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Ring Derivations on Semi-Simple Commutative Banach Algebras

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Introduction.

Let A be a commutative Banach algebra. An (resp. linear) operator D on A is called a *ring* (resp. *linear*) *derivation* on A if equations $D(f+g)=D(f)+D(g)$ and $D(fg)=fD(g)+D(f)g$ are satisfied for every f and g in A. The image of linear derivation was studied by Singer and Wermer [5] under the hypothesis of continuity of the operator, and Thomas [6] has proved that every linear derivation on a commutative Banach algebra maps into the radical of the algebra. On the other hand there are ring derivations which do not map into the radical (cf. [1]). In this paper we characterize ring derivations on semi-simple commutative Banach algebras. A function algebra is semi-simple and so the results generalize our previous results in [3]. As a consequence of the results it is shown that only the zero operator is a ring derivation on a semi-simple commutative Banach algebra with the carrier space without an isolated point, which is a generalization of a theorem of Nandakumar [4].

1. Lemmata.

LEMMA 1. *Let A be a commutative Banach algebra with the carrier space M_A. Suppose that D is a ring derivation on A. Then $(D(\alpha f))\hat{}=\alpha(D(f))\hat{}$ for every f in A and for every rational number α in the complex number field $\mathbf{C}$, where $\hat{}$ denotes the Gel'fand representation.*

PROOF. If α is a rational real number, then $D(\alpha f)=\alpha D(f)$ by standard argument. So we only show that $(D(if))\hat{}=i(D(f))\hat{}$, where i is the imaginary unit. For every f in A,

$$2fD(f)=D(f^2)=-D((if)^2)=-2ifD(if),$$

so we have $(D(f))\hat{}(x)=-i(D(if))\hat{}(x)$ for every x in M_A with $\hat{f}(x)\neq 0$. When $\hat{f}(x)=0$, choose g in A with $\hat{g}(x)\neq 0$. In the same way we have $(D(g))\hat{}(x)=-i(D(ig))\hat{}(x)$ and $(D(f+g))\hat{}(x)=-i(D(i(f+g)))\hat{}(x)$ since $(f+g)\hat{}(x)=\hat{f}(x)+\hat{g}(x)\neq 0$, so

$$(D(f))\hat{}(x)+(D(g))\hat{}(x)=-i(D(if))\hat{}(x)-i(D(ig))\hat{}(x).$$

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We conclude that $(D(f))^\frown(x) = -i(D(if))^\frown(x)$ even if $\hat{f}(x)=0$. It follows that $i(D(f))^\frown = (D(if))^\frown$ on M_A.

REMARK. If A contains the unit, then $D(\alpha f)=\alpha D(f)$ for every f in A and rational *complex* number α. But it is not the case when A is not unital. Let $\boldsymbol{C}$ be the complex number field. Then $\boldsymbol{C}$ is a radical Banach algebra under the usual scalar multiplication and the usual summation and the multiplication $\times$ defined by $a\times b=0$ with the norm $\|\cdot\|=|\cdot|$. Define D by $D(a)=\bar{a}$, then D is a ring derivation and $D(ia)\neq iD(a)$ if $a\neq 0$.

LEMMA 2. *Let A be a commutative Banach algebra with the carrier space M_A. Suppose that x and y are different points in M_A. Then there is f in A with $\hat{f}(x)=0$ and $\hat{f}(y)=1$.*

Proof is trivial.

LEMMA 3. *Let A be a commutative Banach algebra with the carrier space M_A. Let $\{x_n\}$ be a sequence of distinct points in M_A. Suppose that D is a ring derivation on A. There is f_1 in A which satisfy that $\|f_1\|_A \leqq 1/2$, $\|D(f_1)\|_A \leqq 1/2$ and $\hat{f}_1(x_i)\neq 0$ for every positive integer i. For every positive integer n greater than 1 there is f_n in A which satisfies that $\|f_n\|_A \leqq 1/2$, $\|D(f_n)\|_A \leqq 1/2$, $\hat{f}_n(x_i)=0$ for $1\leqq i<n$ and $\hat{f}_n(x_i)\neq 0$ for $n\leqq i$.*

We can prove Lemma 3 by the same way as in the proof of Lemma 2 in [2].

