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Curvatures of Tangent Bundles with Cheeger-Gromoll Metric

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Although the Sasaki metric [7] of tangent bundles is a "naturally" defined Riemannian metric, it is "extremely rigid" ([6]). For example, O. Kowalski has shown that it is never locally symmetric unless the base metric is locally Euclidean. E. Musso and F. Tricerri [6] have generalized this fact. They have shown that it has a constant scalar curvature if and only if the base metric is flat.

O. Kowalski and the author have shown in [5] there are many other "naturally" defined Riemannian metrics on tangent bundles over Riemannian manifolds. Among the naturally lifted Riemannian metrics of tangent bundles, can we find nicely fitted ones? Concerning this, E. Musso and F. Tricerri [6] have given an explicit expression of a positive definite Riemannian metric of tangent bundles introduced by J. Cheeger and D. Gromoll [3]. They called it the *Cheeger-Gromoll metric*. We can find this metric in the file of a classification of the naturally lifted metrics given by O. Kowalski and the author [5]. In this paper we shall study curvatures of the Cheeger-Gromoll metric of the tangent bundle TM.

We shall give the Levi-Civita connection, the Riemannian curvature and the scalar curvature of the Cheeger-Gromoll metric. Especially, we shall show in Theorem 6.3 that the scalar curvature is nonnegative if the original metric on the base manifold has constant curvature $c \geq -3(n-2)/n$, $n = \dim M$. Since the value of the scalar curvature $\bar{S}$ at $(x, u) \in TM$ depends on the norm of u, $\bar{S}$ is never constant if the original metric on the base manifold has constant curvature.

1. Preliminaries.

First of all, we shall recall briefly lifts of vector fields on Riemannian manifolds to their tangent bundles.

Let g be a Riemannian metric on a manifold M and ∇ the Levi-Civita connection of g. Then the tangent space of the tangent boundle TM at any point $(x, u) \in TM$ splits

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into the horizontal and vertical subspaces with respect to ∇:

$$(TM)_{(x,u)}=H_{(x,u)}\oplus V_{(x,u)}\,.$$

If $X\in M_x$, the *horizontal lift* of X to a point $(x, u)\in TM$ is the unique vector $X^h\in H_{(x,u)}$ such that $p_*X^h=X$, where p denotes the natural projection of TM to M. The *horizontal lift* of a vector field $X\in\mathfrak{X}(M)$ to TM is the vector field X^h whose value at each point (x, u) is the horizontal lift of X_x to (x, u). The *vertical lift* of a vector $X\in M_x$ to (x, u) is the unique vector $X^v\in V_{(x,u)}$ such that $X^v(df)=Xf$ for all functions f on M. Here we consider a 1-form df on M as a function on TM, that is, df is a function defined by $(df)(x, u)=uf$. The *vertical lift* of $X\in\mathfrak{X}(M)$ to TM is the vector field X^v whose value at each point (x, u) is the vertical lift of X_x to (x, u).

Note that the map $X\to X^h$ is an isomorphism between the vector spaces M_x and $H_{(x,u)}$. Also the map $X\to X^v$ is an isomorphism between M_x and $V_{(x,u)}$. Obviously each tangent vector $\bar{Z}\in(TM)_{(x,u)}$ can be written in the form $\bar{Z}=X^h+Y^v$, where $X, Y\in M_x$ are uniquely determined.

If φ be a smooth function on M then

$$X^h(\varphi\circ p)=(X\varphi)\circ p \quad\text{and}\quad X^v(\varphi\circ p)=0 \tag{1.1}$$

hold for all $X\in\mathfrak{X}(M)$.

Let $(U; x^1, x^2, \cdots, x^n)$ be a coordinate system in M, and $(p^{-1}U; x^1, x^2, \cdots, x^n, u^1, u^2, \cdots, u^n)$ a coordinate system in TM. Then the horizontal lift of $X=\sum\xi^i\partial/\partial x^i\in\mathfrak{X}(U)$ is expressed as

$$X^h=\sum\xi^i\frac{\partial}{\partial x^i}-\sum\xi^a u^b\Gamma^i_{ab}\frac{\partial}{\partial u^i}\,, \tag{1.2}$$

where Γ^i_{jk}'s are the local components of ∇. The vertical lift of X is

$$X^v=\sum\xi^i\frac{\partial}{\partial u^i}\,. \tag{1.3}$$

Now let r be the norm of a vector u. Then, for any function f of $\boldsymbol{R}$ to $\boldsymbol{R}$,

$$X^h_{(x,u)}(f(r^2))=0\,, \tag{1.4}$$

$$X^v_{(x,u)}(f(r^2))=2f'(r^2)g_x(X, u)\,. \tag{1.5}$$

These follow from (1.1) and

$$X^h u^i=-\sum\xi^a u^b\Gamma^i_{ab} \quad\text{and}\quad X^v u^i=\xi^i\,. \tag{1.6}$$

