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Convexity Theorems for Riesz Means

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1. M. Riesz proved the so-called convexity theorem between the Riesz means of different orders which was subsequently generalized by several authors. Our purpose here is to give an ultimate form to this theorem. In particular, the main idea in L. S. Bosanquet [1] lies in "translating" the situation so as to be covered by the original M. Riesz theorem. Since there is no convenient "translation" in our case, we are forced to meet this difficulty (case (iii)). The proof is completed by repeated "backward shifts" as in L. S. Bosanquet [1].

To state our results we need the following standard notations.

Let $\sum a_n$ be a given infinite series and let $\{\lambda_n\}$ be an increasing sequence of positive numbers tending to ∞. We define, for x satisfying $\lambda_n \leqq x < \lambda_{n+1}$, $A_\lambda(x) = \sum_{\lambda_\nu \leqq x} a_\nu$. Furthermore, let

$$A_\lambda^k(x) = k\int_0^x (x-t)^{k-1} A(t)dt = \sum_{\lambda_\nu \leqq x} (x-\lambda_\nu)^k a_\nu\,, \qquad k>0\,.$$

We here define $A_\lambda^0(x) = A_\lambda(x)$, and if $x < \lambda_0$, $A_\lambda^k(x) = 0$ for every $k \geqq 0$. $C_\lambda^k(x) = x^{-k} A_\lambda^k(x)$ is called the Riesz mean of order k and type λ of the series $\sum a_\nu$, while $A_\lambda^k(x)$ is called the Riesz sum of the order k and type λ of that series. If $\lim_{x\to\infty} C_\lambda^k(x) = s$ exists and is finite, we say that the series is summable by Riesz mean of order k and type λ, or simply summable (R, λ, k), to the sum s. Let b_n, $B_\lambda(x)$, $B_\lambda^k(x)$ be defined as follows:

$$b_n = \lambda_n a_n\,, \quad B_\lambda(x) = \sum_{\nu=0}^{n} \lambda_\nu a_\nu\,, \qquad (\lambda_n \leqq x < \lambda_{n+1})\,,$$

$$B_\lambda^k(x) = \sum_{\nu=0}^{n} (x-\lambda_\nu)^k \lambda_\nu a_\nu\,, \qquad (k>0,\ \lambda_n \leqq x < \lambda_{n+1})\,.$$

We then have

(1.1) $$B_\lambda^k(x) = xA_\lambda^k(x) - A_\lambda^{k+1}(x)\,.$$

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We omit the suffix λ in the sequel since no confusion will arise.

2. The author [6] proved the following theorem.

THEOREM A. *Let $V(x)$ and $W(x)$ be positive functions defined for $x>0$ such that*
(i) *$x^{\alpha}W(x)$ is non-decreasing for some α, $0\leqq\alpha<1$,*
(ii) *$x^{\beta}V(x)$ is non-decreasing for some β, $\beta\geqq 0$,*
(iii) *$\{W(x)/V(x)\}^{1/\delta}=O(x)$ for some fixed $\delta>0$.*
Then the two extremity conditions

$$A^{\delta}(x)=o(W(x))$$

$$A(x)=O(V(x))$$

together imply, for intermediate γ, $0<\gamma<\delta$,

$$A^{\gamma}(x)=o(V(x)^{1-\gamma/\delta}W(x)^{1/\delta}).$$

This theorem is a generalization of Sunouchi [7] and Bosanquet [1], Theorem A reduces to Riesz convexity theorem [5] by deleting the condition (iii) and putting $\alpha=\beta=0$ (see [6: Theorem 1]). The following Theorem I is an extension of Theorem A with $\alpha=0$. In Theorem A with $\alpha=0$, the condition (ii) shows that $x^{\beta}V(x)$ is a non-decreasing function. Theorem I is a convexity theorem for the class of $V(x)$ with the condition (2.1) (ii) weaker than (ii) in Theorem A.

THEOREM I. *Let $V(x)$ and $W(x)$ be positive functions defined for $x>0$ such that*

(2.1)
(i) *$W(x)$ is non-decreasing,*
(ii) *there exist constants $H\,(>1)$ and η $(0<\eta<1)$ such that $0<x-x'\leqq\eta x$ implies $V(x')/V(x)<H$,*
(iii) *$(W(x)/V(x))^{1/\delta}=O(x)$ for some fixed $\delta>0$.*

Then the two extremity conditions

$$A^{\delta}(x)=o(W(x)) \tag{2.2}$$

$$A(x)=O(V(x)) \tag{2.3}$$

together imply, for intermediate γ, $0<\gamma<\delta$,

$$A^{\gamma}(x)=o(V(x)^{1-\gamma/\delta}W(x)^{1/\delta}). \tag{2.4}$$

Rangachari's theorem [4], which treats Cesàro sums, holds for $0<\delta\leqq 1$, while our Theorem I holds for all $\delta>0$, as will be proved in Section 4. Using Theorem I, we prove the following generalization of Theorem A, whose proof will be given in Section 5.

THEOREM II. *Let $V(x)$ and $W(x)$ be positive functions defined for $x>0$ such that*

(2.5)
(i) $x^{\alpha}W(x)$ *is non-decreasing for some* α, $0 \leqq \alpha < 1$,
(ii) *there exist constants* H (>1) *and* η $(0<\eta<1)$ *such that* $0 < x - x' < \eta x$ *implies* $V(x')/V(x) < H$,
(iii) $\{W(x)/V(x)\}^{1/\delta} = O(x)$ *for some fixed* $\delta > 0$.

Then the two extremity conditions

$$A^{\delta}(x) = o(W(x)) \tag{2.6}$$

$$A(x) = O(V(x)) \tag{2.7}$$

together imply, for intermediate γ, $0 < \gamma < \delta$,

$$A^{\gamma}(x) = o(V(x)^{1-\gamma/\delta} W(x)^{\gamma/\delta}) . \tag{2.8}$$

3. Lemmas.

The following lemmas are needed for the proof of the above theorems.

LEMMA A ([2]). *Let us write, for* $0 < l \leqq 1$, $0 < \xi \leqq x$, $k \geqq 0$,

$$g_{l,k}(\xi, x) = \frac{\Gamma(k+l+1)}{\Gamma(k+1)\Gamma(l)} \int_0^{\xi} (x-t)^{l-1} A^k(t) dt .$$

Then $A^{k+1}(x) = o(W(x))$ *implies*

$$g_{l,k}(\xi, x) = o(W(x)) \qquad \text{uniformly in } \xi . \tag{3.1}$$

LEMMA B ([2]). *For* $k \geqq 0$, $l > 0$ *we have*

$$A^{k+1}(x) = \frac{\Gamma(k+l+1)}{\Gamma(k+1)\Gamma(l)} \int_0^{x} (x-t)^{l-1} A^k(t) dt . \tag{3.2}$$

We also need "backward difference of a function $F(x)$ of order m and step h (>0)" defined as follows:

$$\Delta^0_{-h} F(x) = F(x) , \qquad \Delta^m_{-h} F(x) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} F(x - jh) , \tag{3.3}$$

where m is a positive integer. For fractional α, $0 < \alpha < 1$, put

$$\Delta^{\alpha}_{-h} F(x) = \alpha \int_{x-h}^{x} (x-t)^{\alpha-1} F(t) dt \tag{3.4}$$

and generally for $\delta = m + \alpha$, $\Delta^{\delta}_{-h} F(x) = \Delta^m_{-h}(\Delta^{\alpha}_{-h} F(x))$.

