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Compact Weighted Composition Operators on Certain Subspaces of $C(X, E)$

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§1. Introduction and results.

Let X be a compact Hausdorff space and E a complex Banach space with the norm $\|\cdot\|_E$. By $C(X, E)$ we denote the Banach space of all continuous E-valued functions on X with the usual norm; $\|f\|=\sup\{\|f(x)\|_E : x\in X\}$. When E is the complex field C, we use $C(X)$ in place of $C(X, C)$. Let A be a function algebra on X, that is, a closed subalgebra of $C(X)$ which contains the constants and separates points of X. We define the space $A(X, E)$ by

$$A(X, E)=\{f\in C(X, E) : e^*\circ f\in A \text{ for all } e^*\in E^*\},$$

where E^* is the dual space of E. Clearly $A(X, E)$ is a Banach space relative to the same norm. For example, as a generalization of the disc algebra $A(\bar{D})$ on the closed unit disc $\bar{D}$, we may consider the space $\{f\in C(\bar{D}, E) : f$ is an analytic E-valued function on the open unit disc $D\}$. Here f is said to be analytic on D when it is differentiable at each point of D, in the sense that the limit of the usual difference quotient exists in the norm topology. It is known that this space coincides the following space;

$$\{f\in C(\bar{D}, E) : e^*\circ f\in A(\bar{D}) \text{ for all } e^*\in E^*\}$$

(see [2, p. 126]). The above definition of $A(X, E)$ is abstracted from this property.

We investigate weighted composition operators on $A(X, E)$. A weighted composition operator on $A(X, E)$ is a bounded linear operator T from $A(X, E)$ into itself, which has the form;

$$Tf(x)=w(x)f(\varphi(x)), \qquad x\in X,\ f\in A(X, E),$$

for some selfmap φ of X and some map w from X into $B(E)$, the space of bounded linear operators on E. We write wC_φ in place of T.

Weighted composition operators or composition operators on $C(X, E)$ were studied in [3] and [6], and the case of $E=C$ was considered by Kamowitz [4], Uhlig [8],

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and others. In particular, Theorem 2 of [3] gave the necessary and sufficient conditions for a weighted composition operator on $C(X, E)$ to be compact. In this paper we shall prove an analogue for compact weighted composition operators on $A(X, E)$, which includes results of [7] in the function algebra setting. At the same time, we remove one condition given in [3, Theorem 2]. We also see that there is no compact composition operator on $A(X, E)$, if E is infinite dimensional.

We begin with some notation and terminology on a function algebra A. By M_A we denote the maximal ideal space of A. For each $f \in A$, we put $\hat{f}(m) = m(f)$ for all $m \in M_A$. We consider X as a compact subset of M_A and a selfmap of X as a map from X into M_A. Also we note that M_A is decomposed into (Gleason) parts $\{P_\lambda\}$ for A such that $M_A = \bigcup_\lambda P_\lambda$, and $P_\lambda \cap P_\mu = \emptyset$ $(\lambda \neq \mu)$. For a non-trivial (not a one-point) part P, we consider the following condition;

(α) for any x in P, there are an open neighborhood V of x relative to P and a homeomorphism ρ from a polydisc D^N (N depends on x) onto V such that $\hat{f} \circ \rho$ is analytic on D^N for all $f \in A$ (cf. [5]).

If every non-trivial part for A satisfies the above condition, we say that the associated space $A(X, E)$ has the property (α). See [1] for the details on function algebras.

The main result of this paper is the following theorem.

THEOREM. *Let wC_φ be a weighted composition operator on $A(X, E)$.*

(a) *If wC_φ is compact, then*

(i) *for each connected component C of $S(w) = \{x \in X : w(x) \neq O\}$, there exist an open set U containing C and a part P for A such that $\varphi(U) \subset P$;*

(ii) *the map $w: X \to B(E)$ is continuous in the uniform operator topology, that is, $\|w(x_\lambda) - w(x)\|_{B(E)} \to 0$ as $x_\lambda \to x$;*

(iii) *for any $x \in S(w)$, $w(x)$ is a compact operator on E.*

(b) *In addition, we assume that $A(X, E)$ has the property (α). If wC_φ satisfies the above conditions (i)–(iii), then wC_φ is compact.*

