TOKYO J. MATH.
VOL. 16, NO. 1, 1993

Asymptotic Expansions of Posterior Distributions in a Non-Regular Model

Takeshi KATO

Waseda University

(Communicated by Y. Shimizu)

Abstract. Let f be a density function with respect to Lebesgue measure. We suppose that $f(x)>0$ on $(0, \beta)$, where $0<\beta \leqq +\infty$, and f is uniformly continuous on $(0, \beta)$. Moreover, let $f'(x) \to \alpha$ as $x \to +0$ exist, where $0<\alpha<+\infty$. We consider a non-regular model defined by $f(x, \theta)=f(x-\theta)$, $\theta, x \in \boldsymbol{R}$. In the present paper, under some conditions, it is shown that when θ is regarded as a random variable with a prior density function with respect to Lebesgue measure, there exist asymptotic expansions of centered and scaled posterior distributions of θ.

1. Introduction

Let $f(x)$ be a uniformly continuous density function with respect to Lebesgue measure with its support $(0, \beta)$, $0<\beta \leqq +\infty$, and satisfy $f'(x) \to \alpha$, $0<\alpha<\infty$, as $x \to +0$. And let $\{f(x, \theta) : \theta \in \boldsymbol{R}\}$ be a family of density functions with one parameter defined by $f(x, \theta)=f(x-\theta)$. Moreover, let $x_1, x_2, \cdots, x_n$ be real-valued random variables independently and identically distributed in accordance with $f(x, \theta_0)$. On the occasion of considering the posterior distribution of θ when observations $x_1, \cdots, x_n$ are given, we suppose that the prior distribution has the density function $\rho(\cdot)$ with respect to Lebesgue measure.

The purpose of this paper is to show that, in this non-regular model, we can obtain an asymptotic expansions of the centered and scaled posterior distributions of θ under P_{θ_0}.

Woodroofe [7] has shown the asymptotic normality of $\alpha_n(\theta-\hat{\theta}_n)$ with respect to the marginal distribution of $(x_1, \cdots, x_n)$ in the same non-regular model, where $a_n^2=(\alpha/2)n \log n$, $n \geqq 1$ and $\hat{\theta}_n=\hat{\theta}_n(x_1, \cdots, x_n)$ is the maximum likelihood estimator of θ_0. In a regular model, Johnson [2] has shown that there exist asymptotic expansions of the centered and scaled posterior distributions with probability one. In comparison with that result, ours is concluded in a weaker sense.

Received April 10, 1992

2. Conditions and main theorems.

We shall impose the following conditions. Throughout this paper, let K be any non-negative integer.

(C_1) $\text{supp.} f=\{x: f(x)>0\}=(0, \beta)$, $0<\beta \leqq +\infty$. f is a uniformly continuous density function on its support and satisfies $\lim_{x\to\beta-0} f(x)=0$. Moreover, let $f(x, \theta)=f(x-\theta)$ for every $\theta\in \boldsymbol{R}$.

(C_2) f is $K+3$ times continuously differentiable on $(0, \beta)$, and

$$\lim_{x\to+0} f^{(1)}(x)=\alpha \ (0<\alpha<+\infty)\,, \quad \lim_{x\to+0} x^{j-1} f^{(j)}(x)=0 \qquad \text{for} \quad j=2, 3, \cdots, K+3\,,$$

where $f^{(j)}$ denotes the jth derivative of f.

Let $g(x)=\log f(x)$ and $g^{(j)}(x)=(d^j/dx^j)\log f(x)$ for $0<x<\beta$.

(C_3) For every d $(0<d<\beta)$, there exist $\tau>0$ and $0<\delta<\min\{d, \tau\}$ such that

$$\int_d^{\beta-\tau} \sup_{|t|\leqq\delta} |g^{(j)}(x-t)| f(x)dx<\infty \qquad \text{for} \quad j=2, 3, \cdots, K+3\,,$$

where $\beta-\tau=+\infty$ if $\beta=+\infty$.

(C_4) For every $\delta>0$, $\int_\delta^\beta \{g^{(1)}(x)\}^2 f(x)dx<+\infty$.

(C_5) $\int_0^\beta \{-g(x)\} f(x)dx<+\infty$.

(C_6) Let θ_0 be fixed. And let ρ be a prior density function of θ with respect to Lebesgue measure satisfying $\rho(\theta_0)>0$. ρ is $K+1$ times continuously differentiable in a neighborhood of θ_0.

(C_7) If $\beta<+\infty$, then $\limsup_{x\to\beta-0} g^{(j)}(x)<+\infty$ (may be $-\infty$) for $j=2, 3, \cdots, K+3$. And there are $\tau>0$, $0<\nu<1$, and a sequence of increasing functions $\{h_j\}_{j=2}^{K+3}$ on $(0, \tau\nu^{-1})$ for which

$$g^{(j)}(x)\geqq h_j(\beta-x)\,, \quad \beta-\tau<x<\beta\,, \qquad \text{for} \quad j=2, 3, \cdots, K+3\,,$$

$$\int_{\beta-\tau}^\beta h_j(\nu(\beta-x)) f(x)dx>-\infty\,, \qquad \text{for} \quad j=2, 3, \cdots, K+3\,.$$

(C_8) If $\beta<+\infty$, then, for $j=0, 1, \cdots, K+2$, $f^{(j)}$ is absolutely continuous in a neighborhood of β on $(0, \beta)$.

For example, the density function of Gamma$(2, q)$, where $q>0$ is a scale parameter, satisfies (C_1), (C_2), (C_3), (C_4) and (C_5). As to the case when $\beta<+\infty$, we will prove a lemma which gives sufficient conditions to satisfy (C_7).

Conditions required in Woodroofe [7] are derived from those above. Therefore we can make use of the results in it. In addition, note that these conditions yield the strong consistency of $\hat{\theta}_n$, that is, $\hat{\theta}_n \to \theta_0$ with probability one as $n\to\infty$ (See Wald [5]).

Let

$$\phi = b(\hat{\theta}_n)(\theta-\hat{\theta}_n)\,, \qquad b(\hat{\theta}_n)=\left[-\alpha_n^{-2}\sum_{i=1}^{n}\frac{\partial^2}{\partial\theta^2}\log f(x_i,\theta)\bigg|_{\theta=\hat{\theta}_n}\right]^{1/2},$$

where $\alpha_n^2=(\alpha/2)n\log n$, $n\geqq 1$. We often use a notation b instead of $b(\hat{\theta}_n)$ for simplicity. Denote the posterior distribution function of $\alpha_n\phi$ by F_n. We consider asymptotic expansions of F_n under P_{θ_0}. Our final aim is to show the theorems below. In the following description, when we use "probability", it means P_{θ_0}-probability.

THEOREM 1. *Let* (C_1), (C_2), (C_3), (C_4), (C_5), (C_6) *hold, and* (C_7), (C_8) *be satisfied also if* $\beta<+\infty$. *Then, for any* $\varepsilon>0$ *and* $0<p<1/2$, *there exist a constant* $D_1=D_1(\varepsilon, K)>0$ *and an integer* $N=N(\varepsilon, p, K)>0$ *such that*

$$|\,F_n(\xi)-\Phi(\xi)-\sum_{j=1}^{K}\gamma_j(\xi, X)\alpha_n^{-j}\,|\leqq D_1(\log n)^{-(K+3)/2+p}\,,$$

with probability greater than $1-\varepsilon$ *for* $n>N$, *where* $\Phi(\xi)$ *is the standard normal distribution function,* $X=(x_1,\cdots,x_n)$, *and for each* $j\geqq 1$, $\gamma_j(\xi, X)$ *is a polynomial in* ξ *with stochastically bounded coefficients multiplied by the standard normal density function* $\varphi(\xi)$.

In more detail, the asymptotic expansion of $F_n(\xi)$ can be expressed as follows.

THEOREM 2. *Under the same assumptions as Theorem* 1, *for any* $\varepsilon>0$ *and* $0<p<1/2$, *there exist a constant* $D_2=D_2(\varepsilon, K)>0$ *and an integer* $N=N(\varepsilon, p, K)>0$ *such that*

$$\Bigg|F_n(\xi)-\Phi(\xi)+\varphi(\xi)\Bigg[\sum_{m=1}^{(K+1)/2}\left\{(\log n)^{-(2m+1)/2}\sum_{l=0}^{m}B_{2m+1\,2l}\xi^{2l}\right\}$$
$$+\sum_{m=2}^{(K+1)/2}\left\{(\log n)^{-m}\sum_{l=1}^{m}B_{2m\,2l-1}\xi^{2l-1}\right\}\Bigg]\Bigg|\leqq D_2(\log n)^{-(K+3)/2+p}\,,$$

with probability greater than $1-\varepsilon$ *for* $n>N$, *if* K *is an odd number,*

$$\Bigg|F_n(\xi)-\Phi(\xi)+\varphi(\xi)\Bigg[\sum_{m=1}^{K/2}\left\{(\log n)^{-(2m+1)/2}\sum_{l=0}^{m}B_{2m+1\,2l}\xi^{2l}\right\}$$
$$+\sum_{m=2}^{(K+2)/2}\left\{(\log n)^{-m}\sum_{l=1}^{m}B_{2m\,2l-1}\xi^{2l-1}\right\}\Bigg]\Bigg|\leqq D_2(\log n)^{-(K+3)/2+p}\,,$$

with probability greater than $1-\varepsilon$ *for* $n>N$, *if* K *is an even number,*

$$|F_n(\xi)-\Phi(\xi)|\leqq D_2(\log n)^{-3/2+p},$$

with probability greater than $1-\varepsilon$ *for* $n>N$, *if* $K=0$,
where every $B_{st}=O_p(1)$ *with respect to* P_{θ_0} *as* $n\to\infty$.

3. Auxiliary lemmas.

We will prove a series of lemmas beforehand. The first thing, we quote the following Lemma 1, Lemma 2 and Lemma 3 without proofs.

LEMMA 1 (cf. [7; Lemma 2.1]). *Suppose that* (C_1) *and* (C_2) *are satisfied. Then,*

$$\lim_{n\to\infty} P_{\theta_0}\{n^{1/2}(M_n-\theta_0)>t\}=\exp(-\alpha t^2/2), \qquad \textit{for all } t>0,$$

where $M_n=\min\{x_1, x_2, \cdots, x_n\}$.

From this lemma and Theorems 2.1, 2.2 in [7], we can show that M.L.E. converges to θ_0 strictly faster than M_n.