Next we shall introduce some notations which will be used describing vectors getting from lifted vectors by basic operations on TM. Let T be a tensor of type $(1, s)$ on M. If $X_1, X_2, \cdots, X_{s-1}\in M_x$, then $h\{T(X_1, \cdots, u, \cdots, X_{s-1})\}$ is a horizontal vector at (x, u) which is introduced by the formula

$$h\{T(X_1, \cdots, u, \cdots, X_{s-1})\} = \sum u^a (T(X_1, \cdots, (\partial_a)_x, \cdots, X_{s-1}))^h ,$$

where ∂_a stands for $\partial/\partial x^a$. Also $v\{T(X_1, \cdots, u, \cdots, X_{s-1})\}$ is a vertical vector at (x, u) which is introduced by the formula

$$v\{T(X_1, \cdots, u, \cdots, X_{s-1})\} = \sum u^a (T(X_1, \cdots, (\partial_a)_x, \cdots, X_{s-1}))^v .$$

In particular $U = \sum u^a (\partial_a)^v_{(x,u)} = \sum u^a (\partial/\partial u^a)_{(x,u)}$ is the *canonical vertical vector* at (x, u). Moreover $h\{T(X_1, \cdots, u, \cdots, u, \cdots, X_{s-2})\}$ and $v\{T(X_1, \cdots, u, \cdots, u, \cdots, X_{s-2})\}$ are introduced by the similar way.

The bracket operation of vector fields on the tangent bundle is given by

$$[X^h, Y^h]_{(x,u)} = [X, Y]^h_{(x,u)} - v\{R(X, Y)u\} , \tag{1.7}$$

$$[X^h, Y^v]_{(x,u)} = (\nabla_X Y)^v_{(x,u)} , \tag{1.8}$$

$$[X^v, Y^v]_{(x,u)} = 0 \tag{1.9}$$

for all $X, Y \in \mathfrak{X}(M)$, where R is the Riemannian curvature of g defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} .$$

The Cheeger-Gromoll metric is a positive definite metric on TM which is described in terms of lifted vectors as follows.

DEFINITION 1.1. Let g be a Riemannian metric on a manifold M. Then the *Cheeger-Gromoll metric* is a Riemannian metric $\bar{g}$ on the tangent bundle TM such that

$$\bar{g}_{(x,u)}(X^h, Y^h) = g_x(X, Y) , \qquad \bar{g}_{(x,u)}(X^h, Y^v) = 0 ,$$

$$\bar{g}_{(x,u)}(X^v, Y^v) = \frac{1}{1+r^2}(g_x(X, Y) + g_x(X, u) g_x(Y, u))$$

for all $X, Y \in \mathfrak{X}(M)$. Here r is the norm of u as above.

2. The Levi-Civita connection.

We shall calculate the Levi-Civita connection $\bar{\nabla}$ of TM with Cheeger-Gromoll metric $\bar{g}$. This connection is characterized by the Koszul formula:

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_{\bar{X}} \bar{Y}, \bar{Z}) = {} & \bar{X}(\bar{g}(\bar{Y}, \bar{Z})) + \bar{Y}(\bar{g}(\bar{Z}, \bar{X})) - \bar{Z}(\bar{g}(\bar{X}, \bar{Y})) \\ & - \bar{g}(\bar{X}, [\bar{Y}, \bar{Z}]) + \bar{g}(\bar{Y}, [\bar{Z}, \bar{X}]) + \bar{g}(\bar{Z}, [\bar{X}, \bar{Y}]) \end{aligned}$$

for any $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(TM)$.