LEMMA 4. *Let A be a commutative Banach algebra with the carrier space M_A. If the (not necessarily linear) functional $\phi_x(f)=(D(f))^\frown(x)$ defined on A is not continuous, then for every pair of positive numbers ε and K there exists f in A such that $\|f\|_A<\varepsilon$ and $|(D(f))^\frown(x)|>K$.*

PROOF. Suppose that there are positive number ε_0 and K_0 which satisfy that for every f in A with $\|f\|_A<\varepsilon_0$ we have $|(D(f))^\frown(x)|\leqq K_0$. We will show that ϕ_x is continuous. Let δ be a positive number. Put $\varepsilon=\delta'\varepsilon_0/K'$, where δ' and K' are rational positive numbers such that $\delta'<\delta$ and $K_0<K'$. If $\|f\|_A<\varepsilon$, then $\|(K'/\delta')f\|_A<\varepsilon_0$ so $|(D((K'/\delta')f))^\frown(x)|\leqq K_0$. Since D is linear over rational real number field, which is proven by the standard argument, we have $D((K'/\delta')f)=(K'/\delta')D(f)$ and so $|(D(f))^\frown(x)|<\delta$, which means that ϕ_x is continuous at 0. Thus we see that ϕ_x is continuous since $D(f-g)=D(f)-D(g)$ for every f and g in A.

The following lemma is a version of Theorem 1 in [2] in the case of ring derivations on Banach algebras.

LEMMA 5. *Let A be a commutative Banach algebra with the carrier space M_A. Let D be a ring derivation on A. Then the functional $\phi_x(f)=(D(f))^\frown(x)$ on A is continuous for every x in M_A but a finite exceptions.*

PROOF. Suppose that there are infinite number of points x in M_A at which ϕ_x is not continuous. Choose a sequence $\{x_n\}$ of distinct points at which ϕ_x is discontinuous.

For the sequence $\{x_n\}$, choose a sequence $\{f_n\}$ in A which satisfies the conditions in Lemma 3. Define inductively a sequence $\{F_n\}$ in A as follows. Put $F_1=0$. If $F_1, \cdots, F_{i-1}$ is defined, then put F_i in A satisfying the conditions:

1) $\|F_i\|_A<1$,

2) $|(D(F_i))\hat{}(x_i)|>(i+|(D(\sum_{j=1}^{i-1}(\prod_{\iota=1}^{j}f_\iota^2)F_j))\hat{}(x_i)|)/|\prod_{j=1}^{i}(\hat{f}_j^2(x_i))|$.

We see that $\|D(\prod_{j=1}^{i}f_j^2)\|_A\leqq 1/2$ for every i by induction on i. If $i=1$, then

$$\begin{aligned}\|D(f_1^2)\|_A&=\|2f_1D(f_1)\|_A\\&\leqq 2\|f_1\|_A\|D(f_1)\|_A\\&\leqq 1/2\,.\end{aligned}$$

We will show that $\|D(\prod_{j=1}^{i+1}f_j^2)\|_A\leqq 1/2$ under the hypothesis that $\|D(\prod_{j=1}^{i}f_j^2)\|_A\leqq 1/2$.

$$\begin{aligned}\left\|D\left(\prod_{j=1}^{i+1}f_j^2\right)\right\|_A&=\left\|f_{i+1}^2D\left(\prod_{j=1}^{i}f_j^2\right)+\left(\prod_{j=1}^{i}f_j^2\right)2f_{i+1}D(f_{i+1})\right\|_A\\&\leqq\|f_{i+1}\|_A{}^2\left\|D\left(\prod_{j=1}^{i}f_j^2\right)\right\|_A+2\left(\prod_{j=1}^{i}\|f_j\|_A{}^2\right)\|f_{i+1}\|_A\|D(f_{i+1})\|_A\\&\leqq 1/2\,.\end{aligned}$$