LEMMA 2 (cf. [7; Lemma 2.2]). *Let* $\beta<+\infty$ *and suppose that* (C_1), (C_4) *and* (C_8) *are satisfied. Then,*

$$\lim_{n\to\infty} P_{\theta_0}\{n^{1/2}(\beta-N_n+\theta_0)>t\}=1, \qquad \textit{for all } t>0,$$

where $N_n=\max\{x_1, x_2, \cdots, x_n\}$.

Let

$$G_n(t)=\log\prod_{i=1}^{n} f(x_i, t) \quad \text{and} \quad G_n^{(j)}(t)=\frac{\partial^j}{\partial t^j}G_n(t) \qquad \text{for } j\geqq 1.$$

And put

$$r_n(\theta, k_n)=\sup_{|t|\leqq k_n}|\alpha_n^{-2}G_n^{(2)}(\theta+t\alpha_n^{-1})+1|, \qquad \text{where} \quad k_n=o(\sqrt{\log n}).$$

LEMMA 3 (cf. [7; Lemma 4.1] and [6; Appendix]). *Let* (C_1), (C_2), (C_3) *and* (C_4) *hold, and* (C_7), (C_8) *be satisfied also if* $\beta<+\infty$. *Then*

$$r_n(\hat{\theta}_n, k_n)\to 0, \qquad \textit{in probability as} \quad n\to\infty.$$

Since θ_0 is a location parameter, we will prove lemmas mentioned below on the assumption that $\theta_0=0$ as long as there is not notice in particular.

LEMMA 4. *Let* (C_1), (C_2), (C_3) *and* (C_4) *hold, and* (C_7), (C_8) *be satisfied also if* $\beta<+\infty$. *Then*

$$1/\sqrt{2} \leqq b(\hat{\theta}_n) \leqq \sqrt{3/2}\,, \qquad \textit{in probability as } n \to \infty\,.$$

PROOF. By definitions, $b(\hat{\theta}_n) = [-\alpha_n^{-2} G_n^{(2)}(\hat{\theta}_n)]^{1/2}$. Therefore, by Lemma 3, $|1-b^2| = |1+\alpha_n^{-2} G_n^{(2)}(\hat{\theta}_n)| \to 0$, in probability as $n \to \infty$. This immediately gives the result.

LEMMA 5. *Let* (C_1), (C_2), (C_3) *and* (C_4) *hold, and* (C_7), (C_8) *be satisfied also if* $\beta < +\infty$. *Then*

$$\alpha_n{}^{-2} \log\left(\prod_{i=1}^{n} \frac{f(x_i, \hat{\theta}_n + \phi b^{-1})}{f(x_i, \hat{\theta}_n)} \right) \leqq \frac{-\phi^2}{6} \qquad \textit{for } |\phi| < \frac{k_n}{c_n}\,,$$

$$\textit{in probability as } n \to \infty\,,$$

where $c_n = \sqrt{n \log n}$.

PROOF.

$$\alpha_n^{-2} \log\left(\prod_{i=1}^{n} \frac{f(x_i, \hat{\theta}_n + \phi b^{-1})}{f(x_i, \hat{\theta}_n)} \right) = 2^{-1} (\phi b^{-1})^2 \alpha_n^{-2} \sum_{i=1}^{n} \frac{\partial^2}{\partial \theta^2} \log f(x_i, \theta)\big|_{\theta=\theta_*}\,,$$

where $|\hat{\theta}_n - \theta_*| \leqq |\phi b^{-1}|$. Since $|\phi| < k_n/c_n$, together with Lemma 4, we obtain that $|\hat{\theta}_n - \theta_*| \leqq \sqrt{2}\, k_n/c_n$. The lemma follows from Lemma 3.

LEMMA 6. *Let* (C_1), (C_2), (C_3), (C_4) *and* (C_5) *hold, then for any* $\delta_1 > 0$, *there exists* $\varepsilon_1 > 0$ *such that*

$$\alpha_n^{-2} \log\left(\prod_{i=1}^{n} \frac{f(x_i, \hat{\theta}_n + \phi b^{-1})}{f(x_i, \hat{\theta}_n)} \right) \leqq -\varepsilon_1 \qquad \textit{for } |\phi| \geqq \delta_1\,,$$

$$\textit{in probability as } n \to \infty\,.$$

This lemma is shown in the same way as Lemma 2.3 in Johnson [2]. Therefore, we omit the proof.

Before going to the next step, note that it is obtained from (C_2) that

$$f(x) \sim \alpha x \quad \text{and} \quad g^{(j)}(x) \sim (-1)^{j-1}(j-1)!\,x^{-j} \ (j \geqq 1) \quad \text{as } x \to +0\,.$$

LEMMA 7. *Let* $0 < \delta < \beta$, *and define*

$$z_i = \begin{cases} x_i^{-1} : 0 < x_i < \delta \\ 0 \quad\ : x_i \geqq \delta \end{cases} \qquad (i = 1, 2, \cdots, n)\,.$$

Then, for $j \geqq 3$, *for any* $\varepsilon > 0$, *there exist a constant* $L_1 = L_1(\varepsilon, j) > 0$ *and an integer* $N = N(\varepsilon, j) > 0$ *such that*

$$\left| n^{-j/2} \sum_{i=1}^{n} z_i^j \right| \leqq L_1\,, \qquad \textit{with probability greater than } 1-\varepsilon \textit{ for } n > N\,.$$

PROOF. Put $s=z^j$. Since x has the density function f, the density function u of s is given by

$$u(s)=\frac{s^{-(j+1)/j}f(s^{-1/j})}{j} \quad \text{on } (\delta^{-j}, \infty) \quad \text{and} \quad P_0(s=0)=\int_\delta^\beta f(t)dt\,.$$

Let $\mu(y)=\int_{-y}^{y}(z^j)^2dP_0$, $y>0$. Then, for any $\varepsilon'>0$, we can choose y so large and δ so small that

$$\mu(y)=\int_{y^{-1/j}}^{\delta} t^{-2j}f(t)dt \leqq \alpha(1+\varepsilon')\int_{y^{-1/j}}^{\delta} t^{-2j+1}dt$$

$$=\frac{\alpha(1+\varepsilon')}{2(j-1)}y^{2-2/j}+O(1)\,.$$

Similarly,

$$\mu(y)\geqq\frac{\alpha(1-\varepsilon')}{2(j-1)}y^{2-2/j}+O(1)\,.$$

Therefore

$$\mu(y)\sim\frac{\alpha}{2(j-1)}y^{2-2/j} \quad \text{as } y\to+\infty \qquad \text{for } j\geqq 3\,.$$

Then, by Theorem 2 of [1], page 580, $n^{-j/2}\sum_{i=1}^n z_i^j$ has an asymptotic stable distribution. The lemma follows.

We define

$$r_n^{(j)}(\theta, k_n)=\sup_{|t|\leqq k_n}|n^{-j/2}G_n^{(j)}(\theta+t\alpha_n^{-1})|$$

for $j\geqq 3$.

LEMMA 8. *Let $\beta=+\infty$ and $j\geqq 3$. If (C_1), (C_2) and (C_3) are satisfied, then, for any $\varepsilon>0$, there exist a constant $L_2=L_2(\varepsilon, j)>0$ and an integer $N=N(\varepsilon, j, k_n)>0$ such that*

$$r_n^{(j)}(0, k_n)\leqq L_2\,, \qquad \textit{with probability greater than } 1-\varepsilon \textit{ for } n>N\,.$$

PROOF. This will be proved similarly to Lemma 3.4 in Woodroofe [7]. Let $0<\varepsilon'<1$ be given and $d>0$ be so small that

$$\left|\frac{(-1)^{j-1}x^jg^{(j)}(x)}{(j-1)!}-1\right|<\varepsilon' \qquad \text{for } 0<x<2d\,.$$

And put $q_n=k_n\alpha_n^{-1}$. By Lemma 1, $M_n\geqq q_n/\varepsilon'$ is satisfied in probability as $n\to\infty$. Therefore, for $|t|<k_n$ and $q_n<d$, if j is an odd number, we obtain that

$$n^{-j/2}G_n^{(j)}(t\alpha_n^{-1}) \geqq -(1+\varepsilon')(j-1)!n^{-j/2}\sum_0^d (x_i - t\alpha_n^{-1})^{-j} - n^{-j/2}\sum_d^\infty \sup_{|t|\leqq q_n} |g^{(j)}(x_i - t)|$$

$$\geqq -(1+\varepsilon')(1-\varepsilon')^{-j}(j-1)!n^{-j/2}\sum_0^d x_i^{-j} + o_p(1) \qquad \text{as } n\to\infty ,$$

where $\sum_s^t$ denotes a summation over all $i=1, 2, \cdots, n$ for which $s<x_i<t$. And

$$n^{-j/2}G_n^{(j)}(t\alpha_n^{-1}) \leqq -(1-\varepsilon')(1+\varepsilon')^{-j}(j-1)!n^{-j/2}\sum_0^d x_i^{-j} + o_p(1) \qquad \text{as} \quad n\to\infty .$$

The application of Lemma 7 establishes the existence of lower and upper bounds. Similarly, we can obtain bounds for an even number j. This completes the proof.

LEMMA 9. *Let $\beta<+\infty$ and $j\geqq 3$. If (C_1), (C_2), (C_3), (C_7) and (C_8) are satisfied, then, for any $\varepsilon>0$, there exist a constant $L_2=L_2(\varepsilon, j)>0$ and an integer $N=N(\varepsilon, j, k_n)>0$ such that*

$$r_n^{(j)}(0, k_n) \leqq L_2 , \qquad \textit{with probability greater than } 1-\varepsilon \textit{ for } n>N .$$

PROOF. Let $0<\varepsilon'<1$ be given, let v, τ and h_j be as in the condition (C_7), let $\tau_0=\tau/2$, and d and q_n be as in Lemma 8. Then, in the same way as the previous lemma, for $|t|<k_n$ and $q_n<\min\{d, \tau_0\}$, if j is an odd number, we obtain that

$$n^{-j/2}G_n^{(j)}(t\alpha_n^{-1}) \leqq -(1-\varepsilon')(1+\varepsilon')^{-j}(j-1)!n^{-j/2}\sum_0^d x_i^{-j} + o_p(1)$$

$$-n^{-j/2}\sum_{\beta-\tau_0}^{\beta} g^{(j)}(x_i - t\alpha_n^{-1}) , \qquad \text{as } n\to\infty ,$$

$$n^{-j/2}G_n^{(j)}(t\alpha_n^{-1}) \geqq -(1+\varepsilon')(1-\varepsilon')^{-j}(j-1)!n^{-j/2}\sum_0^d x_i^{-j} + o_p(1)$$

$$-n^{-j/2}\sum_{\beta-\tau_0}^{\beta} g^{(j)}(x_i - t\alpha_n^{-1}) , \qquad \text{as } n\to\infty .$$

Lemma 2 ensures that if $|t|<k_n$, then $\beta-\tau<x_i-t\alpha_n^{-1}<\beta$ and $0<v(\beta-x_i)<\beta-x_i-q_n<\beta-x_i+t\alpha_n^{-1}<\tau v^{-1}$ in probability as $n\to\infty$, for $\beta-\tau_0<x_i<\beta$. Choose $\tau>0$ so small, then by (C_7), there exists a constant $A>0$ such that $A\geqq g^{(j)}(x_i-t\alpha_n^{-1})\geqq h_j(\beta-x_i+t\alpha_n^{-1})\geqq h_j(v(\beta-x_i))$ in probability as $n\to\infty$, for $\beta-\tau_0<x_i<\beta$. Consequently, it follows from Lemma 7 that the bounds exist. In the same way, we can show the case when j is an even number.