PROPOSITION 2.1. *Let $\bar{\nabla}$ be the Levi-Civita connection of TM with Cheeger-Gromoll metric $\bar{g}$. If $X, Y \in \mathfrak{X}(M)$, then*

$$(2.1)\qquad (\bar{\nabla}_{X^h} Y^h)_{(x,u)} = (\nabla_X Y)^h_{(x,u)} - \frac{1}{2} v\{R(X, Y)u\}\,,$$

$$(2.2)\qquad (\bar{\nabla}_{X^h} Y^v)_{(x,u)} = \frac{1}{2(1+r^2)} h\{R(u, Y)X\} + (\nabla_X Y)^v_{(x,u)}\,,$$

$$(2.3)\qquad (\bar{\nabla}_{X^v} Y^h)_{(x,u)} = \frac{1}{2(1+r^2)} h\{R(u, X)Y\}\,,$$

$$(2.4)\qquad \begin{aligned}(\bar{\nabla}_{X^v} Y^v)_{(x,u)} = &-\frac{1}{1+r^2}(\bar{g}(X^v, U)Y^v + \bar{g}(Y^v, U)X^v)_{(x,u)} \\ &+\frac{2+r^2}{1+r^2}\bar{g}_{(x,u)}(X^v, Y^v)U \\ &-\frac{1}{1+r^2}\bar{g}_{(x,u)}(X^v, U)\bar{g}_{(x,u)}(Y^v, U)U\,.\end{aligned}$$

PROOF. Let X, Y and Z be any vector fields on the base manifold M. We calculate $\bar{\nabla}$ using the Koszul formulas for g and for $\bar{g}$.

(1) Direct calculations using (1.7) and (1.8) give

$$\bar{g}_{(x,u)}(\bar{\nabla}_{X^h} Y^h, Z^h) = \bar{g}_{(x,u)}((\nabla_X Y)^h, z^h)\,,$$

$$\bar{g}_{(x,u)}(\bar{\nabla}_{X^h} Y^h, Z^h) = -\frac{1}{2}\bar{g}_{(x,u)}(v\{R(X, Y)u\}, Z^v)\,,$$

which imply (2.1).

(2) Calculations like above give

$$2\bar{g}_{(x,u)}(\nabla_{X^h} Y^h, Z^h) = -\bar{g}_{(x,u)}(Y^v, v\{R(Z, X)u\})\,.$$

Now we claim that

$$(2.5)\qquad \bar{g}_{(x,u)}(Y^v, v\{R(Z, X)u\}) = -\frac{1}{1+r^2}\bar{g}(h\{R(u, Y)X\}, Z^h)\,.$$

In fact, by definitions and the skew-symmetry of R,

$$\begin{aligned}\bar{g}_{(x,u)}(Y^v, v\{R(Z, X)u\}) &= \frac{1}{1+r^2}\sum u^a(g_x(Y, R(Z, X)\partial_a) + g_x(Y, u)g_x(R(Z, X)\partial_a, u)) \\ &= -\frac{1}{1+r^2}g_x(R(u, Y)X, Z) \\ &= -\frac{1}{1+r^2}\bar{g}_{(x,u)}(h\{R(u, Y)X\}, Z^h)\,.\end{aligned}$$

Thus

$$\bar{g}_{(x,u)}(\bar{\nabla}_{X^h} Y^v, Z^h) = \frac{1}{2(1+r^2)} \bar{g}(h\{R(u, Y)X\}, Z^h) . \tag{2.6}$$

Next the Koszul formula $2\bar{g}(\bar{\nabla}_{X^h} Y^v, Z^v)$ reduces to

$$X^h(\bar{g}(Y^v, Z^v)) - \bar{g}(Y^v, (\nabla_X Z)^v) + \bar{g}(Z^v, (\nabla_X Y)^v) .$$

We calculate the first term. Firstly (1.4) implies $X^h(1/(1+r^2)) = 0$. Secondly

$$X^h(g(Y, u) \circ p) = X^h(\sum u^a g(Y, \partial_a) \circ p) = g(\nabla_X Y, u) \circ p$$

by (1.6). Hence

$$X^h(\bar{g}(Y^v, Z^v)) = \bar{g}((\nabla_X Y)^v, Z^v) + \bar{g}(Y^v, (\nabla_X Z)^v) , \tag{2.7}$$

which implies that

$$\bar{g}(\bar{\nabla}_{X^h} Y^v, Z^v) = \bar{g}((\nabla_X Y)^v, Z^v) . \tag{2.8}$$

The formula (2.2) follows from (2.6) and (2.8).

(3) Calculations similar to those in (2) give the formula (2.3).