LEMMA C ([2]). *If* $h > 0$, m *is a positive integer*, $r > 0$ *and* $0 \leqq \beta < 1$, *then*

(3.5) $$h^{m+\beta}A^{r}(x)=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+m+1)}\Delta_{-h}^{m+\beta}A^{r+m}(x)$$
$$+\Delta_{-h}^{\beta}\left(\int_{x-h}^{x}dt_1\int_{t_1-h}^{t_1}dt_2\cdots\int_{t_{m-1}-h}^{t_{m-1}}(A^{r}(x)-A^{r}(t_m))dt_m\right).$$

4. Proof of Theorem I.

From (2.1) (iii), there exists a constant $H'>0$ such that

$$(W(x)/V(x))^{1/\delta}<H'x \qquad (x>0).$$

Given $\varepsilon>0$ such that $\varepsilon^{1/\delta}H'<\eta$, choose x_0 by Lemma A, in such a way that, when $0<l\leqq 1$ and $k\geqq 0$,

(4.1) $$|g_{l,k}(\xi, x)|<\varepsilon W(x) \qquad \text{for} \quad x>x_0 \quad \text{and} \quad 0<\xi\leqq x.$$

We choose x' so that

(4.2) $$x-x'=(\varepsilon W(x)/V(x))^{1/\delta}.$$

Then $x'=x-(\varepsilon W(x)/V(x))^{1/\delta}>x(1-\varepsilon^{1/\delta}H')>x(1-\eta)>0$ and $0<x-x'<\varepsilon^{1/\delta}H'x<\eta x$, so from (2.1) (ii), $V(x')<HV(x)$ holds.

Case (i) $0<\delta\leqq 1$. We write

$$A^{\gamma}(x)=\gamma\left(\int_0^{x'}+\int_{x'}^{x}\right)(x-t)^{\gamma-1}A(t)dt=J_1+J_2, \qquad \text{say}.$$

By (2.1) (ii), (2.3) and (4.2) we have

(4.3) $$|J_2|=\left|\gamma\int_{x'}^{x}(x-t)^{\gamma-1}A(t)dt\right|\leqq\gamma V(x)\left|\int_{x'}^{x}(x-t)^{\gamma-1}dt\right|$$
$$\leqq V(x)(x-x')^{\gamma}=\varepsilon^{\gamma/\delta}V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta}.$$

By the second mean-value theorem, for some u, $0<u\leqq x'$,

$$J_1=\gamma\int_0^{x'}(x-t)^{\gamma-1}A(t)dt=\gamma(x-x')^{\gamma-\delta}\int_u^{x'}(x-t)^{\delta-1}A(t)dt$$
$$=\gamma(x-x')^{\gamma-\delta}\left(\int_0^{x'}-\int_0^{u}\right)(x-t)^{\delta-1}A(t)dt$$
$$=(\gamma/\delta)(x-x')^{\gamma-\delta}(g_{\delta,0}(x', x)-g_{\delta,0}(u, x)).$$

By (4.1) and (4.2), we get

(4.4) $$|J_1|<2(\gamma/\delta)\varepsilon^{(\gamma-\delta)/\delta}(W(x)/V(x))^{(\gamma-\delta)/\delta}\varepsilon W(x)$$

$$=2(\gamma/\delta)\varepsilon^{\gamma/\delta}V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta}.$$

Since ε is arbitrary, (4.3) and (4.4) together imply

$$A^{\gamma}(x)=o(V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta}) \quad \text{as} \quad x\to\infty. \tag{4.5}$$

Case (ii) $\delta>1$ and γ, $0<\gamma<\delta$, is an integer. Put $\delta=p+\beta$, where p is the integral part of δ. We prove our assertion by induction on γ. Since we are interested in the behavior of $A^{\gamma}(x)$ for large x, we may confine ourselves to x large enough, and the indication "as $x\to\infty$" is often omitted.

Let us begin with the case $0<\beta<1$. First assume that

$$|A^{\gamma-1}(x)|<\varepsilon^{(\gamma-1)/\delta}V(x)^{1-(\gamma-1)/\delta}W(x)^{(\gamma-1)/\delta}\cdot D_{\gamma}, \tag{4.6}$$

where $D_{\gamma}=(1+H)^{\gamma}\Gamma(\gamma)2^{p+1}(p+1)^{p+\beta}\Gamma(\beta+1)/\Gamma(\beta+p+1)$. Since $D_1>1$, from (2.3) we have (the case $\gamma=1$ in (4.6))

$$|A(x)|<D_1V(x). \tag{4.7}$$

From (4.6) and (4.2) we have

$$\begin{aligned} |A^{\gamma}(x)-A^{\gamma}(x')| &= \left|\gamma\left(\int_0^x-\int_0^{x'}\right)A^{\gamma-1}(t)dt\right| \\ &\leqq \gamma\int_{x'}^{x}|A^{\gamma-1}(t)|dt \\ &\leqq \gamma(x-x')(\varepsilon^{(\gamma-1)/\delta}V(x)^{1-(\gamma-1)/\delta}W(x)^{(\gamma-1)/\delta}\cdot D_{\gamma} \\ &\leqq \gamma\varepsilon^{\gamma/\delta}V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta}\cdot D_{\gamma}. \end{aligned} \tag{4.8}$$

By (3.5) with $m=p-\gamma$, we obtain, at least for $\gamma<p$,

$$\begin{aligned} h^{p-\gamma+\beta}A^{\gamma}(x) &= \frac{\Gamma(\gamma+1)}{\Gamma(p+1)}\Delta_{-h}^{p-\gamma+\beta}A^{p}(x) \\ &\quad +\Delta_{-h}^{\beta}\left(\int_{x-h}^{x}dt_1\int_{t_1-h}^{t_1}dt_2\cdots\int_{t_{m-1}-h}^{t_{m-1}}(A^{\gamma}(x)-A^{\gamma}(t_m))dt_m\right) \\ &= I_1+I_2, \qquad \text{say}. \end{aligned} \tag{4.9}$$

By (3.4) and (3.3), we have

$$\begin{aligned} I_1 &= \frac{\Gamma(\gamma+1)}{\Gamma(p+1)}\Delta_{-h}^{p-\gamma+\beta}A^{p}(x) \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(p+1)}\beta\int_{x-h}^{x}(x-t)^{\beta-1}\sum_{j=0}^{p-\gamma}(-1)^j\binom{p-\gamma}{j}A^{p}(t-jh)dt \end{aligned}$$

$$=\frac{\Gamma(\gamma+1)}{\Gamma(p+1)}\beta\sum_{j=0}^{p-\gamma}(-1)^j\binom{p-\gamma}{j}\int_{x-(j+1)h}^{x-jh}(x-jh-u)^{\beta-1}A^p(u)du$$

$$=\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{\Gamma(p+\beta+1)}\sum_{j=0}^{p-\gamma}(-1)^j\binom{p-\gamma}{j}$$

$$\times\{g_{\beta,p}(x-jh,\ x-jh)-g_{\beta,p}(x-(j+1)h,\ x-jh)\}\ .$$

Thus each term constituting I_1 does not exceed in absolute value

$$\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{\Gamma(p+\beta+1)}\binom{p-\gamma}{j} \tag{4.10}$$

$$\times\{|g_{\beta,p}(x-jh,\ x-jh)|+|g_{\beta,p}(x-(j+1)h,\ x-jh)|\}\ .$$

Let x satisfy $(\gamma+1)x/(p+1)>x_0$. For $j=0, 1, \cdots, p-\gamma-1$ we have

$$x-jh\geqq x-(p-\gamma)h\geqq\frac{(\gamma+1)x}{p+1}+\frac{(p+\gamma)x'}{p+1}\geqq\frac{(\gamma+1)x}{p+1}>x_0\ .$$