LEMMA 10. *Let $j\geqq 3$. Let (C_1), (C_2), (C_3), (C_4) hold, and (C_7), (C_8) be satisfied also if $\beta<+\infty$. Then, for any $\varepsilon>0$, there exist a constant $L_2=L_2(\varepsilon, j)>0$ and an integer $N=N(\varepsilon, j, k_n)>0$ such that*

$$r_n^{(j)}(\hat{\theta}_n, k_n) \leqq L_2\,, \qquad \textit{with probability greater than } 1-\varepsilon \textit{ for } n>N\,.$$

We omit this proof since it is shown in just the same way as Lemma 4.1 in [7]. Put

$$a_{jn}(\theta)=\alpha_n^{-2}\sum_{i=1}^{n}\frac{(\partial^j/\partial\theta^j)\log f(x_i, \theta)}{j!} \qquad \text{for } j=2, 3, \cdots, K+3\,,$$

$$\rho_k(\theta)=\rho(\hat{\theta}_n)+\rho^{(1)}(\hat{\theta}_n)(\theta-\hat{\theta}_n)+\cdots+\frac{\rho^{(k)}(\hat{\theta}_n)(\theta-\hat{\theta}_n)^k}{k!}\,,$$

where $\rho^{(j)}$ denotes jth derivative of ρ.

LEMMA 11. *Let $j \geqq 3$. Let* (C_1), (C_2), (C_3), (C_4), (C_5), (C_6) *hold, and* (C_7), (C_8) *be satisfied also if $\beta<+\infty$. Then, for any $\varepsilon>0$, there exist a constant $L_3=L_3(\varepsilon, K)>0$ and an integer $N=N(\varepsilon, k_n, K)>0$ such that*

$$\int_{|\phi|<k_n/c_n}\left|\exp\left(\alpha_n^2\sum_{j=2}^{K+3}a_{jn}(\hat{\theta}_n)(\phi b^{-1})^j\right)\rho_K(\hat{\theta}_n+\phi b^{-1})\right.$$
$$\left.-\prod_{i=1}^{n}\frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)}\rho(\hat{\theta}_n+\phi b^{-1})\right|d\phi$$
$$\leqq L_3(\alpha_n^{-(K+2)}+\alpha_n^{-(K+4)}n^{(K+3)/2})\,,$$

with probability greater than $1-\varepsilon$ for $n>N$.

PROOF. The integrand of the left hand side is bounded by

$$\left|\exp\left(\alpha_n^2\sum_{j=2}^{K+3}a_{jn}(\hat{\theta}_n)(\phi b^{-1})^j\right)-\prod_{i=1}^{n}\frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)}\right||\rho_K(\hat{\theta}_n+\phi b^{-1})|$$
$$+\left|\prod_{i=1}^{n}\frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)}\right||\rho_K(\hat{\theta}_n+\phi b^{-1})-\rho(\hat{\theta}_n+\phi b^{-1})|\,.$$

By Lemma 5 the second term has its bounds of the form

$$\frac{\exp(-\alpha_n^2\phi^2/6)|\phi b^{-1}|^{K+1}|\rho^{(K+1)}(\theta_*)|}{(K+1)!}\,,$$

where $|\hat{\theta}_n-\theta_*|\leqq|\phi b^{-1}|$. Now we consider the upper bound of the first term. By Lemma 5, it is bounded by

$$\exp(-\alpha_n^2\phi^2/6)\left|\exp\left[\alpha_n^2\sum_{j=2}^{K+3}a_{jn}(\hat{\theta}_n)(\phi b^{-1})^j\right.\right.$$
$$\left.\left.-\log\left(\prod_{i=1}^{n}\frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)}\right)\right]-1\right||\rho_K(\hat{\theta}_n+\phi b^{-1})|$$

In the meanwhile, Lemma 10 and Lemma 4 yield an inequality that

$$\left|\log\left(\prod_{i=1}^{n}\frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)}\right)-\alpha_n^2\sum_{j=2}^{K+3}a_{jn}(\hat{\theta}_n)(\phi b^{-1})^j\right|$$

$$\leqq |\phi b^{-1}|^{K+3}n^{(K+3)/2}\frac{|n^{-(K+3)/2}G_n^{(K+3)}(\theta^*)|+|n^{-(K+3)/2}G_n^{(K+3)}(\hat{\theta}_n)|}{(K+3)!}$$

$$\leqq n^{(K+3)/2}L_3'|\phi|^{K+3},$$

for some $L_3'>0$, where $|\hat{\theta}_n-\theta^*|\leqq|\phi b^{-1}|$. Therefore, since $|\phi|<k_n/c_n$, the first term is bounded by

$$L_3''n^{(K+3)/2}\exp(-\alpha_n^2\phi^2/6)|\phi|^{K+3}|\rho_K(\hat{\theta}_n+\phi b^{-1})|,$$

for some $L_3''>0$ as $n\to\infty$. Together with Lemma 4, it follows from the continuity of $\rho^{(j)}$ for $j=0, 1, \cdots, K+1$ and the consistency of $\hat{\theta}_n$ that the integrand is bounded by

$$L_3\exp(-\alpha_n^2\phi^2/6)\{|\phi|^{K+1}+n^{(K+3)/2}|\phi|^{K+3}\}$$

for $|\phi|<k_n/c_n$ and some $L_3>0$. Finally, we have only to change variable $y=\alpha_n\phi$.

We quote the following lemma which will be used later.

LEMMA 12 (cf. [4; Lemma 9] and [3; Lemma 1]). *Let $\beta=+\infty$. If* (C$_1$), (C$_2$) *and* (C$_3$) *are satisfied, then for sufficiently small $\delta_2>0$,*

$$\sup_{-\delta_2\leqq t<M_n} n^{-1}G_n^{(3)}(t)<-1, \qquad \textit{in probability as } n\to\infty.$$

For the sake of the case when $\beta<+\infty$, we will prove the following lemma, which is similar to Lemma 12.

LEMMA 13. *Let $\beta<+\infty$. If* (C$_1$), (C$_2$), (C$_3$) *and* (C$_7$) *are satisfied, then there exists $0<\delta_3<1$ such that*

$$\sup_{(N_n-\beta)\delta_3<t<M_n} n^{-1}G_n^{(3)}(t)<-1, \qquad \textit{in probability as } n\to\infty.$$

PROOF. Let ν, τ and h_3 be as in the condition (C$_7$), and let $\tau_0=\tau/2$ and $d>0$ be so small that $g^{(3)}(x)\geqq x^{-3}$ for $0<x<2d$. Suppose that $M_n\leqq\delta_2$ and $\beta-N_n\leqq\delta_2$ for $\delta_2<\min\{d, \tau_0\}$, which are satisfied in probability as $n\to\infty$. If $(N_n-\beta)\delta_3<t<M_n$, then we obtain

$$n^{-1}G_n^{(3)}(t)\leqq -n^{-1}\sum_{0}^{d}(x_i+\delta_2)^{-3}+n^{-1}\sum_{d}^{\beta-\tau_0}\sup_{|t|\leqq\delta_2}|g^{(3)}(x_i-t)|$$

$$-n^{-1}\sum_{\beta-\tau_0}^{\beta}g^{(3)}(x_i-t) \qquad \text{as } n\to\infty.$$

The second term of the right hand side converges as $n\to\infty$ by (C$_3$). And choosing

$\delta_3=1-\nu$, it follows from (C_7) that

$$g^{(3)}(x_i-t)\geqq h_3(\beta-x_i+t)\geqq h_3(\nu(\beta-x_i)) \qquad \text{for} \quad \beta-\tau_0<x<\beta\,.$$

On the other hand, the first term converges to $-\int_0^d (x+\delta_2)^{-3}f(x)dx$ as $n\to\infty$, which diverges to $-\infty$ as $\delta_2\to 0$. The lemma follows.

LEMMA 14. *Let $\beta=+\infty$. Suppose that* (C_1), (C_2), (C_3), (C_4) *and* (C_5) *are satisfied. Then, for any $\varepsilon>0$ and sufficiently small $\delta_1>0$, there exist a constant $c>0$ and an integer $N=N(\varepsilon, \delta_1, k_n)>0$ such that*

$$\log\left(\prod_{i=1}^{n}\frac{f(x_i, \hat\theta_n+\phi b^{-1})}{f(x_i, \hat\theta_n)}\right)\leqq -ck_n^2 \qquad \textit{for} \quad k_n/c_n\leqq|\phi|<\delta_1\,,$$

with probability greater than $1-\varepsilon$ for $n>N$.