(4) Using (1.8), (1.9) and (2.7), it is easily seen that

$$\bar{g}(\bar{\nabla}_{X^v} Y^v, Z^h) = 0 . \tag{2.9}$$

Because of (1.9), the Koszul formula $2\bar{g}(\bar{\nabla}_{X^v} Y^v, Z^v)$ reduces to

$$X^v(\bar{g}(Y^v, Z^v)) + Y^v(\bar{g}(Z^v, X^v)) - Z^v(\bar{g}(X^v, Y^v)) .$$

Since, by (1.1),

$$X^v_{(x,u)}(g(Y, u) \circ p) = X^v_{(x,u)}(\sum u^a g(Y, \partial_a) \circ p) = g_x(X, Y) ,$$

it follows that

$$X^v_{(x,u)}(\bar{g}(Y^v, Z^v)) = -\frac{2}{1+r^2} g_x(X, u)(g_x(Y, Z) + g_x(Y, u) g_x(Z, u))$$

$$+ \frac{1}{1+r^2}(g_x(X, Y) g_x(Z, u) + g_x(X, Z) g_x(Y, u))$$

by (1.5). Hence, by $\bar{g}_{(x,u)}(X^v, U) = g_x(X, u)$,

$$\bar{g}_{(x,u)}(\bar{\nabla}_{X^v} Y^v, Z^v) = -\frac{1}{1+r^2}(\bar{g}(X^v, U)\bar{g}(Y^v, Z^v) + \bar{g}(Y^v, U)\bar{g}(X^v, Y^v))_{(x,u)} \tag{2.10}$$

$$+ \frac{2+r^2}{1+r^2} \bar{g}_{(x,u)}(X^v, Y^v) \bar{g}(Z^v, U)$$

$$-\frac{1}{1+r^2}\bar{g}_{(x,u)}(X^v, U)\bar{g}_{(x,u)}(Y^v, U)\bar{g}_{(x,u)}(Z^v, U)\,.$$

The equations (2.9) and (2.10) give the required formula (2.4). q.e.d.

3. The Riemannian curvature.

We shall calculate the Riemannian curvature tensor of TM with Cheeger-Gromoll metric $\bar{g}$.

PROPOSITION 3.1. *Let $\bar{R}$ be the Riemannian curvature tensor of TM with Cheeger-Gromoll metric $\bar{g}$. If $X, Y, Z \in M_x$, then*

$$\begin{aligned}(3.1)\qquad \bar{R}(X^h, Y^h)Z^h &= (R(X, Y)Z)^h \\ &\quad -\frac{1}{4(1+r^2)}h\{R(u, R(Y, Z)u)X - R(u, R(X, Z)u)Y - 2R(u, R(X, Y)u)Z\} \\ &\quad +\frac{1}{2}v\{(\nabla_Z R)(X, Y)u\}\,,\end{aligned}$$

$$\begin{aligned}(3.2)\qquad \bar{R}(X^h, Y^h)Z^v &= \frac{1}{2(1+r^2)}h\{(\nabla_X R)(u, Z)Y - (\nabla_Y R)(u, Z)X\} + (R(X, Y)Z)^v \\ &\quad -\frac{1}{4(1+r^2)}(v\{R(X, R(u, Z)Y)u - R(Y, R(u, Z)X)u\} + 4\bar{g}(Z^v, U)v\{R(X, Y)u\}) \\ &\quad +\frac{2+r^2}{1+r^2}\bar{g}(v\{R(X, Y)u\}, Z^v)U\,,\end{aligned}$$

$$\begin{aligned}(3.3)\qquad \bar{R}(X^h, Y^v)Z^h &= \frac{1}{2(1+r^2)}h\{(\nabla_X R)(u, Y)Z\} + \frac{1}{2}(R(X, Z)Y)^v \\ &\quad -\frac{1}{4(1+r^2)}(v\{R(X, R(u, Y)Z)u\} + 2\bar{g}(Y^v, U)v\{R(X, Z)u\}) \\ &\quad +\frac{2+r^2}{2(1+r^2)}\bar{g}(v\{R(X, Z)u\}, Y^v)U\,,\end{aligned}$$

$$\begin{aligned}(3.4)\qquad \bar{R}(X^h, Y^v)Z^v &= -\frac{1}{2(1+r^2)}(R(Y, Z)X)^h \\ &\quad +\frac{1}{2(1+r^2)^2}(\bar{g}(Y^v, U)h\{R(u, Z)X\} - \bar{g}(Z^v, u)h\{R(u, Y)X\}) \\ &\quad -\frac{1}{4(1+r^2)^2}h\{R(u, Y)R(u, Z)X\}\end{aligned}$$