Hence, basing on (4.1) and (2.1) (i), we see

$$|g_{\beta,p}(x-(j+1)h,\ x-jh)|<\varepsilon W(x) \tag{4.11}$$

and

$$|g_{\beta,p}(x-jh,\ x-jh)|=|A^\delta(x-jh)|<\varepsilon W(x-jh)<\varepsilon W(x)\ . \tag{4.12}$$

Observing (4.10) we have by (4.11) and (4.12)

$$|I_1|<\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{\Gamma(p+\beta+1)}2^{p-\gamma+1}\cdot\varepsilon W(x) \tag{4.13}$$

$$<\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{\Gamma(p+\beta+1)}2^{p+1}\cdot\varepsilon W(x)\ .$$

By (4.8), (3.4), (2.1) (i) and (2.1) (ii), we have

$$I_2=\Delta^\beta_{-h}\left(\int_{x-h}^{x}dt_1\int_{t_1-h}^{t_1}dt_2\cdots\int_{t_{m-1}-h}^{t_{m-1}}(A^\gamma(x)-A^\gamma(t_m))dt_m\right) \tag{4.14}$$

$$=\Delta^\beta_{-h}\left(\int_0^h dt_1\int_0^h dt_2\cdots\int_0^h(A^\gamma(x)-A^\gamma(x-t_1-t_2-\cdots-t_m))dt_m\right)$$

$$\leqq\beta\int_{x-h}^{x}(x-u)^{\beta-1}\left(\int_0^h dt_1\int_0^h dt_2\cdots\int_0^h|(*)|dt_m\right)du \qquad (m=p-\gamma)$$

$$\left(\begin{matrix}\text{where the abbreviated expression } (*) \text{ designates}\\ A^\gamma(u)-A^\gamma(u-t_1-t_2-\cdots-t_m)\end{matrix}\right)$$

$$\leqq \beta\gamma\varepsilon^{\gamma/\delta}h^{p-\gamma}D_\gamma\int_{x-h}^{x}(x-u)^{\beta-1}V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta}du$$

$$\leqq H^{1-\gamma/\delta}\gamma\varepsilon^{\gamma/\delta}h^{p-\gamma+\beta}D_\gamma V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta}.$$

Hence, by (4.9), (4.13) and (4.14) we have

$$|A^\gamma(x)|<\frac{\Gamma(\gamma+1)\cdot 2^{p+1}h^{-(p-\gamma+\beta)}\cdot\Gamma(\beta+1)}{\Gamma(p+\beta+1)}\varepsilon W(x) \tag{4.15}$$

$$+\varepsilon^{\gamma/\delta}H^{1-\gamma/\delta}(\gamma D_\gamma V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta})$$

$$<\gamma D_\gamma(1+H^{1-\gamma/\delta})\varepsilon^{\gamma/\delta}V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta}.$$

Thus we have the following inequality:

$$|A^\gamma(x)|<D_{\gamma+1}\varepsilon^{\gamma/\delta}(1+H^{1-\gamma/\delta})V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta}. \tag{4.16}$$

This is established, inductively, for $\gamma=1, 2, \cdots, p-1$. The case $\gamma=p$ is treated, without appealing to (3.5), by direct application of (3.4); the computation is much the same as the proof of (4.16) and may be left to the reader. Thus we have seen

$$A^\gamma(x)=o(V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta}). \tag{4.17}$$

Next consider the case $\beta=0$. We make use of induction on γ. Assume that

$$|A^{\gamma-1}(x)|<E_\gamma\varepsilon^{(\gamma-1)/p}V(x)^{1-(\gamma-1)/\delta}W(x)^{(\gamma-1)/\delta} \tag{4.18}$$

where $E_\gamma=\Gamma(\gamma)(1+H)^\gamma 2^p(p+1)^p/\Gamma(p+1)$. Since we see evidently that $E_1>1$, we have from (2.3)

$$|A(x)|<E_1V(x). \tag{4.19}$$

From (4.18) and (4.2),

$$|A^\gamma(x)-A^\gamma(x')|=\left|\gamma\left(\int_0^x-\int_0^{x'}\right)A^{\gamma-1}(t)dt\right| \tag{4.20}$$

$$\leqq\gamma\int_{x'}^{x}|A^{\gamma-1}(t)|dt$$

$$\leqq\gamma(x-x')\varepsilon^{(\gamma-1)/p}V(x)^{1-(\gamma-1)/p}W(x)^{(\gamma-1)/p}\cdot E_\gamma$$

$$=\gamma\varepsilon^{\gamma/p}V(x)^{1-\gamma/p}W(x)^{\gamma/p}\cdot E_\gamma.$$

Since $m=p-\gamma>0$ now, from (3.5) we obtain

$$h^{p-\gamma}A^\gamma(x)=\frac{\Gamma(\gamma+1)}{\Gamma(p+1)}\Delta_{-h}^{p-\gamma}A^p(x) \tag{4.21}$$

$$+\int_{x-h}^{x} dt_1 \int_{t_1-h}^{t_1} dt_2 \cdots \int_{t_{m-1}-h}^{t_{m-1}} (A^{\gamma}(x)-A^{\gamma}(t_m))dt_m$$
$$=I_3+I_4\,, \qquad \text{say}\,.$$

By (3.3) we have

$$|I_3|=\left|\frac{\Gamma(\gamma+1)}{\Gamma(p+1)}\sum_{j=0}^{p-\gamma}(-1)^j\binom{p-\gamma}{j}A^p(x-jh)\right| \tag{4.22}$$
$$\leqq \Gamma(\gamma+1)\cdot(2^{p-\gamma}/\Gamma(p+1))\cdot|A^p(x-jh)|\,.$$

Let x be such that $(\gamma+1)x/(p+1)>x_0$, then for $j=0, 1, 2, \cdots, p-\gamma$, we see

$$x-jh\geqq x-(p-\gamma)h\geqq\frac{(\gamma+1)x}{p+1}+\frac{(p-\gamma)x'}{p+1}\geqq\frac{(\gamma+1)x}{p+1}>x_0\,.$$