Put

$$G=\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}f_j^2\right)F_i$$

and

$$G_p=\sum_{i=p+1}^{\infty}\left(\prod_{j=1,j\neq p+1}^{i}f_j^2\right)F_i\,.$$

Then G and G_p converge in A since $\|f_j\|_A\leqq 1/2$ and $\|F_i\|_A<1$. We see that

$$G=\sum_{i=1}^{p}\left(\prod_{j=1}^{i}f_j^2\right)F_i+f_{p+1}^2G_p\,.$$

We will show that

$$|(D(G))\hat{}(x_p)|\geqq p-1$$

for every positive integer p. This is trivial for $p=1$, so we will prove it for $p\geqq 2$. Since $\hat{f}_{p+1}(x_p)=0$ for every p we have

$$\begin{aligned}(D(f_{p+1}^2G_p))\hat{}(x_p)&=\hat{f}_{p+1}(x_p)(D(f_{p+1}G_p))\hat{}(x_p)\ +\hat{f}_{p+1}(x_p)\hat{G}_p(x_p)(D(f_{p+1}))\hat{}(x_p)\\&=0\,.\end{aligned}$$

Thus

$$
\begin{aligned}
|(D(G))\hat{}(x_p)| &= \left| \left(D\left(\sum_{i=1}^{p} \left(\prod_{j=1}^{i} f_j^2 \right) F_i \right) \right)\hat{}\,(x_p) \right| \\
&\geqq \left| \left(D\left(\left(\prod_{j=1}^{p} f_j^2 \right) F_p \right) \right)\hat{}\,(x_p) \right| - \left| \left(D\left(\sum_{i=1}^{p-1} \left(\prod_{j=1}^{i} f_j^2 \right) F_i \right) \right)\hat{}\,(x_p) \right| \\
&\geqq \left| \left(\prod_{j=1}^{p} \hat{f}_j^2(x_p) \right) (D(F_p))\hat{}(x_p) \right| - \left| \hat{F}_p(x_p) \left(D\left(\prod_{j=1}^{p} f_j^2 \right) \right)\hat{}\,(x_p) \right| \\
&\quad - \left| \left(D\left(\sum_{i=1}^{p-1} \left(\prod_{j=1}^{i} f_j^2 \right) F_i \right) \right)\hat{}\,(x_p) \right| .
\end{aligned}
$$

Then by 2) we have

$$
\begin{aligned}
|(D(G))\hat{}(x_p)| &\geqq p - \| F_p \|_A \left\| D\left(\prod_{j=1}^{p} f_j^2 \right) \right\|_A \\
&\geqq p-1 .
\end{aligned}
$$

We conclude that $|(D(G))\hat{}(x_p)| \geqq p-1$, which is a contradiction since $(D(G))\hat{}$ is a bounded function on M_A.

2. Main results.

In this section we consider the problem on the image of a ring derivation on a commutative Banach algebra. In the case of a radical algebra the image is of course contained in the radical, so we consider the case of the algebra with a non-zero complex homomorphism. Suppose that A is a semi-simple commutative Banach algebra with the carrier space M_A and $x_1, \cdots, x_n$ are isolated points in M_A. Then there are idempotents $e_1, \cdots, e_n$ in A such that $\hat{e}_i(x)=1$ for $x=x_i$ and for otherwise $\hat{e}_i(x)=0$ for each i. (This is a direct consequence of the Šilov idempotent theorem.) Suppose also that $D_1, \dots, D_n$ are ring derivations on C. Then an operator D defined by $D(f)=\sum_{i=1}^{n} D_i(\hat{f}(x_i))e_i$ is a ring derivation on A. We consider the converse of the fact. As a consequence of the following theorem the converse is also true for semi-simple commutative Banach algebras, that is, a ring derivation on a semi-simple commutative Banach algebra has such a representation as above.

THEOREM. *Let A be a commutative Banach algebra with the carrier space M_A. Let D be a ring derivation on A. We assume the following*:

*) $$D(\mathrm{rad}(A)) \subset \mathrm{rad}(A) ,$$

where rad(A) *is the* (*Jacobson*) *radical of A. Then there are at most finite number of isolated points in M_A, say $y_1, \cdots, y_n$, and the same number of ring derivations $D_1, \cdots, D_n$ on the complex number field which satisfy*:

$$D(f) \in \sum_{i=1}^{n} D_i(\hat{f}(y_i))e_i + \mathrm{rad}(A),$$

where e_i is an idempotent such that $\hat{e}_i(x)=1$ for $x=y_i$ and $\hat{e}_i(x)=0$ for $x \neq y_i$ for every i.