$$(3.5)\qquad \bar{R}(X^v, Y^v)Z^h = \frac{1}{1+r^2}(R(X, Y)Z)^h$$

$$+\frac{1}{(1+r^2)^2}(\bar{g}(Y^v, U)h\{R(u, X)Z\} - \bar{g}(X^v, U)h\{R(u, Y)Z\})$$

$$+\frac{1}{4(1+r^2)^2}h\{R(u, X)R(u, Y)Z - R(u, Y)R(u, X)Z\},$$

$$(3.6)\qquad \bar{R}(X^v, Y^v)Z^v = \frac{3+3r^2+r^4}{(1+r^2)^2}(\bar{g}(Y^v, Z^v)X^v - \bar{g}(X^v, Z^v)Y^v)$$

$$-\frac{3+r^2}{(1+r^2)^2}(\bar{g}(Y^v, U)\bar{g}(Z^v, U)X^v - \bar{g}(X^v, U)\bar{g}(Z^v, U)Y^v)$$

$$-\frac{3+r^2}{(1+r^2)^2}(\bar{g}(Y^v, Z^v)\bar{g}(X^v, U) - \bar{g}(X^v, Z^v)\bar{g}(Y^v, U))U.$$

PROOF. Let X, Y and Z be any vector fields on M.

(1) Direct calculations using Proposition 2.1, (1.6), (1.7) and the second Bianchi identity give the required formula (3.1).

(2) Direct calculations also give the formula (3.2). Note that, since $\bar{g}_{(x,u)}(X^v, U) = g_x(X, u)$,

$$\bar{g}_{(x,u)}(v\{R(X, Y)u\}, U) = g_x(R(X, Y)u, u) = 0.$$

(3) Calculations similar to those in (2) give the formula (3.3).

(4) To calculate $\bar{R}(X^h, Y^v)Z^v$, note that (1.5) and $(\bar{\nabla}_{X^h}U)_{(x,u)} = 0$ which follows from (1.6) and (2.2).

(5) Since $[X^v, Y^v] = 0$, calculations to get the formula (3.5) reduce to those of $\bar{\nabla}_{X^v}\bar{\nabla}_{Y^v}Z^h$.

(6) To calculate $\bar{R}(X^v, Y^v)Z^v$, note that

$$\bar{g}(U, U) = r^2 \quad\text{and}\quad \bar{\nabla}_{X^v}U = \frac{1}{1+r^2}(X^v + \bar{g}(X^v, U)U),$$

which follow easily. q.e.d.

4. Sectional curvatures.

Hereafter $\langle\cdot, \cdot\rangle$ stands for the metric g on M, also $|\cdot|$ denotes the norm of vectors with respect to g. For the tangent vectors $X, Y \in M_x$, let $Q(X, Y)$ be the square of the area of the parallelogram with sides X and Y. If X and Y are linearly independent, $K(X, Y) = \langle R(X, Y)Y, X\rangle / Q(X, Y)$ is the sectional curvature of the plane spanned by X

and Y. A bar is used to distinguish objects in TM from the corresponding objects in M.

PROPOSITION 4.1. *If $X, Y \in M_x$, then*

$$\bar{K}(X^h, Y^h) = K(X, Y) - \frac{3|R(X, Y)u|^2}{4(1+r^2)Q(X, Y)}, \tag{4.1}$$

$$\bar{K}(X^h, Y^v) = \frac{|R(u, Y)X|^2}{4(1+r^2)|X|^2(|Y|^2 + \langle Y, u\rangle^2)}, \tag{4.2}$$

$$\bar{K}(X^v, Y^v) = \frac{2r^2}{1+r^2} + \frac{3+r^2}{(1+r^2)^3}\frac{Q(X, Y)}{\bar{Q}(X^v, Y^v)}. \tag{4.3}$$

Here assume that X and Y are linearly independent in (4.1) *and* (4.3).