Hence, for $(\gamma+1)x/(p+1)>x_0$, by (4.1) and (2.1) (i), we have

$$|A^p(x-jh)|<\varepsilon W(x-jh)<\varepsilon W(x)\,. \tag{4.23}$$

Thus an upper bound for $|I_3|$ is obtained from (4.22) and (4.23):

$$|I_3|<\Gamma(\gamma+1)\frac{2^p}{\Gamma(p+1)}\varepsilon W(x)\,. \tag{4.24}$$

On the other hand, by (4.20), (2.1) (i) and (2.1) (ii) we have

$$I_4=\int_{x-h}^{x} dt_1 \int_{t_1-h}^{t_1} dt_2 \cdots \int_{t_{m-1}-h}^{t_{m-1}} (A^{\gamma}(x)-A^{\gamma}(t_m))dt_m \tag{4.25}$$
$$=\int_0^h dt_1\int_0^h dt_2\cdots\int_0^h (A^{\gamma}(x)-A^{\gamma}(x-t_1-t_2-\cdots-t_m))dt_m$$
$$\leqq\int_0^h dt_1\int_0^h dt_2\cdots\int_0^h |A^{\gamma}(x)-A^{\gamma}(x-t_1-t_2-\cdots-t_m)|dt_m$$
$$\leqq\gamma h^{p-\gamma}\varepsilon^{\gamma/p}V(x)^{1-\gamma/p}W(x)^{\gamma/p}\cdot E_{\gamma}\,.$$

Combining the estimates of I_3, (4.24), and I_4, (4.25), we obtain

$$|A^{\gamma}(x)|<\Gamma(\gamma+1)\frac{2^p}{\Gamma(p+1)}\cdot h^{-(p-\gamma)}\varepsilon W(x)+\varepsilon^{\gamma/p}\gamma\cdot E_{\gamma}\cdot V(x)^{1-\gamma/p}W(x)^{\gamma/p} \tag{4.26}$$
$$<2\gamma E_{\gamma}\varepsilon^{\gamma/p}V(x)^{1-\gamma/p}W(x)^{\gamma/p}$$
$$<E_{\gamma+1}\varepsilon^{\gamma/p}V(x)^{1-\gamma/p}W(x)^{\gamma/p}\,.$$

Thus, by induction, (4.26) holds for $\gamma=1, 2, \cdots, p-1$. Since ε can be taken arbitrarily small, this means

$$A^{\gamma}(x)=o(V(x)^{1-\gamma/p}W(x)^{\gamma/p}) \qquad \text{as} \quad x\to\infty\,. \tag{4.27}$$

Case (iii) $\delta>1$ and $0<\gamma<\delta$ but γ is not an integer. Let $\delta=p+\beta$ where p is the integral part of δ. We begin with the subcase $0<\beta<1$. By Lemma B, we obtain

$$\begin{aligned} A^{\delta-1}(x)&=\frac{\Gamma(p+\beta)}{\Gamma(p)\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}A^{p-1}(t)dt \\ &=\frac{\Gamma(p+\beta)}{\Gamma(p)\Gamma(\beta)}\left(\int_0^{x'}+\int_{x'}^{x}\right)(x-t)^{\beta-1}A^{p-1}(t)dt \\ &=K_1+K_2\,, \qquad \text{say}\,. \end{aligned} \tag{4.28}$$

Replacing γ in (4.16) by $p-1$, we have

$$|A^{p-1}(x)|<D_p\varepsilon^{(p-1)/\delta}V(x)^{1-(p-1)/\delta}W(x)^{(p-1)/\delta}\,. \tag{4.29}$$

This gives

$$\begin{aligned} |K_2|&\leqq\frac{\Gamma(p+\beta)}{\Gamma(p)\Gamma(\beta)}\int_{x'}^{x}(x-t)^{\beta-1}|A^{p-1}(t)|dt \\ &\leqq\frac{\Gamma(p+\beta)\varepsilon^{(p-1)/\delta}}{\Gamma(p)\Gamma(\beta)}\cdot D_p\cdot\int_{x'}^{x}(x-t)^{\beta-1}V(t)^{1-(p-1)/\delta}W(t)^{(p-1)/\delta}dt\,. \end{aligned}$$

By (2.1) (i), (2.1) (ii) and (4.2) we have

$$\begin{aligned} |K_2|&\leqq\frac{\Gamma(p+\beta)\varepsilon^{(p-1)/\delta}}{\Gamma(p)\Gamma(\beta)\beta}\cdot D_p\cdot V(x)^{1-(p-1)/\delta}W(x)^{(p-1)/\delta}(x-x')^{\beta} \\ &<\frac{\Gamma(p+\beta)}{\Gamma(p)\Gamma(\beta)\beta}\varepsilon^{(\delta-1)/\delta}D_pV(x)^{1-(\delta-1)/\delta}W(x)^{(\delta-1)/\delta}\,. \end{aligned} \tag{4.30}$$

Since ε is arbitrary, we have

$$K_2=o(V(x)^{1-(\delta-1)/\delta}W(x)^{(\delta-1)/\delta})\,. \tag{4.31}$$

As to K_1, an integration by parts shows

$$K_1=\frac{\Gamma(p+\beta)}{\Gamma(p)\Gamma(\beta)\cdot p}\left\{[(x-t)^{\beta-1}A(t)]_0^{x'}+(\beta-1)\int_0^{x'}(x-t)^{\beta-2}A^{p}(t)dt\right\}.$$

Writing

$$T_1=\frac{\Gamma(p+\beta)}{\Gamma(p)\Gamma(\beta)\cdot p} \quad \text{and} \quad T_2=\frac{\beta-1}{p+\beta}\,,$$

we have

$$K_1 = T_1 \cdot (x-x')^{\beta-1} A^p(x') + T_2 \cdot \frac{\Gamma(p+\beta+1)}{\Gamma(p+1)\Gamma(\beta)} \int_0^{x'} \frac{(x-t)^{\beta-1}}{x-t} A^p(t)dt .$$

By the second mean-value theorem, for some v_1, $0 < v_1 \leqq x'$,

$$|K_1| \leqq T_1 \cdot (x-x')^{\beta-1} |A^p(x)| + \left| T_2 \cdot \frac{\Gamma(p+\beta+1)}{\Gamma(p+1)\Gamma(\beta)(x-x')} \int_{v_1}^{x'} (x-t)^{\beta-1} A^p(t)dt \right| \tag{4.32}$$

$$= J_3 + J_4 , \qquad \text{say} .$$

Replacing γ in (4.16) by p, we have

$$|A^p(x)| < D_{p+1} \varepsilon^{p/\delta} V(x)^{1-p/\delta} W(x)^{p/\delta} . \tag{4.33}$$

(4.2) and (4.33) together imply

$$J_3 < T_1 \cdot D_{p+1} (\varepsilon W(x)/V(x))^{(\beta-1)/\delta} \varepsilon^{p/\delta} V(x')^{1-p/\delta} W(x')^{p/\delta} .$$

Thus, (2.1) (i) and (2.1) (ii) yield

$$J_3 < T_1 D_{p+1} H^{(\delta-p)/\delta} \varepsilon^{(\delta-1)/\delta} V(x)^{1-(\delta-1)/\delta} W(x)^{(\delta-1)/\delta} . \tag{4.34}$$

On the other hand, (4.2) and (4.1) give

$$J_4 < 2T_2 \varepsilon^{(\delta-1)/\delta} V(x)^{1-(\delta-1)/\delta} W(x)^{(\delta-1)/\delta} . \tag{4.35}$$