PROOF. Let $\varepsilon>0$ be fixed. Put $t=\hat\theta_n+\phi b^{-1}$ and $q_n=k_n\alpha_n^{-1}$. Then,

$$\hat\theta_n-b^{-1}\delta_1<t\leqq\hat\theta_n-\sqrt{\alpha/2}\,b^{-1}q_n \qquad \text{if} \quad -\delta_1<\phi\leqq -k_n/c_n\,,$$

$$\hat\theta_n+\sqrt{\alpha/2}\,b^{-1}q_n\leqq t<\hat\theta_n+b^{-1}\delta_1 \qquad \text{if} \quad k_n/c_n\leqq\phi<\delta_1\,.$$

It follows from Theorem 2.1 in [7] and Lemma 1 that

$$\begin{aligned}-(1+\sqrt{2}\,)\delta_1<\hat\theta_n-b^{-1}\delta_1<\hat\theta_n-\sqrt{\alpha/2}\,b^{-1}q_n<\hat\theta_n\\ <\hat\theta_n+\sqrt{\alpha/2}\,b^{-1}q_n<M_n<\hat\theta_n+b^{-1}\delta_1\,,\end{aligned}$$

with probability greater than $1-\varepsilon$ as $n\to\infty$. By Lemma 3.1 in [7] and Lemma 12,

$$\begin{aligned}\sup_{\hat\theta_n+\sqrt{\alpha/2}b^{-1}q_n\leqq t<\hat\theta_n+\delta_1 b^{-1}} G_n(t)&=\sup_{\hat\theta_n+\sqrt{\alpha/2}b^{-1}q_n\leqq t<M_n} G_n(t)\\ &\leqq G_n(\hat\theta_n+\sqrt{\alpha/2}\,b^{-1}q_n)\leqq G_n(\hat\theta_n)+4^{-1}\alpha b^{-2}q_n^2G_n^{(2)}(\hat\theta_n) \qquad \text{as } n\to\infty\,,\end{aligned}$$

$$\sup_{\hat\theta_n-\delta_1 b^{-1}<t\leqq\hat\theta_n-\sqrt{\alpha/2}b^{-1}q_n} G_n(t)\leqq G_n(\hat\theta_n)+4^{-1}\alpha b^{-2}q_n^2G_n^{(2)}(\hat\theta_n-\sqrt{\alpha/2}\,b^{-1}q_n) \qquad \text{as } n\to\infty\,.$$

Since $\sqrt{2/3}\leqq b^{-1}(\hat\theta_n)\leqq\sqrt{2}$ as $n\to\infty$, the application of Lemma 3 yields that

$$G_n(t)-G_n(\hat\theta_n)\leqq -ck_n^2 \qquad \text{for } k_n/c_n\leqq|\phi|<\delta_1\,,$$

with probability greater than $1-\varepsilon$ as $n\to\infty$ for some $c>0$.

LEMMA 15. *Let $\beta<+\infty$. Suppose that* (C_1), (C_2), (C_3), (C_4), (C_7) *and* (C_8) *are satisfied. Then, for any $\varepsilon>0$, $a>0$ and $\delta_4>0$, there exist a constant $c>0$ and an integer $N=N(\varepsilon, a, \delta_4, k_n)>0$ such that*

$$\log\left(\prod_{i=1}^{n}\frac{f(x_i, \hat\theta_n+\phi b^{-1})}{f(x_i, \hat\theta_n)}\right)\leqq -ck_n^2$$

$$\textit{for}\quad \frac{k_n}{c_n} \leqq \phi < \frac{(\log n)^{\delta_4}}{\sqrt{n}} \quad \textit{or} \quad -\frac{a}{\sqrt{n}} < \phi \leqq -\frac{k_n}{c_n},$$

with probability greater than $1-\varepsilon$ *for* $n>N$.

PROOF. Let $\varepsilon>0$ be fixed and $0<\delta_3<1$ that of Lemma 13. Put $t=\hat{\theta}_n+\phi b^{-1}$ and $q_n=k_n\alpha_n^{-1}$. Then,

$$\hat{\theta}_n-\frac{ab^{-1}}{\sqrt{n}}<t\leqq\hat{\theta}_n-\sqrt{\alpha/2}\,b^{-1}q_n \qquad \text{if} \quad -\frac{a}{\sqrt{n}}<\phi\leqq-\frac{k_n}{c_n},$$

$$\hat{\theta}_n+\sqrt{\alpha/2}\,b^{-1}q_n\leqq t<\hat{\theta}_n+\frac{b^{-1}(\log n)^{\delta_4}}{\sqrt{n}} \qquad \text{if} \quad \frac{k_n}{c_n}\leqq\phi<\frac{(\log n)^{\delta_4}}{\sqrt{n}}.$$

Theorem 2.2 in [7], Lemma 1 and Lemma 2 imply that

$$(N_n-\beta)\delta_3<\hat{\theta}_n-\frac{ab^{-1}}{\sqrt{n}}<\hat{\theta}_n-\sqrt{\alpha/2}\,b^{-1}q_n<\hat{\theta}_n$$

$$<\hat{\theta}_n+\sqrt{\alpha/2}\,b^{-1}q_n<M_n<\hat{\theta}_n+b^{-1}\frac{(\log n)^{\delta_4}}{\sqrt{n}}$$

with probability greater than $1-\varepsilon$ as $n\to\infty$. By Lemma 3.1 in [7] and Lemma 13,

$$\sup_{\hat{\theta}_n-ab^{-1}/\sqrt{n}<t\leqq\hat{\theta}_n-\sqrt{\alpha/2}\,b^{-1}q_n} G_n(t)$$

$$\leqq G_n(\hat{\theta}_n)+4^{-1}\alpha b^{-2}q_n^2G_n^{(2)}(\hat{\theta}_n-\sqrt{\alpha/2}\,b^{-1}q_n) \qquad \text{as } n\to\infty,$$

$$\sup_{\hat{\theta}_n+\sqrt{\alpha/2}\,b^{-1}q_n\leqq t<\hat{\theta}_n+b^{-1}(\log n)^{\delta_4}/\sqrt{n}} G_n(t)=\sup_{\hat{\theta}_n+\sqrt{\alpha/2}\,b^{-1}q_n\leqq t<M_n} G_n(t)$$

$$\leqq G_n(\hat{\theta}_n)+4^{-1}\alpha b^{-2}q_n^2G_n^{(2)}(\hat{\theta}_n) \qquad \text{as } n\to\infty.$$

The application of Lemma 3 immediately yields the result.

LEMMA 16. *Let* $\beta=+\infty$. *If* (C_1), (C_2), (C_3), (C_5) *and* (C_4) *are satisfied, then for any* $\varepsilon>0$, $\delta_5>0$, *sufficiently small* $\delta_1>0$ *and every* $k>0$, *there exist a constant* $a_0=a_0(k)>0$ *and an integer* $N=N(\varepsilon,\delta_1,\delta_5,k)>0$ *such that*

$$\prod_{i=1}^{n}\frac{f(x_i,\hat{\theta}_n+\phi b^{-1})}{f(x_i,\hat{\theta}_n)}\leqq n^{-k} \qquad \textit{for} \quad \frac{a_0(\log n)^{\delta_5}}{\sqrt{n}}\leqq|\phi|<\delta_1,$$

with probability greater than $1-\varepsilon$ *for* $n>N$.

PROOF. Let $\varepsilon>0$ be fixed and $a_0>0$ some constant. Put $t=\hat{\theta}_n+\phi b^{-1}$ and $q_n=(\log n)^{\delta_5}/\sqrt{n}$. It follows from Theorem 2.1 in [7] and Lemma 1 that $\hat{\theta}_n+a_0b^{-1}q_n>M_n$

with probability greater than $1-\varepsilon$ as $n\to\infty$. Therefore, if $a_0(\log n)^{\delta_5}/\sqrt{n}\leqq\phi<\delta_1$, then $M_n<t$ with probability greater than $1-\varepsilon$ as $n\to\infty$. Consequently, all we have to do is show the case when $-\delta_1<\phi\leqq-a_0(\log n)^{\delta_5}/\sqrt{n}$. Since

$$-(1+\sqrt{2})\delta_1<\hat{\theta}_n-b^{-1}\delta_1<\hat{\theta}_n-a_0b^{-1}q_n<\hat{\theta}_n<M_n$$

with probability greater than $1-\varepsilon$ as $n\to\infty$, as in the previous lemma,

$$\sup_{\hat{\theta}_n-\delta_1b^{-1}<t\leqq\hat{\theta}_n-a_0q_nb^{-1}} G_n(t)\leqq G_n(\hat{\theta}_n)+2^{-1}a_0^2b^{-2}q_n^2G_n^{(2)}(\hat{\theta}_n-a_0b^{-1}q_n)$$

$$\leqq G_n(\hat{\theta}_n)-\frac{\alpha a_0^2}{12}(\log n)^{1+2\delta_5}\leqq G_n(\hat{\theta}_n)-\frac{\alpha a_0^2}{12}\log n \qquad \text{as } n\to\infty\,.$$

We have used Lemma 3 in the second inequality. Let $k>0$ be fixed and define $a_0>0$ so as to satisfy $\alpha a_0^2/12\geqq k$. The lemma follows easily.

LEMMA 17. *Let $\beta<+\infty$. If (C_1), (C_2), (C_3), (C_4), (C_5), (C_7) and (C_8) are satisfied, then for any $\varepsilon>0$, $\delta_1>0$ and every $k>0$, there exist a constant $a_0=a_0(k)>0$ and an integer $N=N(\varepsilon,\delta_1,k)>0$ such that*

$$\prod_{i=1}^{n}\frac{f(x_i,\hat{\theta}_n+\phi b^{-1})}{f(x_i,\hat{\theta}_n)}\leqq n^{-k} \qquad \text{for} \quad -\delta_1<\phi\leqq-\frac{a_0}{\sqrt{n}}\,,$$

with probability greater than $1-\varepsilon$ for $n>N$.

PROOF. Let $\varepsilon>0$ be fixed and $0<\delta_3<1$ that of Lemma 13. It follows from Theorem 2.2 in [7], Lemma 2 and the consistency of $\hat{\theta}_n$ that $\hat{\theta}_n-\delta_1b^{-1}<N_n-\beta<(N_n-\beta)\delta_3<\hat{\theta}_n-a_0b^{-1}/\sqrt{n}$ with probability greater than $1-\varepsilon$ as $n\to\infty$ for $a_0>0$. By Lemma 3.1 in [7] and Lemma 13,

$$\sup_{\hat{\theta}_n-\delta_1b^{-1}<t\leqq\hat{\theta}_n-a_0b^{-1}/\sqrt{n}} G_n(t)=\sup_{N_n-\beta<t\leqq\hat{\theta}_n-a_0b^{-1}/\sqrt{n}} G_n(t)\leqq G_n(\hat{\theta}_n-a_0b^{-1}/\sqrt{n})$$

$$\leqq G_n(\hat{\theta}_n)+(2n)^{-1}a_0^2b^{-2}G_n^{(2)}(\hat{\theta}_n-a_0b^{-1}/\sqrt{n}) \qquad \text{as } n\to\infty\,.$$

Then, we can prove this lemma analogously to Lemma 16.

Now that we have finished proving Lemmas for the sake of main theorems, we will approach the establishment of those conclusions.