Before proving the theorem, we make a few remarks on a weighted composition operator wC_φ on $A(X, E)$. For each $e \in E$, let f_e be the constant e function, i.e., $f_e(x) = e$ for all $x \in X$. Since $wC_\varphi f_e$ belongs to $A(X, E)$, it follows that $\sup\{\|w(x)e\|_E : x \in X\} = \sup\{\|wC_\varphi f_e(x)\|_E : x \in X\} = \|wC_\varphi f_e\| < +\infty$. By the uniform boundedness principle, we have

$$|||w||| = \sup\{\|w(x)\|_{B(E)} : x \in X\} < +\infty .$$

Moreover, if $\{x_\lambda\}$ is a net in X with $x_\lambda \to x$, then we have

$$\|w(x_\lambda)e - w(x)e\|_E = \|wC_\varphi f_e(x_\lambda) - wC_\varphi f_e(x)\|_E \to 0 ,$$

as $x_\lambda \to x$. It means that the map $w: X \to B(E)$ is continuous in the strong operator topology. (Note that w is not necessarily continuous in the uniform operator topology. See [3] for example.) This continuity of w shows that $S(w)=\{x\in X: w(x)\neq O\}$ is open in X. Also, we see that φ is continuous on $S(w)$. This is the consequence of the fact that $wC_\varphi f$ is continuous on X for all $f\in A(X,E)$. But φ is not necessarily continuous on $X\setminus S(w)$, because $wC_\varphi f$ is zero on $X\setminus S(w)$ even if φ is anyhow defined.

§2. Proof of the theorem.

Let wC_φ be a weighted composition operator on $A(X,E)$. We may assume that w is not identically zero, otherwise there is nothing to prove.

We first show the part (a) of the theorem. Suppose that wC_φ is compact. Since the proof of (ii) and (iii) is similar to that of the same part of [3, Theorem 2], we only show (i). For this purpose, we observe that for each $x\in S(w)$, there are a neighborhood U of x and a part P for A such that $\varphi(U)\subset P$.

If not, there exist a point x_0 in $S(w)$ and a part P_0 containing $\varphi(x_0)$ such that $\varphi(U)\not\subset P_0$ for any neighborhood U of x_0. Choose $e\in E$ so that $\delta=\|w(x_0)e\|_E>0$, and let $U_1=\{x\in X: \|w(x)e\|_E>\delta/2\}$. Since U_1 is an open neighborhood of x_0, it follows that $\varphi(U_1)\not\subset P_0$. Hence we find $x_1\in U_1$ with $\varphi(x_1)\notin P_0$, and we have $F_1\in A$ such that

$$\|F_1\|\leq 1\,,\quad F_1(\varphi(x_0))=0\,,\quad F_1(\varphi(x_1))>\frac{3}{4}.$$

Next put $U_2=\{x\in U_1: |F_1(\varphi(x))|<1/4\}$. Since U_2 is an open neighborhood of x_0, it follows that $\varphi(U_2)\not\subset P_0$. So we find $x_2\in U_2$ with $\varphi(x_2)\notin P_0$ and $F_2\in A$ such that

$$\|F_2\|\leq 1\,,\quad F_2(\varphi(x_0))=0\,,\quad F_2(\varphi(x_2))>\frac{3}{4}\,.$$

Here we note that $|F_1(\varphi(x_2))|<1/4$. Continuing this process, we obtain a sequence $\{x_n\}$ in U_1 and a sequence $\{F_n\}$ in A such that

$$\|F_n\|\leq 1\,,\quad F_n(\varphi(x_0))=0\,,\quad F_n(\varphi(x_n))>\frac{3}{4}\,,$$

$$|F_k(\varphi(x_n))|<\frac{1}{4}\qquad (k=1,\cdots,n-1)\,.$$

Set $f_n(x)=F_n(x)e$ ($x\in X$, $n=1,2,\cdots$). Since $\{f_n\}$ is a bounded sequence in $A(X,E)$, the compactness of wC_φ implies that $\{wC_\varphi f_n\}$ has a subsequence $\{wC_\varphi f_{n'}\}$ converging uniformly. But, for any m', n' ($m'<n'$),

$$\begin{aligned}\|wC_\varphi f_{m'}-wC_\varphi f_{n'}\| &\geq \|w(x_{n'})f_{m'}(\varphi(x_{n'}))-w(x_{n'})f_{n'}(\varphi(x_{n'}))\|_E\\ &=\|w(x_{n'})F_{m'}(\varphi(x_{n'}))e-w(x_{n'})F_{n'}(\varphi(x_{n'}))e\|_E\end{aligned}$$

$$=|F_{m'}(\varphi(x_{n'}))-F_{n'}(\varphi(x_{n'}))|\cdot\|w(x_{n'})e\|_E>\left(\frac{3}{4}-\frac{1}{4}\right)\cdot\frac{\delta}{2}=\frac{\delta}{4}.$$

This is a contradiction.