Combining (4.34) and (4.35) we observe that

$$|K_1| < (T_1 D_{p+1} H^{(\delta-p)/\delta} + 2T_2) \varepsilon^{(\delta-1)/\delta} V(x)^{1-(\delta-1)/\delta} W(x)^{(\delta-1)/\delta} . \tag{4.36}$$

Since ε is arbitrary, this implies

$$K_1 = o(V(x)^{1-(\delta-1)/\delta} W(x)^{(\delta-1)/\delta}) \qquad \text{as} \quad x \to \infty . \tag{4.37}$$

By (4.28), (4.31) and (4.37), we have

$$A^{\delta-1}(x) = o(V(x)^{1-(\delta-1)/\delta} W(x)^{(\delta-1)/\delta}) . \tag{4.38}$$

Now the hypotheses of Theorem I with $\delta = 1$ are satisfied with

$A^{\delta-1}(x)$ in place of $A(x)$ and
$V(x)^{1-(\delta-1)/\delta} W(x)^{(\delta-1)/\delta}$ in place of $V(x)$,
$W(x)$ being unchanged.

In fact,

$$W(x)/\{V(x)^{1-(\delta-1)/\delta} W(x)^{(\delta-1)/\delta}\} = \{W(x)/V(x)\}^{1/\delta} = O(x) ,$$

and, by (2.1) (ii), for x' satisfying $0 < x-x' < \eta x \quad (0 < \eta < 1)$,

$$\frac{V(x')^{1-(\delta-1)/\delta} W(x')^{(\delta-1)/\delta}}{V(x)^{1-(\delta-1)/\delta} W(x)^{(\delta-1)/\delta}} = \left(\frac{V(x')}{V(x)}\right)^{1/\delta} \left(\frac{W(x')}{W(x)}\right)^{1-1/\delta} < H^{1/\delta} .$$

Therefore, by the case (i) ($\delta=1$) already established, for γ satisfying $\delta-1<\gamma<\delta$,

$$A^{\gamma}(x)=o((V(x)^{1-(\delta-1)/\delta}W(x)^{(\delta-1)/\delta})^{1-(\gamma-(\delta-1))}W(x)^{\gamma-(\delta-1)}) \tag{4.39}$$
$$=o(V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta}).$$

Since in our present case γ is not an integer, it can be written as $\gamma=r+\sigma>1$, where r is the integral part of γ. By Lemma B,

$$A^{\gamma-1}(x)=\frac{\Gamma(r+\sigma)}{\Gamma(r)\Gamma(\sigma)}\int_0^x (x-t)^{\sigma-1}A^{r-1}(t)dt \tag{4.40}$$
$$=\frac{\Gamma(r+\sigma)}{\Gamma(r)\Gamma(\sigma)}\left\{\int_0^{x'}+\int_{x'}^{x}\right\}(x-t)^{\sigma-1}A^{r-1}(t)dt$$
$$=M_1+M_2\,,\qquad \text{say}\,.$$

Let us consider M_2 first. Replacing γ in (4.16) by $r-1$, we have

$$|A^{r-1}(x)|<D_r\varepsilon^{(r-1)/\delta}V(x)^{1-(r-1)/\delta}W(x)^{(r-1)/\delta}\,. \tag{4.41}$$

From (4.41), the following inequality is obtained:

$$|M_2|\leqq\frac{\Gamma(r+\sigma)}{\Gamma(r)\Gamma(\sigma)}\int_{x'}^{x}(x-t)^{\sigma-1}|A^{r-1}(t)|dt$$
$$=\frac{\Gamma(r+\sigma)\varepsilon^{(r-1)/\delta}}{\Gamma(r)\Gamma(\sigma)}D_r\int_{x'}^{x}(x-t)^{\sigma-1}V(t)^{1-(r-1)/\delta}W(t)^{(r-1)/\delta}dt\,.$$

Since $W(x)$ is non-decreasing and $V(t)<HV(x)$ by (2.1) (ii),

$$|M_2|\leqq\frac{\Gamma(r+\sigma)H^{1-(r-1)/\delta}\varepsilon^{(r-1)/\delta}}{\Gamma(r)\Gamma(\sigma)}D_r\cdot V(x)^{1-(r-1)/\delta}W(x)^{(r-1)/\delta}\int_{x'}^{x}(x-t)^{\sigma-1}dt$$
$$=\frac{\Gamma(r+\sigma)H^{1-(r-1)/\delta}\varepsilon^{(r-1)/\delta}}{\sigma\Gamma(r)\Gamma(\sigma)}D_r\cdot V(x)^{1-(r-1)/\delta}W(x)^{(r-1)/\delta}(x-x')^{\sigma}\,.$$

From the choice of x' (see (4.2)), this gives

$$|M_2|\leqq\frac{\Gamma(r+\sigma)H^{1-(r-1)/\delta}\varepsilon^{(r-1)/\delta}}{\sigma\Gamma(r)\Gamma(\sigma)}D_r\cdot V(x)^{1-(\gamma-1)/\delta}W(x)^{(\gamma-1)/\delta}\,. \tag{4.42}$$

Since ε is arbitrary, (4.42) means

$$M_2=o(V(x)^{1-(\gamma-1)/\delta}W(x)^{(\gamma-1)/\delta})\,. \tag{4.43}$$

Next, we consider the term M_1 in (4.40), separating the two subcases: $p\leqq\gamma<\delta$ and $\delta-1<\gamma<p$.

Let γ satisfy $p\leqq\gamma<\delta$. Integrating by parts, we see

$$M_1 = \frac{\Gamma(r+\sigma)}{\Gamma(r)\Gamma(\sigma)r}\left\{[(x-t)^{\sigma-1}A^r(t)]_0^{x'} + (\sigma-1)\int_0^{x'}(x-t)^{\sigma-2}A^r(t)dt\right\}.$$

Write, for the sake of brevity,

$$T_3 = \frac{\Gamma(r+\sigma)}{\Gamma(r)\Gamma(\sigma)r} \quad \text{and} \quad T_4 = \frac{\Gamma(r+\sigma)\Gamma(\delta-r)(\sigma-1)}{\Gamma(\sigma)\Gamma(\delta+1)}.$$

We have

$$M_1 = T_3\cdot(x-x')^{\sigma-1}A^r(x') + T_4\cdot\frac{\Gamma(\delta+1)}{\Gamma(r+1)\Gamma(\delta-r)}\int_0^{x'}(x-t)^{\gamma-\delta-1}A^r(t)dt.$$

By the second mean-value theorem, we get, for some v_2, $0<v_2\leqq x'$,

$$\begin{aligned} M_1 &= T_3\cdot(x-x')^{\sigma-1}A^r(x') + T_4\cdot\frac{\Gamma(\delta+1)(x-x')^{\gamma-\delta-1}}{\Gamma(r+1)\Gamma(\delta-r)}\int_{v_2}^{x'}(x-t)^{\delta-r-1}A^r(t)dt \\ &= T_3\cdot(x-x')^{\sigma-1}A^r(x') + T_4(x-x')^{\gamma-\delta-1}\{g_{\delta-r,r}(x', x) - g_{\delta-r,r}(v_2, x)\}, \end{aligned}$$

so that

$$\begin{aligned} |M_1| &\leqq T_3\cdot(x-x')^{\sigma-1}|A^r(x')| + T_4(x-x')^{\gamma-\delta-1}\{|g_{\delta-r,r}(x', x)| + |g_{\delta-r,r}(v_2, x)|\} \\ &= J_5 + J_6, \qquad \text{say}. \end{aligned}$$