4. Proofs of main theorems

Define a function $\psi_{Kn}(z)$ for each K and n by

$$\psi_{Kn}(z)=\sum_{j=3}^{K+3}b^{-j}a_{jn}(\hat{\theta}_n)z^{j-3}\,.$$

Recalling that $b^2=-2a_{2n}(\hat{\theta}_n)$, we write

$$\exp\left(\alpha_n^2 \sum_{j=2}^{K+3} a_{jn}(\hat{\theta}_n)(\phi b^{-1})^j\right)\rho_K(\hat{\theta}_n+\phi b^{-1})$$
$$=\exp(-\alpha_n^2\phi^2/2)[\rho_K(\hat{\theta}_n+\phi b^{-1})\exp\{\alpha_n^2\phi^3\psi_{Kn}(\phi)\}]\,.$$

Let $P_K(w, z, X)=\rho_K(\hat{\theta}_n+zb^{-1})\exp\{w\psi_{Kn}(z)\}$, then

$$P_K(w, z, X)=\sum_{l,m} c_{lm}(X)w^l z^m\,,$$

where $c_{lm}(X)=(l!m!)^{-1}(\partial^{l+m}/\partial w^l\partial z^m)P_K(w, z, X)|_{w=0,z=0}$. And put $P_K^t(w, z, X)=\sum_{l+m\leqq K} c_{lm}(X)w^l z^m$. The detailed expression of $c_{lm}(X)$ is given in Appendix. The following lemma gives the evaluation of $c_{lm}(X)$.

LEMMA 18. *Let* (C_1), (C_2), (C_3), (C_4), (C_5), (C_6) *hold, and* (C_7), (C_8) *be satisfied also if* $\beta<+\infty$. *Then, for any* $\varepsilon>0$ *and every* $l\geqq 1$, *there exist constants* $L_4=L_4(K)>0$, $L_5=L_5(\varepsilon, K)>0$ *and an integer* $N=N(\varepsilon, K)>0$ *such that*

$$|c_{lm}(X)|(\log n)^l n^{-(l+m)/2}\leqq\frac{1}{l!}(\sqrt{2})^K L_4\left\{\frac{2}{3!\,\alpha}(\sqrt{2})^3 L_5(K+1)\right\}^l \qquad \textit{for } 0\leqq m\leqq K\,,$$

$$|c_{lm}(X)|(\log n)^l n^{-(l+m)/2}\leqq\frac{1}{l!}(\sqrt{2})^K L_4\left\{\frac{2}{3!\,\alpha}(\sqrt{2})^{K+3} L_5(K+1)\right\}^l \qquad \textit{for } K+1\leqq m\leqq(l+1)K\,,$$

with probability greater than $1-\varepsilon$ *for* $n>N$.

PROOF. We prove the case where $0\leqq m\leqq K$ with $l\geqq 3$. The other cases can be shown similarly. First of all, by (C_6) and the consistency of $\hat{\theta}_n$, note that there is a constant $L_4=L_4(K)>0$ such that $|\rho^{(m-r)}(\hat{\theta}_n)|\leqq L_4$, in probability as $n\to\infty$ for $0\leqq r\leqq m$, $0\leqq m\leqq K$. Then, from the expression of $c_{lm}(X)$ (see Appendix), together with Lemma 4,

$$|c_{lm}(X)|(\log n)^l n^{-(l+m)/2}\leqq\frac{1}{l!}(\sqrt{2})^{m+3l}L_4\sum_{r=0}^{m}\sum_{i_1=0}^{r}\sum_{i_2=0}^{i(1)}\sum_{i_3=0}^{i(2)}$$
$$\cdots\sum_{i_{l-1}=0}^{i(l-2)}\frac{1}{(m-r)!}\prod_{s=1}^{l}|a_{i_s+3n}(\hat{\theta}_n)|(\log n)^l n^{-(l+r)/2}\,,$$

where $i(s)$ and i_s are as in Appendix. Let $\varepsilon>0$ be fixed. Since $r=\sum_{s=1}^{l} i_s$ and $a_{jn}(\hat{\theta}_n)=\alpha_n^{-2}G_n^{(j)}(\hat{\theta}_n)/j!$, application of Lemma 10 gives that there exists $L_5=L_5(\varepsilon, K)>0$ such that

$$\prod_{s=1}^{l}|a_{i_s+3n}(\hat{\theta}_n)|(\log n)^l n^{-(l+r)/2}\leqq\prod_{s=1}^{l}\left|\frac{2}{\alpha(i_s+3)!}n^{-(i_s+3)/2}G_n^{(i_s+3)}(\hat{\theta}_n)\right|\leqq\left(\frac{2L_5}{3!\,\alpha}\right)^l$$

with probability greater than $1-\varepsilon$ as $n\to\infty$. Taking the range of $m, r, i_1, i_2, \cdots, i_{l-1}$

into account, we obtain the result.

Now, since $k_n/\sqrt{\log n}\to 0$ as $n\to\infty$,

$$(4.1)\quad \int_{|\phi|<k_n/c_n} \exp(-\alpha_n^2\phi^2/2)\left|\sum_{l+m\geqq K+1} c_{lm}(X)(\alpha_n^2\phi^3)^l\phi^m\right| d\phi$$

$$\leqq \int_{|\phi|<k_n/c_n} \exp(-\alpha_n^2\phi^2/2)\sum_{l+m\geqq K+1}\left(\frac{\alpha}{2}\right)^l |c_{lm}(X)|(\log n)^l n^{-(l+m)/2} \times\left(\frac{k_n}{\sqrt{\log n}}\right)^{2l+(l+m)} d\phi$$

$$\leqq\left(\frac{k_n}{\sqrt{\log n}}\right)^{K+1}\int_{|\phi|<k_n/c_n}\exp(-\alpha_n^2\phi^2/2)\sum_{l+m\geqq K+1}\left(\frac{\alpha}{2}\right)^l|c_{lm}(X)| \times(\log n)^l n^{-(l+m)/2}\left(\frac{k_n}{\sqrt{\log n}}\right)^{2l} d\phi\,.$$

Noting that $c_{lm}(X)=0$ for $m\geqq(l+1)K+1$, $l=0, 1, \cdots$ (see Appendix), we obtain

$$\sum_{l+m\geqq K+1}\left(\frac{\alpha}{2}\right)^l|c_{lm}(X)|(\log n)^l n^{-(l+m)/2}\left(\frac{k_n}{\sqrt{\log n}}\right)^{2l}$$

$$\leqq\sum_{l=1}^{\infty}\sum_{m=0}^{K}\left(\frac{\alpha}{2}\right)^l|c_{lm}(X)|(\log n)^l n^{-(l+m)/2}\left(\frac{k_n}{\sqrt{\log n}}\right)^{2l} +\sum_{l=1}^{\infty}\sum_{m=K+1}^{(l+1)K}\left(\frac{\alpha}{2}\right)^l|c_{lm}(X)|(\log n)^l n^{-(l+m)/2}\left(\frac{k_n}{\sqrt{\log n}}\right)^{2l}$$

$$\leqq\left(\frac{k_n}{\sqrt{\log n}}\right)^2\left[\sum_{l=1}^{\infty}\sum_{m=0}^{K}\left(\frac{\alpha}{2}\right)^l|c_{lm}(X)|(\log n)^l n^{-(l+m)/2} +\sum_{l=1}^{\infty}\sum_{m=K+1}^{(l+1)K}\left(\frac{\alpha}{2}\right)^l|c_{lm}(X)|(\log n)^l n^{-(l+m)/2}\right]$$

$$\leqq\left(\frac{k_n}{\sqrt{\log n}}\right)^2(\sqrt{2})^K L_4\left[\sum_{l=1}^{\infty}\sum_{m=0}^{K}\frac{\{(\sqrt{2})^3L_5(K+1)/3!\}^l}{l!} +\sum_{l=1}^{\infty}\sum_{m=K+1}^{(l+1)K}\frac{\{(\sqrt{2})^{K+3}L_5(K+1)/3!\}^l}{l!}\right]$$

with probability greater than $1-\varepsilon$ for any fixed $\varepsilon>0$ as $n\to\infty$, where we have used Lemma 18 in the final step. Since the summations in the final brackets converge, by changing variable $t=\alpha_n\phi$, there exist a constant $L_6'=L_6'(\varepsilon, K)>0$ and an integer $N=N(\varepsilon, K, k_n)>0$ such that (4.1) is bounded by

$$L_6' n^{-1/2} (\log n)^{-(k+4)/2} k_n^{K+3}$$

with probability greater than $1-\varepsilon$ for $n>N$. Moreover, for any fixed $p\in(0, 1/2)$, put $s=2p/(K+3)$ and define $k_n=(\log n)^{s/2}$. Then, $k_n=o(\sqrt{\log n})$ as $n\to\infty$, and there exist a constant $L_6=L_6(\varepsilon, K)>0$ and an integer $N=N(\varepsilon, p, K)>0$ such that (4.1) is bounded by

$$L_6 n^{-1/2} (\log n)^{-(K+4)/2+p}$$

with probability greater than $1-\varepsilon$ for $n>N$.

Similarly, by changing variable $t=\alpha_n\phi$ and using Lemma 18, there exist a constant $L_7=L_7(\varepsilon, K)>0$ and an integer $N=N(\varepsilon, K)>0$ such that

$$\begin{aligned}(4.2)\qquad &\int_{|\phi|\geqq k_n/c_n} \exp(-\alpha_n^2\phi^2/2)|P_K^t(\alpha_n^2\phi^3, \phi, X)|d\phi \\ &\leqq 2\alpha_n^{-1} \sum_{l+m\leqq K} |c_{lm}(X)|(\log n)^l n^{-(l+m)/2} \\ &\quad\times\left(\frac{\alpha}{2}\right)^{-(l+m)/2} (\log n)^{-(3l+m)/2} \int_{\sqrt{\alpha/2}\,k_n}^{\infty} \exp(-t^2/2) t^{3l+m} dt \\ &\leqq 2\alpha_n^{-1} \sum_{l+m\leqq K} (\sqrt{2})^K L_4 \frac{\{2(\sqrt{2})^3 L_5(K+1)/(3!\alpha)\}^l}{l!} \\ &\quad\times\left(\frac{\alpha}{2}\right)^{-(l+m)/2} (\log n)^{-(3l+m)/2} \exp(-\alpha k_n^2/8) \int_{\sqrt{\alpha/2}\,k_n}^{\infty} \exp(-t^2/4) t^{3l+m} dt \\ &\leqq L_7 \alpha_n^{-1} \exp(-\alpha k_n^2/8)\end{aligned}$$

with probability greater than $1-\varepsilon$ for $n>N$.