PROOF. Let $\{y_1, \cdots, y_n\}$ be a set of points x in M_A at which the functional ϕ_x is discontinuous, then the set is finite by Lemma 5. First we show that $(D(f))\hat{}$ vanishes off $\{y_1, \cdots, y_n\}$ for every f in A and each y_i is an isolated point in M_A. Put $\hat{A}=\{\hat{f}: f \in A\}$. Then $\hat{A}$ is a semi-simple Banach algebra, with respect to the quotient norm induced by $A/\mathrm{rad}(A)$, of which the carrier space is M_A. Put $K=$the closure of $M_A-\{y_1, \cdots, y_n\}$ in M_A. Then $\hat{A}|K$ is a Banach algebra with respect to the quotient norm. We define an operator $\tilde{D}$ on $\hat{A}|K$ by $\tilde{D}(\varphi)=(D(f))\hat{}|K$, where $\varphi=\hat{f}|K$ for some f in A. Then $\tilde{D}$ is well defined and is a ring derivation on $\hat{A}|K$. We will show that $\tilde{D}$ is well defined. Suppose that $\{y_{i(1)}, \cdots, y_{i(l)}\}=M_A-K$. Then $\{y_{i(1)}, \cdots, y_{i(l)}\}$ is a subset of $\{y_1, \cdots, y_n\}$ and each $y_{i(j)}$ is an isolated point in M_A and so for every j there is an idempotent $e_{i(j)}$ in A such that $\hat{e}_{i(j)}(x)=0$ for $x \neq y_{i(j)}$ and $\hat{e}_{i(j)}(y_{i(j)})=1$. Suppose that $\hat{f}|K=\hat{g}|K$. Then we see that

$$f-g=\sum_{j=1}^{l} (\hat{f}(y_{i(j)})-\hat{g}(y_{i(j)}))e_{i(j)}+r,$$

where r is in $\mathrm{rad}(A)$. So we have

$$\begin{aligned} D(f-g) &= \sum_{j=1}^{l} D((\hat{f}(y_{i(j)})-\hat{g}(y_{i(j)}))e_{i(j)})+D(r) \\ &= \sum_{j=1}^{l} D((\hat{f}(y_{i(j)})-\hat{g}(y_{i(j)}))e_{i(j)}^2)+D(r) \\ &= \sum_{j=1}^{l} D((\hat{f}(y_{i(j)})-\hat{g}(y_{i(j)}))e_{i(j)})e_{i(j)}+D(r), \end{aligned}$$

since $e_{i(j)}=e_{i(j)}^2$ and $D(e_{i(j)})=0$. (If e is an idempotent in A, then $D(e)=0$ since $2eD(e)=D(e^2)=D(e)$ and $2eD(e)=2e^2D(e)=eD(e^2)=eD(e)$.) It follows by *) that

$$(D(f-g))\hat{}=\sum_{j=1}^{l} (D((\hat{f}(y_{i(j)})-\hat{g}(y_{i(j)}))e_{i(j)}))\hat{}\,\hat{e}_{i(j)}$$

and so we have $(D(f))\hat{}|K=(D(g))\hat{}|K$, that is, $\tilde{D}$ is well defined. The fact that $\tilde{D}$ is a ring derivation is easy to prove. If we can prove that $\tilde{D}$ is linear, then since $\hat{A}|K$ is semi-simple we have $\tilde{D}=0$ by the fact that there are no nonzero continuous linear derivations on semi-simple commutative Banach algebras (cf. [5, Theorem 1], [2, Theorem 2], [6]). It follows that $(D(f))\hat{}|K=0$ for every f in A. We then see that $\phi_x=0$, that is, ϕ_x is continuous for every x in K. We also conclude that

$$\{y_1, \cdots, y_n\}=\{y_{i(1)}, \cdots, y_{i(l)}\}.$$

Therefore each y_i is an isolated point in M_A and $(D(f))\hat{}$ vanishes off $\{y_1, \cdots, y_n\}$ for every f in A.