Replacing γ in (4.16) by r, we have

$$|A^r(x)| < D_{r+1}\varepsilon^{r/\delta}V(x)^{1-r/\delta}W(x)^{r/\delta}. \tag{4.44}$$

From (4.2) and (4.44) we see

$$J_5 < T_3 D_{r+1}\{\varepsilon W(x)/V(x)\}^{(\sigma-1)/\delta}\varepsilon^{r/\delta}V(x)^{1-r/\delta}W(x)^{r/\delta}.$$

By (2.1) (i) and (2.1) (ii), the following inequality is obtained:

$$J_5 < T_3 D_{r+1}H^{(\delta-r)/\delta}\varepsilon^{(\gamma-1)/\delta}V(x)^{1-(\gamma-1)/\delta}W(x)^{(\gamma-1)/\delta}. \tag{4.45}$$

From (4.1) and (4.2), we obtain

$$J_6 < 2T_4\varepsilon^{(\gamma-1)/\delta}V(x)^{1-(\gamma-1)/\delta}W(x)^{(\gamma-1)/\delta}. \tag{4.46}$$

Combining (4.45) and (4.46), we observe that

$$|M_1| < C(r, \delta, H)\varepsilon^{(\gamma-1)/\delta}V(x)^{1-(\gamma-1)/\delta}W(x)^{(\gamma-1)/\delta}, \tag{4.47}$$

where

$$C(r, \delta, H) = (T_3 D_{r+1}H^{(\delta-r)/\delta} + 2T_4).$$

Since ε is arbitrary, (4.47) implies

$$M_1=o(V(x)^{1-(\gamma-1)/\delta}W(x)^{(\gamma-1)/\delta})\,. \tag{4.48}$$

Then, by (4.40), (4.36) and (4.48), we have

$$A^{\gamma-1}(x)=o(V(x)^{1-(\gamma-1)/\delta}W(x)^{(\gamma-1)/\delta})\,. \tag{4.49}$$

Now let γ be such that $\delta-1<\gamma\leqq p$. Integrating by parts twice, we have

$$\begin{aligned}M_1&=\frac{\Gamma(r+\sigma)}{\Gamma(r)\Gamma(\sigma)r}\left\{[(x-t)^{\sigma-1}A^r(t)]_0^{x'}+(\sigma-1)\int_0^{x'}(x-t)^{\sigma-2}A^r(t)dt\right\}\\&=\frac{\Gamma(r+\sigma)}{\Gamma(r)\Gamma(\sigma)r}\left\{(x-x')^{\sigma-1}A^r(x')+\frac{\sigma-1}{r+1}[(x-t)^{\sigma-2}A^{r+1}(t)]_0^{x'}\right.\\&\quad\left.+\frac{(\sigma-1)(\sigma-2)}{r+1}\int_0^{x'}(x-t)^{\sigma-3}A^{r+1}(t)dt\right\}.\end{aligned}$$

Writing

$$T_5=\frac{\Gamma(r+\sigma)}{\Gamma(r)\Gamma(\sigma)r}\,,\qquad T_6=\frac{\Gamma(r+\sigma)(\sigma-1)}{\Gamma(r)\Gamma(\sigma)r}\quad\text{and}$$

$$T_7=\frac{(\sigma-1)(\sigma-2)\Gamma(\delta-r-1)\Gamma(r+\sigma)}{\Gamma(\delta+1)\Gamma(\sigma)}\,,$$

we have

$$\begin{aligned}M_1&=T_5(x-x')^{\sigma-1}A^r(x')+T_6(x-x')^{\sigma-2}A^{r+1}(x')\\&\quad+T_7\frac{\Gamma(\delta+1)}{\Gamma(r+2)\Gamma(\delta-r-1)}\int_0^{x'}(x-t)^{\gamma-\delta-1}(x-t)^{\delta-r-2}A^{r+1}(t)dt\,.\end{aligned}$$

By the second mean-value theorem, we have, for some v_3, $0<v_3\leqq x'$,

$$\begin{aligned}M_1&=T_5(x-x')^{\sigma-1}A^r(x')+T_6(x-x')^{\sigma-2}A^{r+1}(x')\\&\quad+T_7\frac{\Gamma(\delta+1)}{\Gamma(r+2)\Gamma(\delta-r-1)}(x-x')^{\gamma-\delta-1}\int_{v_3}^{x'}(x-t)^{\delta-r-1-1}A^{r+1}(t)dt\,,\\|M_1|&\leqq T_5(x-x')^{\sigma-1}|A^r(x')|+T_6(x-x')^{\sigma-2}|A^{r+1}(x')|\\&\quad+T_7(x-x')^{\gamma-\delta-1}\{|g_{\delta-r-1,r+1}(x',x)|+|g_{\delta-r-1,r+1}(v_3,x)|\}\\&=J_7+J_8+J_9\,,\qquad\text{say}\,.\end{aligned}$$

By (4.2) and (4.44), we have

$$J_7<T_5D_{r+1}\left(\frac{\varepsilon W(x)}{V(x)}\right)^{(\sigma-1)/\delta}\varepsilon^{r/\delta}V(x')^{1-r/\delta}W(x')^{r/\delta}\,.$$

This implies, in view of (2.1) (i) and (2.1) (ii),

$$J_7 < T_5 D_{r+1} H^{(\delta-r)/\delta} \varepsilon^{(\gamma-1)/\delta} V(x)^{1-(\gamma-1)/\delta} W(x)^{(\gamma-1)/\delta}. \tag{4.50}$$

Replacing γ in (4.16) by $r+1$, we have

$$|A^{r+1}(x)| < D_{r+2} \varepsilon^{(r+1)/\delta} V(x)^{1-(r+1)/\delta} W(x)^{(r+1)/\delta}. \tag{4.51}$$

(4.2) and (4.51) together imply

$$J_8 < T_6 D_{r+2} \left(\frac{\varepsilon W(x)}{V(x)}\right)^{(\sigma-2)/\delta} \varepsilon^{(r+1)/\delta} V(x')^{1-(r+1)/\delta} W(x')^{(r+1)/\delta}.$$

From (2.1) (i) and (2.1) (ii) we obtain

$$J_8 < T_6 D_{r+2} H^{(\delta-r-1)/\delta} \varepsilon^{(\gamma-1)/\delta} V(x)^{1-(\gamma-1)/\delta} W(x)^{(\gamma-1)/\delta}. \tag{4.52}$$

On the other hand, (4.1) and (4.2) give

$$J_9 < 2T_7 \varepsilon^{(\gamma-1)/\delta} V(x)^{1-(\gamma-1)/\delta} W(x)^{(\gamma-1)/\delta}. \tag{4.53}$$