If $\beta<+\infty$, then, Lemma 1, Lemma 4 and Theorem 2.2 in [7] lead us to inequalities that $M_n<\hat{\theta}_n+\phi b^{-1}$ for $(\log n)^{\delta_4}/\sqrt{n}\leqq\phi$, $\delta_4>0$ and $\beta<N_n-(\hat{\theta}_n+\phi b^{-1})$ for $-\delta_1\geqq\phi$, $\delta_1>0$ with probability greater than $1-\varepsilon$ as $n\to\infty$. When $\beta<+\infty$,

$$\begin{aligned}&\int_{-\infty}^{\infty} \left|\rho(\hat{\theta}_n+\phi b^{-1}) \prod_{i=1}^n \frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)} - \exp(-\alpha_n^2\phi^2/2) P_K^t(\alpha_n^2\phi^3, \phi, X)\right| d\phi \\ &\leqq \int_{k_n/c_n}^{(\log n)^{\delta_4}/\sqrt{n}} \rho(\hat{\theta}_n+\phi b^{-1}) \prod_{i=1}^n \frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)} d\phi \\ &\quad+ \int_{-a_0/\sqrt{n}}^{-k_n/c_n} \rho(\hat{\theta}_n+\phi b^{-1}) \prod_{i=1}^n \frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)} d\phi \\ &\quad+ \int_{-\delta_1}^{-a_0/\sqrt{n}} \rho(\hat{\theta}_n+\phi b^{-1}) \prod_{i=1}^n \frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)} d\phi\end{aligned}$$

$$+\int_{|\phi|\geqq k_n/c_n} \exp(-\alpha_n^2\phi^2/2)|P_K^t(\alpha_n^2\phi^3, \phi, X)|d\phi$$

$$+\int_{|\phi|<k_n/c_n} \left|\rho(\hat{\theta}_n+\phi b^{-1})\prod_{i=1}^{n}\frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)}\right.$$

$$\left. -\exp(-\alpha_n^2\phi^2/2)P_K(\alpha_n^2\phi^3, \phi, X)\right|d\phi$$

$$+\int_{|\phi|<k_n/c_n} \exp(-\alpha_n^2\phi^2/2)\left|\sum_{l+m\geqq K+1} c_{lm}(X)(\alpha_n^2\phi^3)^l\phi^m\right|d\phi .$$

Therefore, it follows from Lemmas 11, 15, 17, the evaluation of (4.1) and (4.2), the continuity of ρ and the consistency of $\hat{\theta}_n$ that for any $\varepsilon>0$, $0<p<1/2$, there exist a constant $B_1=B_1(\varepsilon, K)>0$ and an integer $N=N(\varepsilon, p, K)>0$ such that

$$(4.3)\quad \int_{-\infty}^{\infty}\left|\rho(\hat{\theta}_n+\phi b^{-1})\prod_{i=1}^{n}\frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)}-\exp(-\alpha_n^2\phi^2/2)P_K^t(\alpha_n^2\phi^3, \phi, X)\right|d\phi$$
$$\leqq B_1 n^{-1/2}(\log n)^{-(K+4)/2+p}$$

with probability greater than $1-\varepsilon$ for $n>N$.

When $\beta=+\infty$, we divide the integration of

$$\rho(\hat{\theta}_n+\phi b^{-1})\sum_{i=1}^{n}\frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)}$$

in the range $|\phi|\geqq k_n/c_n$ into three parts, that in the range $|\phi|\geqq\delta_1$, $\delta_1>|\phi|\geqq a_0(\log n)^{\delta_5}/\sqrt{n}$ and $a_0(\log n)^{\delta_5}/\sqrt{n}>|\phi|\geqq k_n/c_n$. Then, by the application of Lemmas 6, 14 and 16 instead of Lemmas 15 and 17, we obtain (4.3) in this case, too.

In the same way, there exist a constant $B_2=B_2(\varepsilon, K)>0$ and an integer $N=N(\varepsilon, p, K)>0$ such that

$$(4.4)\quad \int_{-\infty}^{\xi\alpha_n^{-1}}\left|\rho(\hat{\theta}_n+\phi b^{-1})\prod_{i=1}^{n}\frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)}-\exp(-\alpha_n^2\phi^2/2)P_K^t(\alpha_n^2\phi^3, \phi, X)\right|d\phi$$
$$\leqq B_2 n^{-1/2}(\log n)^{-(k+4)/2+p}$$

with probability greater than $1-\varepsilon$ for $n>N$, where (4.4) is uniform in ξ. Put

$$\eta_j(\xi, X)=\sum_{l=0}^{j} c_{lj-l}(X)\int_{-\infty}^{\xi}\exp(-y^2/2)y^{2l+j}dy \quad\text{and}\quad \beta_j(X)=\eta_j(\infty, X)$$

for $j=0, 1, \cdots, K$. (4.3) and (4.4) imply that

$$(4.5)\quad \left|\int_{-\infty}^{\infty}\rho(\hat{\theta}_n+\phi b^{-1})\prod_{i=1}^{n}\frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)}d\phi-\sum_{j=0}^{K}\beta_j(X)\alpha_n^{-(j+1)}\right|$$
$$\leqq B_1 n^{-1/2}(\log n)^{-(k+4)/2+p},$$

(4.6)
$$\left|\int_{-\infty}^{\xi\alpha_n^{-1}} \rho(\hat{\theta}_n+\phi b^{-1}) \prod_{i=1}^{n} \frac{f(x_i, \hat{\theta}_n+\phi b^{-1})}{f(x_i, \hat{\theta}_n)} d\phi - \sum_{j=0}^{K} \eta_j(\xi, X)\alpha_n^{-(j+1)}\right| \leqq B_2 n^{-1/2}(\log n)^{-(K+4)/2+p},$$

respectively.

The result of Theorem 1 in the case when $K=0$ is derived easily from (4.5) and (4.6).

Define a sequence of functions $\{\gamma_j(\xi, X)\}_{j=0}^{K}$ so as to satisfy $\gamma_0(\xi, X)=0$ and

$$\eta_j(\xi, X)=\beta_0(X)\gamma_j(\xi, X)+\sum_{l=1}^{j} \gamma_{j-l}(\xi, X)\beta_l(X)+\beta_j(X)\Phi(\xi) \qquad \text{for } j=1, 2, \cdots, K.$$

It follows from $\eta_0(\xi, X)=\sqrt{2\pi}\,\rho(\hat{\theta}_n)\Phi(\xi)$ and $\beta_0(X)=\sqrt{2\pi}\,\rho(\hat{\theta}_n)$ that

(4.7)
$$\frac{\sum_{j=0}^{K}\eta_j(\xi, X)\alpha_n^{-j}}{\sum_{j=0}^{K}\beta_j(X)\alpha_n^{-j}}=\Phi(\xi)+\sum_{j=1}^{K}\gamma_j(\xi, X)\alpha_n^{-j} - \frac{\sum_{j=K+1}^{2K}\{\sum_{l=j-K}^{K}\gamma_{j-l}(\xi, X)\beta_l(X)\}\alpha_n^{-j}}{\sum_{j=0}^{K}\beta_j(X)\alpha_n^{-j}}.$$

We will prove the following lemmas regarding the evaluation of $\eta_j(\xi, X)$, $\beta_j(X)$ and $\gamma_j(\xi, X)$.

LEMMA 19. *Let* (C_1), (C_2), (C_3), (C_4), (C_5), (C_6) *hold, and* (C_7), (C_8) *be satisfied also if* $\beta<\infty$. *Then, for any* $\varepsilon>0$ *and* $1\leqq j\leqq K$, *there exist constants* $L_8=L_8(\varepsilon, K)>0$, $L_9=L_9(\varepsilon, K)>0$ *and an integer* $N=N(\varepsilon, K)>0$ *such that*

$$|\eta_j(\xi, X)|n^{-j/2}\log n\leqq L_8 \qquad \textit{uniformly in } \xi,$$

with probability greater than $1-\varepsilon$ *for* $n>N$, *and*

$$|\beta_j(X)|n^{-j/2}\log n\leqq L_9$$

with probability greater than $1-\varepsilon$ *for* $n>N$.

PROOF.

$$|\beta_j(X)|\leqq\sum_{l=0}^{j}|c_{lj-l}(X)|\int_{-\infty}^{\infty}\exp(-y^2/2)|y|^{2l+j}dy.$$

Since $c_{0j}(X)=\rho^{(j)}(\hat{\theta}_n)b^{-j}/j!$, by ($C_6$) and Lemma 4, there exists a constant $L'_9>0$ such that $|c_{0j}(X)|\leqq L'_9$ in probability as $n\to\infty$ for $1\leqq j\leqq K$. In addition, Lemma 18 implies that

$$|c_{lj-l}(X)|\leqq\frac{1}{l!}(\sqrt{2})^K L_4\left\{\frac{2}{3!\,\alpha}(\sqrt{2})^3 L_5(K+1)\right\}^l(\log n)^{-l}n^{j/2}$$

for some $L_4=L_4(K)>0$, $L_5=L_5(\varepsilon, K)>0$ with probability greater than $1-\varepsilon$ as $n\to\infty$, for $1\leqq l\leqq j$ and $1\leqq j\leqq K$. The lemma follows easily. Similarly, we can show the inequality for $\eta_j(\xi, X)$.

LEMMA 20. *Let* (C_1), (C_2), (C_3), (C_4), (C_5), (C_6) *hold, and* (C_7), (C_8) *be satisfied also if* $\beta<+\infty$. *Then, for any* $\varepsilon>0$ *and* $1\leqq j\leqq K$, *there exist a constant* $L_{10}=L_{10}(\varepsilon, K)>0$ *and an integer* $N=N(\varepsilon, K)>0$ *such that*

$$|\gamma_j(\xi, X)|n^{-j/2}\log n\leqq L_{10} \qquad \text{uniformly in } \xi,$$

with probability greater than $1-\varepsilon$ *for* $n>N$.

PROOF. When $j=1$, $|\gamma_1(\xi, X)|=|\eta_1(\xi, X)|/|\beta_0(X)|$. Since $\beta_0(X)=\sqrt{2\pi}\,\rho(\hat{\theta}_n)$, $\beta_0(X)>\varepsilon'$ for some $\varepsilon'>0$, in probability as $n\to\infty$ by (C_6). The application of Lemma 19 yields the result in this case. Then we use induction. Suppose that the assertion holds until j. $\gamma_{j+1}(\xi, X)$ is expressed in the form

$$\gamma_{j+1}(\xi, X)=\left\{\eta_{j+1}(\xi, X)-\sum_{l=1}^{j+1}\gamma_{j+1-l}(\xi, X)\beta_l(X)-\beta_{j+1}(X)\Phi(\xi)\right\}\Big/\beta_0(X).$$

The assumption and Lemma 19 immediately give the desired result.

