$ be the cut-off function in Remark 2.1. In any case we see that $b \in S_A^\alpha(V_2, \Delta_2 \cup -\Delta_2)$, $d'=_{\mathrm{def}}\omega \cdot d \in S_A^\alpha(V_2, \boldsymbol{R}^2, \Delta_2 \cup -\Delta_2)$ and that $\omega(b-d')$ defines a non-zero symbol in $S_A^\alpha(V_2, \boldsymbol{R}^2, -\Delta_2)$. The arguments for the proof in case I are also valid in this case, and so we have the desired result. This establishes the proof of Lemma 3.1. Q.E.D.

§ 4. Proof of the main result (II).

In this section we shall prove the theorem.

Let U be an open non-empty set in D. Let $f \in C_0^\infty(D)$ and assume that $f=Af=0$ in U. We show $f\equiv 0$ in $(U+\Gamma)\cap D$. Since U is open, for $x_0 \in U$ given, we can find $\varepsilon>0$ with $B(x_0, \varepsilon)\equiv\{x; |x-x_0|<\varepsilon\}\subset\subset U$. We have only to show

(1) for every $x_0 \in U$ and any proper open convex cone $\Gamma'\subset\subset\Gamma$,

$$f\equiv 0 \qquad \text{in} \quad (x_0+\Gamma')\cap D\,.$$

Indeed, since $x_0 \in U$ and $\Gamma'\subset\subset\Gamma$ are arbitrary in (1) and since f is C^∞ in D, we obtain

(2) $f\equiv 0$ in $(U+\Gamma)\cap D$.

Further we may assume $x_0=0$ for simplicity.

To "sweep out" the first quadrant $\left(=\operatorname{span}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)\right)$, we choose a family $(F_\lambda)_{\lambda>0}$ of curves *"folium cartesii"* on it, which are defined by

$$\begin{aligned} F_\lambda(x_1, x_2) &= x_1^3+x_2^3-3\lambda x_1x_2 \qquad (\lambda>0)\\ &=0\,. \end{aligned}$$

These curves can be expressed as

$$\left\{(x_1(t), x_2(t))\,;\; x_1(t)=\frac{3\lambda t}{1+t^3},\; x_2(t)=\frac{3\lambda t^2}{1+t^3}\;(0<t<+\infty)\right\}\,.$$

Let T be the linear transformation which maps Γ' to the first quadrant. Such a transformation exists and can be represented by the matrix $M=\begin{pmatrix}a_1 & a_2\\ b_1 & b_2\end{pmatrix}^{-1}$, where $\begin{pmatrix}a_1\\ b_1\end{pmatrix}$, $\begin{pmatrix}a_2\\ b_2\end{pmatrix}$ are unit vectors on each edge of Γ'. Let $G_\lambda(x_1, x_2) =_{\text{def}} T^*F_\lambda(x_1, x_2)$. Then $G_\lambda=\{(x_1, x_2)\in \boldsymbol{R}^2; G_\lambda(x_1, x_2)=0\}$ is a (closed) analytic curve in Γ' which fills Γ' as λ runs from 0 to $+\infty$. Moreover we observe that the set $\{((x_1, x_2); dG_\lambda(x_1, x_2)); G_\lambda(x_1, x_2)=0\}$ of outer normal vectors is contained in $G_\lambda\times(\boldsymbol{R}^2-\{0\}-(-\Gamma^\circ))$ for every $\lambda\in(0, +\infty)$.

If $\lambda>0$ is taken sufficiently small, the interior of G_λ is contained in $B(0, \varepsilon)$, and hence both f and Af vanish there by the assumption. We may employ so-called "sweeping-out method": Now

(i) $\{G_\lambda; 0<\lambda<+\infty\}$ is a family of analytic curves in Γ' which sweeps Γ'.

(ii) f and Af vanish in the closure of $\cup_{\lambda'<\lambda} G_{\lambda'}$ for small $\lambda>0$ (since f and Af are C^∞).

Assume that $\lambda_0=\sup\{\lambda\in(0, +\infty); f$ and Af vanish in $\cup_{\lambda'<\lambda}G_{\lambda'}\}<+\infty$.

Take arbitrary $z \in G_{\lambda_0}$. Then the assumption of Lemma 3.1 is satisfied with $x=z$ and $dS(x)=dG_{\lambda_0}(z)=\xi \in \boldsymbol{R}^2-\{0\}-(-\Gamma^\circ)$, and so $z \notin \operatorname{supp} f$. This in turn implies $z \notin \operatorname{supp} Af$ in view of (A1) and Lemma 3.2.

Since z is arbitrary in a compact set $K \subset\subset G_{\lambda_0}-B(0, \varepsilon/2)$, and since f and Af vanish in $B(0, \varepsilon) \supset\supset B(0, \varepsilon/2)$, this implies that $f=Af=0$ in $\cup_{\lambda'<\lambda_0+\eta} G_{\lambda'}$ for some $\eta>0$. This contradicts the definition of λ_0. Hence $\lambda_0=+\infty$. This proves (1) and the proof of the theorem is established.

Q.E.D.

Appendix.

Consider the pseudodifferential operator A corresponding to the symbol $a(\xi)$ which satisfies the assumptions (A1)~(A3) in § 2. Generalizing the arguments in § 3 and using the same argument as in § 4, we can prove the following

PROPOSITION. *Assume that the open convex cone in* (A3) *satisfies* angle $\Gamma^* \geqq \pi/2$. *If a satisfies the condition*

(AP1) $$a(-\xi)-e^{-i\alpha\pi}a(\xi) \neq 0 \qquad \text{for all} \quad \xi \in S^1-(-\bar{\Gamma}^*),$$

then A has $(\Gamma^*)^\circ$-[AL]. Here $(\Gamma^*)^\circ$ denotes the dual cone of Γ^*.

Indeed the symbol $a(\xi)$ of stable generator with index $\alpha \in (0, 1) \cup (1, 2)$ whose Lévy measure is supported on $\Gamma=(\Gamma^*)^\circ$ (in this case angle$(\Gamma^*)^\circ \leqq \pi/2$) satisfies the condition (AP1), since

$$\begin{aligned} &e^{-i\alpha\pi}a(\xi) \\ &= -Ce^{-i3\alpha\pi/2}\int_{S^1\cap\Gamma\cap\{\theta;\langle x,\xi\rangle>0\}}|\langle x, \xi\rangle|^\alpha d\theta - Ce^{-i\alpha\pi/2}\int_{S^1\cap\Gamma\cap\{\theta;\langle x,\xi\rangle<0\}}|\langle x, \xi\rangle|^\alpha d\theta \\ &\neq -Ce^{i\alpha\pi/2}\int_{S^1\cap\Gamma\cap\{\theta;\langle x,\xi\rangle>0\}}|\langle x, \xi\rangle|^\alpha d\theta - Ce^{-i\alpha\pi/2}\int_{S^1\cap\Gamma\cap\{\theta;\langle x,\xi\rangle<0\}}|\langle x, \xi\rangle|^\alpha d\theta \\ &= a(-\xi) \qquad (\xi \in S^1-(-\bar{\Gamma}^*)) . \end{aligned}$$

When the angle of $\Gamma^* < \pi/2$, the sufficient condition for $(\Gamma^*)^\circ$-antilocality will be more complicated.

ACKNOWLEDGEMENT. The author would like to express his hearty thanks to the referee for his invaluable advices.

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