PROOF. Direct calculations give

$$\bar{Q}(X^h, Y^h) = Q(X, Y), \tag{4.4}$$

$$\bar{Q}(X^h, Y^v) = \frac{1}{1+r^2}|X|^2(|Y|^2 + \langle Y, u\rangle^2), \tag{4.5}$$

$$\begin{aligned}\bar{Q}(X^v, Y^v) = \frac{1}{(1+r^2)^2}(Q(X, Y) + |X|^2\langle Y, u\rangle^2 + |Y|^2\langle X, u\rangle^2 \\ -2\langle X, Y\rangle\langle X, u\rangle\langle Y, u\rangle).\end{aligned} \tag{4.6}$$

(1) By the first Bianchi identity and the skew-symmetry of R, it follows from (3.1) that

$$\bar{g}(\bar{R}(X^h, Y^h)Y^h, X^h) = \langle R(X, Y)Y, X\rangle - \frac{3}{4(1+r^2)}|R(X, Y)u|^2.$$

This together with (4.4) gives (4.1).

(2) The equation (4.2) follows from (4.5) and

$$\bar{g}(\bar{R}(X^h, Y^v)Y^v, X^h) = \frac{1}{4(1+r^2)^2}|R(u, Y)X|^2,$$

which follows from (3.2) and (4.2).

(3) Calculations using (3.3) show that

$$\bar{g}(\bar{R}(X^v, Y^v)Y^v, X^v) = \frac{2r^2}{1+r^2}\bar{Q}(X^v, Y^v) + \frac{3+r^2}{(1+r^2)^3}Q(X, Y),$$

hence (4.3) follows from (4.6). q.e.d.

THEOREM 4.2. *Let (M, g) be a space of constant curvature c, and $\bar{K}$ the sectional curvature function of the tangent bundle TM with Cheeger-Gromoll metric $\bar{g}$. Then*

$\bar{K}(X^h, Y^h)$ is nonnegative if $0 \leq c \leq 4/3$, $\bar{K}(X^h, Y^v)$ and $\bar{K}(X^v, Y^v)$ are nonnegative if $c \geq 0$. Here, assume, for $\bar{K}(X^h, Y^h)$ and $\bar{K}(X^v, Y^v)$, that X and Y are linearly independent.

PROOF. (1) Let $\{X, Y\}$ be an orthonormal basis for a tangent plane to M at x. Then, since M has constant curvature c,

$$Q(X, Y)=1\,, \quad K(X, Y)=c\,, \quad R(X, Y)u=c(\langle Y, u\rangle X-\langle X, u\rangle Y)\,.$$

If $u \neq 0$, then, by (4.1),

$$\bar{K}(X^h, Y^h)=c-\frac{3c^2(\langle Y, u\rangle^2+\langle X, u\rangle^2)}{4(1+r^2)}\,.$$

Let $\{E_1, E_2, \cdots, E_n\}$ be an orthonormal basis for M_x such that $E_1=X$ and $E_2=Y$. Then

$$\langle X, u\rangle^2+\langle Y, u\rangle^2 \leq \sum_i \langle E_i, u\rangle^2=|u|^2\,,$$

which together with $|u|^2=r^2 \leq 1+r^2$ implies that $\bar{K}(X^h, Y^h)$ is nonnegative if $0 \leq c \leq 4/3$.

If $u=0$, then clearly $\bar{K}(X^h, Y^h)=c \geq 0$ at $(x, 0)$.

(2) The assertion for $\bar{K}(X^h, Y^v)$ and for $\bar{K}(X^v, Y^v)$ is clear by (4.2) and (4.3).

q.e.d.

REMARK. I would like to thank Professor K. Ogiue for some discussions concerning the above proof.

COROLLARY 4.3. *If the base manifold (M, g) is flat, then the Cheeger-Gromoll metric of the tangent bundle TM has the nonnegative sectional curvatures, which are never constant.*

PROOF. Let $\{X, Z\}$ and $\{Y, W\}$ be two pairs of linearly independent vector fields on M. Then Propositions 3.1 and 4.1 imply that $\bar{K}(X^h+Y^v, Z^h+W^v)$ is non-negative if the base manifold is flat. q.e.d.

REMARK. Musso-Tricerri [6] have shown that the Cheeger-Gromoll metric $\bar{g}$ on TM has the nonnegative sectional curvatures if the base manifold is the sphere with standard metric. Does the metric $\bar{g}$ have nonnegative sectional curvatures when the base manifold is a space of constant curvature c with $0 \leq c \leq 4/3$?