Now let C be a connected component of $S(w)$. If we fix $x_0\in C$, then $\varphi(x_0)$ belongs to some part P for A. Put $U=\{x\in S(w):\varphi(x)\in P\}$. Then the above observation shows that U is open and closed in $S(w)$, and the connectedness of C implies that $C\subset U$. Thus we obtain the condition (i).

Conversely, assume that wC_φ satisfies the conditions (i)–(iii). Using the property (α), we must show that wC_φ is compact. Let $\{f_n\}$ be a sequence in $A(X, E)$ with $\|f_n\|\le 1$, and $\varepsilon>0$ given. Set $U_0=\{x\in X: \|w(x)\|_{B(E)}<\varepsilon/2\}$. Then, by (ii), U_0 is an open set. For any $x\in U_0$, and $m, n=1, 2, \cdots$,

$$\begin{aligned}\|wC_\varphi f_m(x)-wC_\varphi f_n(x)\|_E&=\|w(x)(f_m(\varphi(x))-f_n(\varphi(x)))\|_E\\ &\le\|w(x)\|_{B(E)}(\|f_m(\varphi(x))\|_E+\|f_n(\varphi(x))\|_E)\\ &\le 2\|w(x)\|_{B(E)}<2\cdot\frac{\varepsilon}{2}=\varepsilon.\end{aligned}\tag{1}$$

We next show that every $x\in X\setminus U_0$ has an open neighborhood $U(x)$ such that

$$\|wC_\varphi f_n(x)-wC_\varphi f_n(y)\|_E<\frac{\varepsilon}{3}\quad\text{for all } y\in U(x),\quad\text{and } n=1, 2, \cdots.\tag{2}$$

Let P be the part containing $\varphi(x)$. If P is a one-point part, we take

$$U(x)=\left\{y\in S(w):\varphi(y)=\varphi(x),\ \|w(x)-w(y)\|_{B(E)}<\frac{\varepsilon}{3}\right\}.$$

By (i) and (ii), $U(x)$ is an open neighborhood of x, and we have

$$\begin{aligned}\|wC_\varphi f_n(x)-wC_\varphi f_n(y)\|_E&=\|w(x)f_n(\varphi(x))-w(y)f_n(\varphi(x))\|_E\\ &\le\|w(x)-w(y)\|_{B(E)}\|f_n(\varphi(x))\|_E\le\|w(x)-w(y)\|_{B(E)}<\frac{\varepsilon}{3},\end{aligned}$$

for all $y\in U(x)$, and $n=1, 2, \cdots$.

On the other hand, if P is non-trivial, then there are a neighborhood V of $\varphi(x)$ and a homeomorphism ρ from D^N onto V in the property (α). Hence for any $e^*\in E^*$ with $\|e^*\|=1$, $\{(e^*\circ f_n)\hat{}\circ\rho\}$ is a bounded sequence of analytic functions on D^N, and so a normal family in the sense of Montel. Consequently, we find an open neighborhood $W\subset D^N$ of $\zeta=\rho^{-1}(\varphi(x))$ such that

$$|(e^*\circ f_n)\hat{}\circ\rho(\zeta)-(e^*\circ f_n)\hat{}\circ\rho(\eta)|<\frac{\varepsilon}{6|||w|||},$$

for all $\eta\in W$ and $n=1, 2, \cdots$. Now let

$$U(x)=\left\{y\in S(w) : \varphi(y)\in\rho(W),\ \|w(x)-w(y)\|_{B(E)}<\frac{\varepsilon}{6}\right\}.$$

Using (i), (ii), and (α), we can easily check that $U(x)$ is an open neighborhood of x. Furthermore, for any $y\in U(x)$ and $n=1, 2, \cdots$, we have

$$\begin{aligned}\|wC_\varphi f_n(x)-wC_\varphi f_n(y)\|_E &= \|w(x)f_n(\varphi(x))-w(y)f_n(\varphi(y))\|_E\\ &\le \|w(x)-w(y)\|_{B(E)}\cdot\|f_n(\varphi(x))\|_E\\ &\quad+\|w(y)\|_{B(E)}\cdot\|f_n(\varphi(x))-f_n(\varphi(y))\|_E\\ &\le \|w(x)-w(y)\|_{B(E)}+|||w|||\cdot\|f_n(\varphi(x))-f_n(\varphi(y))\|_E .\end{aligned}$$