Combining (4.50), (4.52) and (4.53) we observe that

$$|M_1| \leqq \text{Const.} \times \varepsilon^{(\gamma-1)/\delta} V(x)^{1-(\gamma-1)/\delta} W(x)^{(\gamma-1)/\delta}, \tag{4.54}$$

where the exact value of the "Const." is

$$T_5 D_{r+1} H^{(\delta-r)/\delta} + T_6 D_{r+2} H^{(\delta-r-1)/\delta} + 2T_7.$$

Since ε is arbitrary, (4.54) means

$$M_1 = o(V(x)^{1-(\gamma-1)/\delta} W(x)^{(\gamma-1)/\delta}). \tag{4.55}$$

(4.40), (4.43), (4.49) and (4.53) together imply

$$A^{\gamma-1}(x) = o(V(x)^{1-(\gamma-1)/\delta} W(x)^{(\gamma-1)/\delta}). \tag{4.56}$$

It remains to consider the case $\beta=0$. By the same arguments as in Case (ii), we have

$$A^{p-1}(x) = o(V(x)^{1-(p-1)/p} W(x)^{(p-1)/p}). \tag{4.57}$$

Now the hypotheses of Case (i) are satisfied with $A^{p-1}(x)$ $(p>1)$ in place of $A(x)$ and $V(x)^{1-(p-1)/p} W(x)^{(p-1)/p}$ in place of $V(x)$ ($W(x)$ unchanged). In fact,

$$\frac{W(x)}{V(x)^{1-(p-1)/p} W(x)^{(p-1)/p}} = \left(\frac{W(x)}{V(x)}\right)^{1/p} = O(x) \quad \text{and},$$

by (2.1) (ii) for x' satisfying $0 < x - x' < \eta x$ $(0 < \eta < 1)$,

$$\frac{V(x')^{1-(p-1)/p} W(x')^{(p-1)/p}}{V(x)^{1-(p-1)/p} W(x)^{(p-1)/p}} = \left(\frac{V(x')}{V(x)}\right)^{1/p} \left(\frac{W(x')}{W(x)}\right)^{1-1/p} < H^{1/p}.$$

Therefore, by the Case (i) ($\delta = 1$) already established, for γ such that $p-1<\gamma<p$, we see

$$A^{\gamma}(x)=o(\{V(x)^{1-(p-1)/p}W(x)^{(p-1)/p}\}^{1-(\gamma-(p-1))}W(x)^{\gamma-(p-1)}) \tag{4.58}$$
$$=o(V(x)^{1-\gamma/p}W(x)^{\gamma/p})\,.$$

Let $\gamma=r+\sigma$, r being the integral part of γ. We have only to consider the case $0<\sigma<1$. By Lemma B, we obtain

$$A^{\gamma-1}(x)=\frac{\Gamma(r+\sigma)}{\Gamma(r)\Gamma(\sigma)}\int_0^x (x-t)^{\sigma-1}A^{r-1}(t)dt \tag{4.59}$$
$$=\frac{\Gamma(r+\sigma)}{\Gamma(r)\Gamma(\sigma)}\left\{\int_0^{x'}+\int_{x'}^{x}\right\}(x-t)^{\sigma-1}A^{r-1}(t)dt$$
$$=N_1+N_2\,,\qquad \text{say}\,.$$

Replacing γ (resp. δ) by $r-1$ (resp. p) in (4.26), we have

$$|N_2|\leqq\frac{\Gamma(r+\sigma)}{\Gamma(r)\Gamma(\sigma)}\int_{x'}^{x}(x-t)^{\sigma-1}|A^{r-1}(t)|dt \tag{4.60}$$
$$\leqq\frac{\Gamma(r+\sigma)\varepsilon^{(r-1)/p}}{\Gamma(r)\Gamma(\sigma)}E_r\int_{x'}^{x}(x-t)^{\sigma-1}V(t)^{1-(r-1)/p}W(t)^{(r-1)/p}dt\,.$$

Since $W(x)$ is non-decreasing, and the choice of x' implies $0<x-t<x-x'<\eta x$ so that $V(t)<HV(x)$, we have

$$|N_2|\leqq\frac{\Gamma(r+\sigma)\cdot H^{1-(r-1)/p}\cdot\varepsilon^{(r-1)/p}}{\Gamma(r)\Gamma(\sigma)}\cdot E_r$$
$$\times V(x)^{1-(r-1)/p}W(x)^{(r-1)/p}\int_{x'}^{x}(x-t)^{\sigma-1}dt$$
$$=\frac{\Gamma(r+\sigma)\cdot H^{1-(r-1)/p}\cdot\varepsilon^{(r-1)/p}}{\Gamma(r)\Gamma(\sigma)\sigma}\cdot E_r$$
$$\times V(x)^{1-(r-1)/p}W(x)^{(r-1)/p}(x-x')^{\sigma}\,.$$

From (4.2), we get

$$|N_2|=\frac{\Gamma(r+\sigma)\cdot H^{1-(r-1)/p}\cdot\varepsilon^{(r-1)/p}}{\Gamma(r)\Gamma(\sigma)\sigma}\cdot E_r \tag{4.61}$$
$$\times V(x)^{1-(r-1)/p}W(x)^{(r-1)/p}\times(\varepsilon W(x)/V(x))^{\sigma/p}$$
$$=\frac{\Gamma(r+\sigma)\cdot H^{1-(r-1)/p}\cdot\varepsilon^{(\gamma-1)/p}}{\Gamma(r)\Gamma(\sigma)\sigma}\cdot E_r$$
$$\times V(x)^{1-(\gamma-1)/p}W(x)^{(\gamma-1)/p}\,.$$

Since ε is arbitrary, this implies

$$N_2=o(V(x)^{1-(\gamma-1)/p}W(x)^{(\gamma-1)/p}). \tag{4.62}$$

As to N_1, integrating by parts we have

$$N_1=\frac{\Gamma(r+\sigma)}{\Gamma(r)\Gamma(\sigma)r}\left\{[(x-t)^{\sigma-1}A^r(t)]_0^{x'}+(\sigma-1)\int_0^{x'}(x-t)^{\sigma-2}A^r(t)dt\right\}.$$

Writing

$$T_8=\frac{\Gamma(r+\sigma)}{\Gamma(r)\Gamma(\sigma)r}\quad(=T_5)\quad\text{and}\quad T_9=\frac{\Gamma(r+\sigma)\Gamma(p-r)(\sigma-1)}{\Gamma(\sigma)\Gamma(p+1)}$$

we have

$$\begin{aligned}N_1&=T_8(x-x')^{\sigma-1}A^r(x')\\&\quad+T_9\frac{\Gamma(p+1)}{\Gamma(r+1)\Gamma(p-r)}\int_0^{x'}(x-t)^{\gamma-p-1}(x-t)^{p-r-1}A^r(t)dt.\end{aligned}$$

By the second mean-value theorem, we have for some v_4, $0<v_4<x'$,

$$\begin{aligned}N_1&=T_8(x-x')^{\sigma-1}A^r(x')\\&\quad+T_9\cdot\frac{\Gamma(p+1)(x-x')^{\gamma-p-1}}{\Gamma(r+1)\Gamma(p-r)}\int_{v_4}^{x'}(x-t)^{p-r-1}A^r(t)dt,\quad\text{and}\end{aligned}$$

$$\begin{aligned}|N_1|&\leqq T_8(x-x')^{\sigma-1}|A^r(x')|+T_9(x-x')^{\gamma-p-1}\{|g_{p-r,r}(x',x)+|g_{p-r,r}(v_4,x)|\}\\&=J_{10}+J_{11},\qquad\text{say}.\end{aligned}$$