5. The scalar curvature.

Let (x, u) be a point of TM which is not in the zerosection, and $\{E_1, E_2, \cdots, E_n\}$ an orthonormal basis for M_x such that $E_1=u/r$. Here r is the norm of u as before. Then, putting

$$F_i=E_i^h\ (i=1, 2, \cdots, n)\,, \quad F_{1*}=E_1^v\,, \quad E_{p*}=\sqrt{1+r^2}\,E_p^v\ (p=2, 3, \cdots, n)\,,$$

we get an orthonormal basis $\{F_1, F_2, \cdots, F_n, F_{1*}, F_{2*}, \cdots, F_{n*}\}$ for the tangent space

$(TM)_{(x,u)}$. Let

$$K_{ij}=K(E_i, E_j) \quad (i, j=1, 2, \cdots, n;\ i\neq j)$$
$$\bar{K}_{AB}=\bar{K}(F_A, F_B) \quad (A, B=1, 2, \cdots, n, 1^*, 2^*, \cdots, n^*;\ A\neq B).$$

Then the following proposition is a direct consequence of Proposition 5.1.

PROPOSITION 5.1. *Under the notations above,*

$$\bar{K}_{ij}=K_{ij}-\frac{3}{4(1+r^2)}|R(E_i, E_j)u|^2,$$

$$\bar{K}_{i1^*}=0,$$

$$\bar{K}_{ip^*}=\frac{1}{4}|R(u, E_p)E_i|^2,$$

$$\bar{K}_{1^*p^*}=\frac{3+3r^2+2r^4}{1+r^2},$$

$$\bar{K}_{p^*q^*}=3(1+r^2)^2.$$

Here i and j run over $\{1, 2, \cdots, n\}$; p and q run over $\{2, 3, \cdots, n\}$.

REMARK. For any orthonormal basis $\{E_1, E_2, \cdots, E_n\}$ for M_x, putting $F_i=E_i^h(x, 0)$, $F_{i^*}=E_i^v(x, 0)$ $(i=1, 2, \cdots, n)$ we get an orthonormal basis $\{F_1, F_2, \cdots, F_n, F_{1^*}, F_{2^*}, \cdots, F_{n^*}\}$ for $(TM)_{(x,0)}$. Although the equations in Proposition 5.1 are obtained at a point (x, u), $u\neq 0$, these still hold for the above basis at a point in the zerosection.

PROPOSITION 5.2. *Let S and $\bar{S}$ be the scalar curvatures of (M, g) and $(TM, \bar{g})$, respectively. Then*

$$\bar{S}=S-\frac{3}{2(1+r^2)}\sum_{i<j}|R(E_i, E_j)u|^2+\frac{1}{2}\sum_{i,j}|R(u, E_i)E_j|^2$$
$$+\frac{n-1}{1+r^2}(2(3+3r^2+2r^4)+3(1+r^2)^3(n-2)).$$

PROOF. Since $S=2\sum_{i<j}K_{ij}$ and

$$\bar{S}=2\sum_{i<j}\bar{K}_{ij}+2\sum_{i,j}\bar{K}_{ij^*}+2\sum_{i<j}\bar{K}_{i^*j^*},$$

the assertion can be easily obtained from Proposition 5.1. q.e.d.

THEOREM 5.3. *If the base manifold M has constant curvature $c\geq -3(n-2)/n$, then its tangent bundle TM with Cheeger-Gromoll metric has nonnegative scalar curvature.*

PROOF. Let $\{E_1, E_2, \cdots, E_n\}$ be an orthonormal basis for M_x as above. If M has constant curvature c, then

$$|R(E_i, E_j)u|^2 = c^2r^2(\delta_{i1} + \delta_{j1}) \qquad \text{for} \quad i \neq j\,,$$

and

$$|R(u, E_i)E_j|^2 = c^2r^2(\delta_{ij} - 2\delta_{i1}\delta_{j1} + \delta_{j1})\,.$$

Hence, by Proposition 5.2,

$$\bar{S} = n(n-1)c + \frac{(n-1)c^2r^2}{2(1+r^2)}(2r^2+1) + \frac{n-1}{1+r^2}(2(3+3r^2+2r^4) + 3(1+r^2)(n-2))\,, \tag{5.1}$$

from which the assertion follows since the minimal value of (5.1) attains on the zerosection. q.e.d.

COROLLARY 5.4. *If the base manifold has constant curvature, then its tangent bundle TM with Cheeger-Gromoll metric is not (curvature) homogeneous.*

PROOF. The equation (5.1) implies that the scalar curvature $\bar{S}$ is never constant if c is a constant. Hence the assertion follows. q.e.d.

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