Here we take $e_n^*\in E^*$ with $\|e_n^*\|\le 1$ such that $\|f_n(\varphi(x))-f_n(\varphi(y))\|_E=|e_n^*(f_n(\varphi(x))-f_n(\varphi(y)))|$, and put $\eta=\rho^{-1}(\varphi(y))$. Then we have

$$\begin{aligned}\|f_n(\varphi(x))-f_n(\varphi(y))\|_E &= |e_n^*(f_n(\varphi(x)))-e_n^*(f_n(\varphi(y)))|\\ &=|e_n^*\circ f_n\circ\rho(\zeta)-e_n^*\circ f_n\circ\rho(\eta)|<\frac{\varepsilon}{6|||w|||},\end{aligned}$$

and so

$$\|wC_\varphi f_n(x)-wC_\varphi f_n(y)\|_E\le\frac{\varepsilon}{6}+|||w|||\cdot\frac{\varepsilon}{6|||w|||}=\frac{\varepsilon}{3}.$$

Thus we obtain an open neighborhood $U(x)$ of x satisfying (2). Since X is a compact set, we can find a finite set $\{x_1, \cdots, x_M\}$ in $X\setminus U_0$ such that $X=U_0\cup\bigcup_{i=1}^M U(x_i)$. For each i, $\{f_n(\varphi(x_i))\}_{n=1}^\infty$ is a bounded sequence in E, and $w(x_i)$ is a compact operator on E by (iii). Consequently we have a subsequence $\{f_{n'}\}$ of $\{f_n\}$ such that

$$\begin{aligned}&\|wC_\varphi f_{m'}(x_i)-wC_\varphi f_{n'}(x_i)\|_E\\ &\qquad=\|w(x_i)f_{m'}(\varphi(x_i))-w(x_i)f_{n'}(\varphi(x_i))\|_E<\frac{\varepsilon}{3},\end{aligned}$$

for all m', n' and $i=1, \cdots, M$. Hence, for any $x\in X\setminus U_0$, taking x_i so that $x\in U(x_i)$, we have

$$\begin{aligned}\|wC_\varphi f_{m'}(x)-wC_\varphi f_{n'}(x)\|_E &\le \|wC_\varphi f_{m'}(x)-wC_\varphi f_{m'}(x_i)\|_E\\ &\quad+\|wC_\varphi f_{m'}(x_i)-wC_\varphi f_{n'}(x_i)\|_E+\|wC_\varphi f_{n'}(x_i)-wC_\varphi f_{n'}(x)\|_E\\ &<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\end{aligned}$$

for all m', n'. Together with (1), we see that $\{f_{n'}\}$ is a subsequence of $\{f_n\}$ such that

(3) $$\|wC_\varphi f_{m'}-wC_\varphi f_{n'}\|<\varepsilon \quad \text{for any} \quad m', n' .$$

Now we choose a first subsequence $\{f_{1,n}\}$ of $\{f_n\}$ satisfying (3) as $\varepsilon=1$, and inductively a $k+1$-th subsequence $\{f_{k+1,n}\}$ of $\{f_{k,n}\}$ satisfying (3) as $\varepsilon=1/k$. The Cantor diagonal process shows that the sequence $\{wC_\varphi f_n\}$ has a subsequence which is a Cauchy sequence in $A(X, E)$. Hence the completeness of $A(X, E)$ establishes the compactness of wC_φ, and the proof of the theorem is completed.

§3. Applications.

We here apply the theorem to various spaces. When $A=C(X)$, then $A(X, E)=C(X, E)$. Notice that every part for $C(X)$ is one-point. Our theorem yields the following corollary, which says that the condition (2.5) in [3, Theorem 2] is removable.

COROLLARY 1. *Let wC_φ be a weighted composition operator on $C(X, E)$. Then wC_φ is compact if and only if* (i) *for each connected component C of $S(w)=\{x\in X: w(x)\neq O\}$, there exists an open set U containing C such that φ is constant on U*; (ii) *the map w is continuous in the uniform operator topology*; *and* (iii) *for each $x\in S(w)$, $w(x)$ is a compact operator on E.*

We next consider the case of $E=\mathbf{C}$. Then the space $A(X, \mathbf{C})$ is a function algebra A on X, and the conditions (ii) and (iii) in the theorem are automatically satisfied. Consequently we obtain results of [7].

Finally we remark on composition operators on $A(X, E)$. Let I_E be the identity operator on E, and define w by $w(x)=I_E$ for all $x\in X$. A weighted composition operator wC_φ on $A(X, E)$ induced by this map w is said to be a composition operator. If E is an infinite dimensional Banach space, I_E is not compact, and so the above map w does not satisfy the condition (iii) in the theorem. Hence the part (a) of the theorem shows the following corollary (cf. [6]):

COROLLARY 2. *If E is infinite dimensional, then there is no compact composition operator on $A(X, E)$.*

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