Replacing δ by p in (4.2) and γ by r in (4.26), we have

$$x-x'=\left(\frac{\varepsilon W(x)}{V(x)}\right)^{1/p}, \tag{4.63}$$

$$|A^r(x)|<E_{r+1}\varepsilon^{r/p}V(x)^{1-r/p}W(x)^{r/p}. \tag{4.64}$$

From (4.63) and (4.64) we obtain

$$J_{10}<T_8E_{r+1}\left(\frac{\varepsilon W(x)}{V(x)}\right)^{(\sigma-1)/p}\varepsilon^{r/p}V(x')^{1-r/p}W(x')^{r/p}.$$

By (2.1) (i) and (2.1) (ii) we have

$$J_{10}\leqq T_8E_{r+1}H^{(p-r)/p}\varepsilon^{(\gamma-1)/p}V(x)^{1-(\gamma-1)/p}W(x)^{(\gamma-1)/p}. \tag{4.65}$$

From (4.63) and (4.1), we get

$$J_{11}<2T_9\varepsilon^{(\gamma-1)/p}V(x)^{1-(\gamma-1)/p}W(x)^{(\gamma-1)/p}. \tag{4.66}$$

Combining (4.65) and (4.66) we confirm that

$$|N_1| \leqq \text{Const.}\, \varepsilon^{(\gamma-1)/p} V(x)^{1-(\gamma-1)/p} W(x)^{(\gamma-1)/p}, \tag{4.67}$$

where Const. is the abbreviation of the expression

$$T_8 E_{r+1} H^{(p-r)/p} + 2T_9 .$$

Since ε is arbitrary, (4.67) implies

$$N_1 = o(V(x)^{1-(\gamma-1)/p} W(x)^{(\gamma-1)/p}) . \tag{4.68}$$

Combining (4.59), (4.62) and (4.68), we have finally

$$A^{\gamma-1}(x) = o(V(x)^{1-(\gamma-1)/p} W(x)^{(\gamma-1)/p}) . \tag{4.69}$$

Thus, Theorem I holds for such $\gamma > 1$ that $\delta - 1 \leqq \gamma < \delta$ and it holds also for $\gamma - 1$ in place of γ, so the theorem is proved for all γ satisfying $0 < \gamma < \delta$ by induction.

5. Proof of Theorem II.

From (2.5) (i) we get, for $x \geqq 1$, $W(1)x^{-\alpha} \leqq W(x)$ $(0 \leqq \alpha < 1)$. Therefore,

$$\int_0^x W(t)dt \to \infty .$$

Using (2.6), it follows;

$$\begin{aligned} A^{\delta+1}(x) &= (\delta+1) \int_0^x A^\delta(t)dt = o\left(\int_0^x W(t)dt \right) \\ &= o\left(x^\alpha W(x) \int_0^x t^{-\alpha} dt \right) = o(xW(x)) . \end{aligned} \tag{5.1}$$

Combining (1.1), (2.6) and (5.1) we have

$$B^\delta(x) = xA^\delta(x) - A^{\delta+1}(x) = o(xW(x)) . \tag{5.2}$$

From (2.7) we obtain

$$xA(x) = O(xV(x)) . \tag{5.3}$$

The function $xW(x) = x^{1-\alpha} \cdot x^\alpha W(x)$ is non-decreasing by (2.5) (i). From (2.5) (ii) and (2.5) (iii), we see

$$\frac{x'V(x')}{xV(x)} < H \qquad \text{for} \quad 0 < x - x' < \eta x ,$$

$$\left(\frac{xW(x)}{xV(x)}\right)^{1/\delta}=O(x)\,.$$

Thus, the hypotheses of Theorem I are satisfied with

$A(x)$	replaced by	$xA(x)$
$A^\delta(x)$		$B^\delta(x)$
$W(x)$		$xW(x)$
$V(x)$		$xV(x)$.

Therefore, by Theorem I, for γ satisfying $0<\gamma<\delta$, we have

$$\begin{aligned}B^\gamma(x)&=o((xV(x))^{1-\gamma/\delta}(xW(x))^{\gamma/\delta})\\&=o(x\cdot V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta})\,,\qquad \text{as}\quad x\to\infty\,.\end{aligned}\tag{5.4}$$

Now suppose that $\gamma>0$ and $\delta-1\leqq\gamma<\delta$. If $\gamma>\delta-1$, we have, by (2.6), Lemma B and (2.5) (i)

$$\begin{aligned}A^{\gamma+1}(x)&=\frac{\Gamma(\gamma+2)}{\Gamma(\delta+1)\Gamma(\gamma-\delta+1)}\int_0^x(x-t)^{\gamma-\delta}A^\delta(t)dt\\&=o\left(\int_0^x(x-t)^{\gamma-\delta}W(t)dt\right)\\&=o\left(x^\alpha W(x)\int_0^x(x-t)^{\gamma-\delta}t^{-\alpha}dt\right)\\&=o(x^{\gamma-\delta+1}W(x))\,.\end{aligned}\tag{5.5}$$

Since (5.5) reduces to (2.6) for $\gamma=\delta-1$, (5.5) does hold for $\delta-1\leqq\gamma<\delta$. By (1.1), (5.4), (5.5) and (2.5) (iii), we have

$$\begin{aligned}A^\gamma(x)&=x^{-1}\{B^\gamma(x)+A^{\gamma+1}(x)\}\\&=o(x^{-1}\{xV(x)\}^{1-\gamma/\delta}W(x)^{\gamma/\delta}+x^{\gamma-\delta+1}W(x))\\&=o\left(V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta}\left\{1+x^{\gamma-\delta}\left(\frac{W(x)}{V(x)}\right)^{1-\gamma/\delta}\right\}\right)\\&=o(V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta})\qquad\text{as}\quad x\to\infty\,.\end{aligned}\tag{5.6}$$

Thus (2.8) holds for $\delta-1\leqq\gamma<\delta$.

Now, suppose $\delta-2\leqq\gamma<\delta-1$ so that $\delta-1\leqq\gamma+1<\delta$. We have, by (2.5) (iii) and (5.4),

$$
\begin{aligned}
(5.7) \qquad A^{\gamma}(x) &= x^{-1}(B^{\gamma}(x)+A^{\gamma+1}(x)) \\
&= o(x^{-1}\{(xV(x))^{1-\gamma/\delta}W(x)^{\gamma/\delta}+V(x)^{1-(\gamma+1)/\delta}W(x)^{(\gamma+1)/\delta}\}) \\
&= o(V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta}+x^{-1}V(x)^{1-\gamma/\delta}W(x)^{\gamma/\delta}(W(x)/V(x))^{1/\delta}) \\
&= o(V(x)^{1-\gamma/\delta}W(x))^{\gamma/\delta})\,.
\end{aligned}
$$

Thus the validity of Theorem II for $\gamma+1$ induces the validity for γ, and the theorem is proved completely by induction.

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