By Lemma 19, we can show that $\sum_{j=0}^{K}\eta_j(\xi, X)\alpha_n^{-j}$ and $\sum_{j=0}^{K}\beta_j(X)\alpha_n^{-j}$ are bounded. Therefore it follows from (4.5), (4.6) that for any $\varepsilon>0$ and $0<p<1/2$, there exist a constant $D'_1=D'_1(\varepsilon, K)>0$ and an integer $N=N(\varepsilon, p, K)>0$ such that

$$\left|F_n(\xi)-\frac{\sum_{j=0}^{K}\eta_j(\xi, X)\alpha_n^{-j}}{\sum_{j=0}^{K}\beta_j(X)\alpha_n^{-j}}\right|\leqq D'_1(\log n)^{-(K+3)/2+p},$$

with probability greater than $1-\varepsilon$ for $n>N$. And by (4.7),

$$\left|F_n(\xi)-\Phi(\xi)-\sum_{j=1}^{K}\gamma_j(\xi, X)\alpha_n^{-j}\right|\leqq D'_1(\log n)^{-(K+3)/2+p}$$
$$+\left|\frac{\sum_{j=K+1}^{2K}\{\sum_{l=j-K}^{K}\gamma_{j-l}(\xi, X)\beta_l(X)\}\alpha_n^{-j}}{\sum_{j=0}^{K}\beta_j(X)\alpha_n^{-j}}\right|,$$

with probability greater than $1-\varepsilon$ for $n>N$. The second term of the right hand side is bounded by $D''_1(\log n)^{-(K+5)/2}$ for some $D''_1=D''_1(\varepsilon, K)>0$ with probability greater than $1-\varepsilon$ as $n\to\infty$. This completes the proof of Theorem 1.

We will prove the following lemma to establish Theorem 2.

LEMMA 21. *Let* (C_1), (C_2), (C_3), (C_4), (C_5), (C_6) *hold, and* (C_7), (C_8) *be satisfied also if* $\beta<+\infty$. *Then, for any* $\varepsilon>0$, $L>0$ *and* $1\leqq j\leqq K$, *there exists an integer* $N=N(\varepsilon, K)>0$ *such that*

$$\gamma_j(\xi, X)\alpha_n^{-j}=-\varphi(\xi)\sum_{m=1}^{j}\left\{(\log n)^{-(j+2m)/2}\sum_{l=0}^{(j-1+2m)/2}B_{j+2m\,2l}^{(j)}\xi^{2l}\right\}+o_p((\log n)^{-L})$$

with probability greater than $1-\varepsilon$ *for* $n>N$, *if* j *is an odd number,* *and*

$$\gamma_j(\xi, X)\alpha_n^{-j} = -\varphi(\xi) \sum_{m=1}^{j} \left\{ (\log n)^{-(j+2m)/2} \sum_{l=1}^{(j+2m)/2} B_{j+2m\,2l-1}^{(j)} \xi^{2l-1} \right\} + o_p((\log n)^{-L})$$

with probability greater than $1-\varepsilon$ *for* $n > N$, *if* j *is an even number,*

where $B_{st}^{(j)} = O_p(1)$ *with respect to* P_{θ_0} *for every* j *as* $n \to \infty$ *and* $\varphi(\xi)$ *is the standard normal density function.*

PROOF. We prove the case where j is an even number. For simplicity, denote $\eta_j(\xi, X)$, $\beta_j(X)$, $\gamma_j(\xi, X)$, $c_{ij}(X)$ by η_j, β_j, γ_j, c_{ij} respectively. Since $c_{0j} = \rho^{(j)}(\hat{\theta}_n) b^{-j}/j!$ for $0 \leqq j \leqq K$,

$$|c_{0j}|\alpha_n^{-j} = o_p((\log n)^{-L}) \quad \text{for any } L > 0 \quad \text{as } n \to \infty\,.$$

And by Lemma 18, for any $\varepsilon > 0$, $1 \leqq l \leqq j$ and $1 \leqq j \leqq K$, there exist $B_l > 0$ such that

$$|c_{lj-l}|\alpha_n^{-j} \leqq B_l(\log n)^{-(2l+j)/2}\,,$$

with probability greater than $1-\varepsilon$ as $n \to \infty$. Let $j = 2$. Since

$$\gamma_2 = -\varphi(\xi)\{c_{20}(\xi^5 + 5\xi^3 + 15\xi) + c_{11}(\xi^3 + 3\xi) + c_{02}\xi\}/\rho(\hat{\theta}_n)\,,$$

from what mentioned above, the lemma follows easily in this case.

General cases can be proved inductively. Suppose that the conclusion hold until j. When $j+2$,

$$\gamma_{j+2}\alpha_n^{-(j+2)} = \{\eta_{j+2} - \beta_{j+2}\Phi(\xi)\}\alpha_n^{-(j+2)}/\beta_0 - \sum_{l=1}^{j+2} \gamma_{j+2-l}\beta_l\alpha_n^{-(j+2)}/\beta_0\,.$$

The first term of the right hand side is equal to

$$-\varphi(\xi) \sum_{l=0}^{j+2} c_{lj+2-l}\alpha_n^{-(j+2)} \left[\xi^{2l+j+1} + \sum_{s=1}^{l+j/2} \prod_{i=1}^{s} \{2(l-i)+j+3\}\xi^{2(l-s)+j+1} \right] \Big/ \rho(\hat{\theta}_n)\,,$$

which is expressed in the form

$$-\varphi(\xi) \sum_{l=1}^{j+2} \left\{ (\log n)^{-(j+2l+2)/2} \sum_{s=1}^{(j+2l+2)/2} B_{j+2l+2\,2s-1}\xi^{2s-1} \right\} + o_p((\log n)^{-L})$$

for some $B_{j+2l+2\,2s-1} = O_p(1)$ and any $L > 0$ with probability greater than $1-\varepsilon$ as $n \to \infty$. Now we consider the second term. Since $\beta_l = 0$ if l is an odd number, by the assumption of induction, the second term is written in the form

$$-\varphi(\xi) \sum_{l=1}^{j/2} \left[\sum_{m=1}^{j-2l+2} \left\{ (\log n)^{-(j-2l+2m+2)/2} \sum_{s=1}^{(j-2l+2m+2)/2} B_{j-2l+2m+2\,2s-1}\xi^{2s-1} \right\} + o_p((\log n)^{-L}) \right] \beta_{2l}\alpha_n^{-2l}$$

for some $B_{j-2l+2m+2\,2s-1}=O_p(1)$ as $n\to\infty$. Moreover, since

$$\beta_{2l}\alpha_n^{-2l}=\sum_{s=1}^{2l} B_s(\log n)^{-(l+s)}+o_p((\log n)^{-L}),$$

for some B_s as $n\to\infty$, the second term is written in the form

$$-\varphi(\xi)\sum_{l=1}^{j/2}\left[\sum_{m=1}^{j-2l+2}\sum_{s=1}^{2l}(\log n)^{-(j+2m+2s+2)/2}\sum_{k=1}^{(j-2l+2m+2)/2} B_{j+2m+2s+2\,2k-1}\xi^{2k-1}\right]$$
$$+o_p((\log n)^{-L})$$

for some $B_{j+2m+2s+2\,2k-1}=O_p(1)$ with probability greater than $1-\varepsilon$ as $n\to\infty$. Both of terms are polynomials in ξ with only odd degrees multiplied by the standard normal density function. Consequently, by putting together terms with the same order of $(\log n)^{-i/2}$, $i=j+4, j+6, \cdots, 3j+6$, the conclusion holds for $j+2$, too.

The case when j is an odd number is shown similarly.

In Lemma 21, we can see that each $\gamma_j(\xi, X)\alpha_n^{-j}$ is a polynomial in ξ with stochastically bounded coefficients $B_{st}^{(j)}$, multiplied by $(\log n)^{-i/2}$, $i=3, 4, \cdots, 3j$, and the standard normal density function $\varphi(\xi)$. In an expansion

$$\left|F_n(\xi)-\Phi(\xi)-\sum_{j=1}^{K}\gamma_j(\xi, X)\alpha_n^{-j}\right|\leqq D_1(\log n)^{-(K+3)/2+p},$$

polynomials in ξ multiplied by $(\log n)^{-i/2}$, $i=3, 4, \cdots, K+2$, are essentially needed. From this point of view, we obtain the conclusion of Theorem 2.

The following lemma gives sufficient conditions to satisfy (C_7).

LEMMA 22. *Let $\beta<+\infty$ and f satisfy* (C_1), (C_2) *and* (C_8). *If*

$$\lim_{x\to\beta-0} f^{(j)}(x)=0 \qquad \text{for} \quad j=0, 1, \cdots, K+2,$$

and

$$f^{(K+3)}(x)=(\beta-x)^{\gamma}L(\beta-x)$$

where $\gamma>-1$ and $L(x)$ satisfies $L(tx)/L(t)\to 1$ as $t\to 0$ for every fixed $x>0$, then (C_7) *holds.*

PROOF. We can show this lemma analogously to Lemma 3.6 in [7]. It follows from Theorem 1 in [1], page 281, that

$$t^{\gamma+1}L(t)\Big/\int_0^t yL(y)dy\to\gamma+1 \qquad \text{as } t\to 0 \quad \text{for } \gamma>-1.$$

By making use of this fact, we can show inductively that

$$f^{(K+3-j)}(x) \sim \frac{(-1)^j(\beta-x)^{\gamma+j}L(\beta-x)}{\prod_{i=1}^{j}(\gamma+i)} \qquad \text{as } x\to\beta-0\,, \quad \text{for } j=1,\cdots,K+3\,.$$

Therefore

$$g^{(j)}(x) \sim -\frac{(j-1)!(\gamma+K+3)}{(\beta-x)^j} \qquad \text{as } x\to\beta-0\,, \quad \text{for } j=2,\cdots,K+3\,.$$

This implies that $\limsup_{x\to\beta-0} g^{(j)}(x) \leqq 0$ for $j=2,\cdots,K+3$. Moreover, define $h_j(x)=-c/x^j$ for fixed $c>0$ for $j=2,\cdots,K+3$. Clearly $h_j(x)$ is an increasing function on $x>0$, and satisfies $g^{(j)}(x)\geqq h_j(\beta-x)$ for $\beta-\tau<x<\beta$ for sufficiently large c and sufficiently small τ. Finally, note that

$$\int_{\beta-\tau}^{\beta} h_j(v(\beta-x))f(x)dx = -cv^{-j}\int_{\beta-\tau}^{\beta}(\beta-x)^{-j}f(x)dx\,.$$

By adjusting the lemma in [1], page 280, to this case, we can show that the last expression is finite for all $0<v<1$.