We will prove that $\tilde{D}$ is linear. Let x be a point in $M_A-\{y_1, \cdots, y_n\}$. We show that ϕ_x is linear, that is, $\phi_x(\alpha f)=\alpha\phi_x(f)$ for every complex number α and f in A. Choose a sequence $\{\alpha_n\}$ of rational complex numbers such that $\alpha_n\to\alpha$. Then $\phi_x((\alpha-\alpha_n)f)\to 0$ since ϕ_x is continuous. On the other hand

$$\begin{aligned}\phi_x((\alpha-\alpha_n)f)&=(D((\alpha-\alpha_n)f))\hat{}(x)\\&=(D(\alpha f))\hat{}(x)-(D(\alpha_n f))\hat{}(x)\\&=(D(\alpha f))\hat{}(x)-\alpha_n(D(f))\hat{}(x)\\&=\phi_x(\alpha f)-\alpha_n\phi_x(f)\end{aligned}$$

by Lemma 1. Since $\alpha_n\phi_x(f)\to\alpha\phi_x(f)$ we conclude that $\phi_x(\alpha f)=\alpha\phi_x(f)$. Thus we have $(D(\alpha f))\hat{}(x)=\alpha(D(f))\hat{}(x)$ on $M_A-\{y_1, \cdots, y_n\}$, and so on K. It follows that $\tilde{D}$ is a linear derivation.

For $1\leqq i\leqq n$ define the ring derivation D_i on the complex number field by

$$D_i(\alpha)=(D(\alpha e_i))\hat{}(y_i)\,,$$

where e_i is an idempotent in A such that $\hat{e}_i(y_i)=1$ and $\hat{e}_i(x)=0$ for $x\neq y_i$. Note that D_i is well defined since $(D(\alpha e_i))\hat{}=(D(\alpha e_i'))\hat{}$ holds for idempotents e_i and e_i' in A with $\hat{e}_i=\hat{e}_i'$ by the condition $*$). Since $D(e)=0$ for an idempotent e we see that $D(f-\sum_{i=1}^n fe_i)$ is in rad(A) for every f in A. For $(D(f-\sum_{i=1}^n fe_i))\hat{}$ vanishes off $\{y_1, \cdots, y_n\}$ and

$$\begin{aligned}\left(D\left(f-\sum_{i=1}^n fe_i\right)\right)\hat{}(y_j)&=(D(f))\hat{}(y_j)-\sum_{i=1}^n (D(fe_i))\hat{}(y_j)\\&=(D(f))\hat{}(y_j)-\sum_{i=1}^n (D(f))\hat{}(y_j)\hat{e}_i(y_j)\\&=0\end{aligned}$$

for $1\leqq j\leqq n$, we have that $(D(f-\sum_{i=1}^n fe_i))\hat{}$ vanishes on M_A. We have $D(fe_i-\hat{f}(y_i)e_i)$ is in rad(A) since $fe_i-\hat{f}(y_i)e_i$ is in rad(A) and the condition $*$) holds. We also see that

$$D(\hat{f}(y_i)e_i)-(D(\hat{f}(y_i)e_i))\hat{}(y_i)e_i$$

is in the radical of A. It follows that

$$\begin{aligned}D(f)=&D\left(f-\sum_{i=1}^n fe_i\right)+D\left(\sum_{i=1}^n (fe_i-\hat{f}(y_i)e_i)\right)\\&+\sum_{i=1}^n \{D(\hat{f}(y_i)e_i)-(D(\hat{f}(y_i)e_i))\hat{}(y_i)e_i\}+\sum_{i=1}^n (D(\hat{f}(y_i)e_i))\hat{}(y_i)e_i\end{aligned}$$

is in

$$\sum_{i=1}^n (D(\hat{f}(y_i)e_i))\hat{}(y_i)e_i+\text{rad}(A)=\sum_{i=1}^n D_i(\hat{f}(y_i))e_i+\text{rad}(A)\,.$$

COROLLARY 1. *Let A be a semi-simple commutative Banach algebra. Let D be a ring derivation on A. Then there exist at most finite number of isolated points $y_1, \cdots, y_n$ in the carrier space M_A and the same number of ring derivations $D_1, \cdots, D_n$ on the complex number field which satisfy that $D(f)=\sum_{i=1}^{n} D_i(\hat{f}(y_i))e_i$ for every f in A, where e_i is the idempotent in A such that $\hat{e}_i(y_i)=1$ and $\hat{e}_t(x)=0$ for $x \neq y_i$.*

Since a function algebra is a semi-simple commutative Banach algebra we see that every ring derivation on a function algebra is represented as in the same way as in Corollary 1 (cf. [3], [4]).

COROLLARY 2. *Let A be a semi-simple commutative Banach algebra with the carrier space without isolated points. Then only the zero operator is the ring derivation on A.*

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