Take the density function of Beta(2, q), $q>K+3$, for instance. In fact, it satisfies not only (C_3), (C_4) and (C_5) but also conditions required in Lemma 22.

5. Examples of expansions.

As a matter of fact, it is too difficult to calculate stochastically bounded coefficients $B_{st}^{(j)}$ specifically in general cases. But it is possible if K is not so large. Put $A_j=(\alpha n/2)^{-j/2}b(\hat{\theta}_n)^{-j}G_n^{-(j)}(\hat{\theta}_n)/j!$. Then the expansions for $K=1,2,\cdots,6$ are given as follows.

(i) $K=1$

$$|F_n(\xi)-\Phi(\xi)+\varphi(\xi)(\log n)^{-3/2}A_3(\xi^2+2)| \leqq D_{21}(\log n)^{-2+p}\,.$$

(ii) $K=2$

$$\begin{aligned}|F_n(\xi)-\Phi(\xi)+\varphi(\xi)\{&(\log n)^{-3/2}A_3(\xi^2+2)\\ &+(\log n)^{-2}A_4(\xi^3+3\xi)\}| \leqq D_{22}(\log n)^{-5/2+p}\,.\end{aligned}$$

(iii) $K=3$

$$\begin{aligned}|F_n(\xi)-\Phi(\xi)+\varphi(\xi)\{&(\log n)^{-3/2}A_3(\xi^2+2)+(\log n)^{-2}A_4(\xi^3+3\xi)\\ &+(\log n)^{-5/2}(A_5/2)(\xi^4+4\xi^2+8)\}| \leqq D_{23}(\log n)^{-3+p}\,.\end{aligned}$$

(iv) $K=4$

$$\begin{aligned}|F_n(\xi)-\Phi(\xi)+\varphi(\xi)\{&(\log n)^{-3/2}A_3(\xi^2+2)\\ &+(\log n)^{-2}A_4(\xi^3+3\xi)+(\log n)^{-5/2}(A_5/2)(\xi^4+4\xi^2+8)\\ &+(\log n)^{-3}(A_6+A_3^2/2)(\xi^5+5\xi^3+15\xi)\}| \leqq D_{24}(\log n)^{-7/2+p}\,.\end{aligned}$$

(v) $K=5$

$$\begin{aligned}|F_n(\xi)-\Phi(\xi)+\varphi(\xi)[(\log n)^{-3/2}A_3(\xi^2+2)+(\log n)^{-2}A_4(\xi^3+3\xi)\\ +(\log n)^{-5/2}(A_5/2)(\xi^4+4\xi^2+8)+(\log n)^{-3}(A_6+A_3^2/2)(\xi^5+5\xi^3+15\xi)\\ +(\log n)^{-7/2}\{(A_7+A_3A_4)\xi^6+6(A_7+A_3A_4)\xi^4+(24A_7+27A_3A_4)\xi^2\\ +48A_7+54A_3A_4\}]|\leqq D_{25}(\log n)^{-4+p}.\end{aligned}$$

(iv) $K=6$

$$\begin{aligned}|F_n(\xi)-\Phi(\xi)+\varphi(\xi)[(\log n)^{-3/2}A_3(\xi^2+2)+(\log n)^{-2}A_4(\xi^3+3\xi)\\ +(\log n)^{-5/2}(A_5/2)(\xi^4+4\xi^2+8)+(\log n)^{-3}(A_6+A_3^2/2)(\xi^5+5\xi^3+15\xi)\\ +(\log n)^{-7/2}\{(A_7+A_3A_4)\xi^6+6(A_7+A_3A_4)\xi^4\\ +(24A_7+27A_3A_4)\xi^2+48A_7+54A_3A_4\}\\ +(\log n)^{-4}\{(A_8+A_5A_3+A_4^2/2)\xi^7+7(A_8+A_5A_3+A_4^2/2)\xi^5\\ +(35(A_8+A_5A_3+A_4^2/2)+3A_4^2)\xi^3\\ +(105(A_8+A_5A_3+A_4^2/2)+9A_4^2)\xi\}]|\leqq D_{26}(\log n)^{-9/2+p}.\end{aligned}$$

ACKNOWLEDGMENT. I would like to thank the referee for his careful reading of the paper and useful comments. And I am also grateful to Professor Takeru Suzuki for his helpful proposal and proper advice.

Appendix.

In the proof of some lemmas, it is required to know the detailed expressions of $c_{lm}(X)$ defined in section 4. They are given as follows.

$$c_{0m}(X)=\begin{cases}\dfrac{1}{m!}\rho^{(m)}(\hat{\theta}_n)b^{-m} & (0\leqq m\leqq K)\\ 0 & (K+1\leqq m)\end{cases}$$

$$c_{1m}(X)=\begin{cases}b^{-(m+3)}\displaystyle\sum_{r=0}^{m}\frac{1}{(m-r)!}\rho^{(m-r)}(\hat{\theta}_n)a_{r+3n}(\hat{\theta}_n) & (0\leqq m\leqq K)\\ b^{-(m+3)}\displaystyle\sum_{r=0}^{K-1}\frac{1}{(K-r)!}\rho^{(K-r)}(\hat{\theta}_n)a_{r+m-K+3n}(\hat{\theta}_n)I(r\,|\,r+m\leqq 2K) & (K+1\leqq m\leqq 2K)\\ 0 & (2K+1\leqq m)\end{cases}$$

$$c_{2m}(X)=\begin{cases} \dfrac{1}{2}b^{-(m+6)}\sum_{r=0}^{m}\sum_{s=0}^{r}\dfrac{1}{(m-r)!}\rho^{(m-r)}(\hat{\theta}_n)a_{r-s+3n}(\hat{\theta}_n)a_{s+3n}(\hat{\theta}_n) & (0\leqq m\leqq K) \\ \dfrac{1}{2}b^{-(m+6)}\sum_{r=0}^{K}\sum_{s=0}^{s(0)}\dfrac{1}{(K-r)!}\rho^{(K-r)}(\hat{\theta}_n)a_{r+m-K-s+3n}(\hat{\theta}_n)a_{s+3n}(\hat{\theta}_n) & \\ \qquad\times I((r,s)\mid r+m-s\leqq 2K)\,, & \\ \quad\text{where } s(0)=\min\{K, r+m-K\} & (K+1\leqq m\leqq 3K) \\ 0 & (3K+1\leqq m) \end{cases}$$

$$c_{lm}(X)= \;(l\geqq 3)\begin{cases} \dfrac{1}{l!}b^{-(m+3l)}\sum_{r=0}^{m}\sum_{i_1=0}^{r}\sum_{i_2=0}^{i(1)}\sum_{i_3=0}^{i(2)}\cdots\sum_{i_{l-1}=0}^{i(l-2)}\dfrac{1}{(m-r)!}\rho^{(m-r)}(\hat{\theta}_n) & \\ \qquad\times a_{i_l+3n}(\hat{\theta}_n)a_{i_{l-1}+3n}(\hat{\theta}_n)\times\cdots\times a_{i_2+3n}(\hat{\theta}_n)a_{i_1+3n}(\hat{\theta}_n)\,, & \\ \quad\text{where } i(1)=r-i_1,\ i(2)=r-i_1-i_2,\ \cdots,\ i(l-2)=r-\sum_{s=1}^{l-2}i_s\,, & \\ \quad i_l=r-\sum_{s=1}^{l-1}i_s & (0\leqq m\leqq K) \\ \dfrac{1}{l!}b^{-(m+3l)}\sum_{r=0}^{K}\sum_{i_1=0}^{m(0)}\sum_{i_2=0}^{m(1)}\sum_{i_3=0}^{m(2)}\cdots\sum_{i_{l-1}=0}^{m(l-2)}\dfrac{1}{(K-r)!}\rho^{(K-r)}(\hat{\theta}_n) & \\ \qquad\times a_{i_l+m-K+3n}(\hat{\theta}_n)a_{i_{l-1}+3n}(\hat{\theta}_n)\times\cdots\times a_{i_2+3n}(\hat{\theta}_n)a_{i_1+3n}(\hat{\theta}_n) & \\ \qquad\times I\left((r,i_1,i_2,\cdots,i_{l-1})\,\middle|\, r+m-\sum_{s=1}^{l-1}i_s\leqq 2K\right), & \\ \quad\text{where } m(0)=\min\{K, r+m-K\},\ m(1)=\min\{K, r+m-K-i_1\}\,, & \\ \quad m(2)=\min\{K, r+m-K-i_1-i_2\},\ \cdots, & \\ \quad m(l-2)=\min\{K, r+m-K-\sum_{s=1}^{l-2}i_s\}\,, & \\ \quad i_l=r-\sum_{s=1}^{l-1}i_s & (K+1\leqq m\leqq (l+1)K) \\ 0 & ((l+1)K+1\leqq m) \end{cases}$$

References

[1] W. FELLER, *An Introduction to Probability Theory and its Applications, II*, Second ed., Wiley Series in Probab. and Math. Statist., 1971.

[2] R. A. JOHNSON, Asymptotic expansions associated with posterior distributions, Ann. Math. Statist., **41** (1970), 851–864.

[3] T. MATSUDA, The Berry-Esseen bound for maximum likelihood estimates of translation parameters of truncated distribution asymptotic properties of posterior distributions, Osaka J. Math., **20** (1983), 185–195.

[4] T. MATSUDA, Asymptotic properties of posterior distributions in a truncated case, Osaka J. Math., **20** (1983), 307–320.

[5] A. WALD, Note on the consistency of the maximum likelihood estimate, Ann. Math. Statist., **20** (1949), 595–601.

[6] L. WEISS and J. WOLFOWITZ, Maximum likelihood estimation of a translation parameter of a truncated distribution, Ann. Statist., **1** (1973), 944–947.

[7] M. WOODROOFE, Maximum likelihood estimation of translation parameter of a truncated distribution, Ann. Math. Statist., **43** (1972), 113–122.

Present Address:
DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE AND ENGINEERING
WASEDA UNIVERSITY
OHKUBO, SHINJUKU-KU, TOKYO 169–50, JAPAN