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The Theta-Curve Cobordism Group Is Not Abelian

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Introduction.

A spatial theta-curve $f:\theta\to S^3$ is an embedding of a theta-curve with its vertices and edges labelled. Given two spatial theta-curves f and g, we can define a new spatial theta-curve $f\,\#\,g$, the vertex connected sum of f and g, up to ambient isotopy [7]. K. Taniyama [6] defines cobordism between spatial theta-curves, and observes that (1) the cobordism classes form a group Θ under vertex connected sum: the cobordism inverse of a theta-curve f is represented by the reflected inverse $f!$ of f; (2) a theta-curve is slice if and only if an associated 2-component parallel link is slice (i.e. bounds disjoint disks in the 4-ball). He investigates the theta-curve cobordism group Θ through constituent knots of theta-curves, but the following fundamental question is left open in [6].

QUESTION 1. *Is Θ an abelian group?*

This note presents an example answering the question in the negative. The proof consists of showing that certain 2-component links are not slice using the refinement of the Casson-Gordon technique due to P. Gilmer [2].

Finally we raise intriguing questions below.

QUESTION 2. (1) *Does Θ contain the free group of infinite rank?*
(2) *What is the center of Θ?*

1. Statement of results.

We use the same notation as in [6], e.g. i-th parallel link $l_i(f)$, reflected inverse $f!$ of a spatial theta-curve f, theta-curve cobordism group Θ. Given a knot K and $q\in\boldsymbol{R}$, $\sigma_{(q)}(K)$ is the signature of the matrix $(1-e^{2\pi iq})V+(1-e^{-2\pi iq})V^T$ where V is a Seifert matrix for K.

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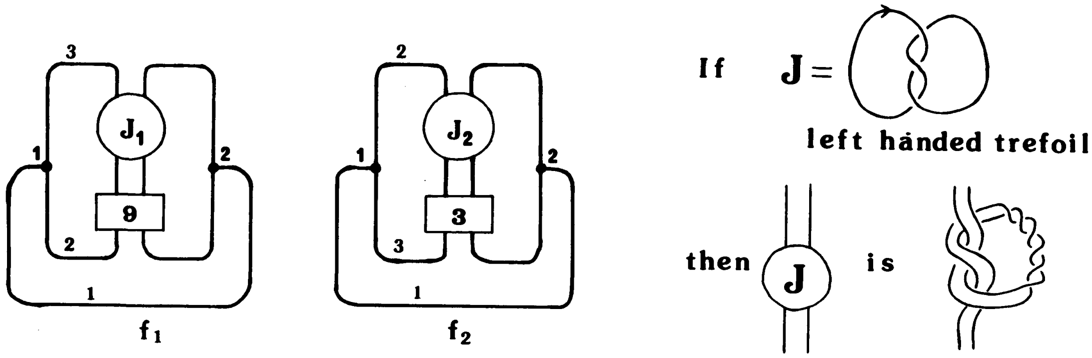
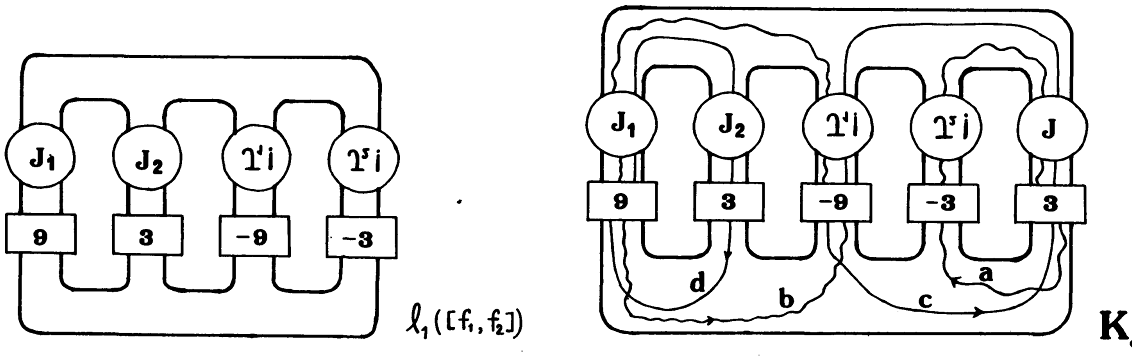
Let f_1 and f_2 be the theta-curves given in Figure 1(a). The bands are tied in knots J_i without twisting (cf. Figure 1(b)), and the integers in the boxes indicate the numbers of half-twists.



(a) (b)

FIGURE 1

PROPOSITION 3. *If* $[f_1]$, $[f_2] \in \Theta$ *commute, then* $\sigma_{(1/3)}(J_1)=0$, -2 *or* -4. *Consequently, if* J_1 *is a left handed trefoil knot* (*indicated in Figure* 1(b)) *and* J_2 *is arbitrary, then* $[f_1]$ *and* $[f_2]$ *do not commute.*



(a) (b)

FIGURE 2

Assume that $[f_1]$ and $[f_2]$ commute. Then by [6, Theorem 5] the first parallel link $l_1([f_1, f_2])$ is slice where $[f_1, f_2]$ denotes the theta-curve $f_1 \# f_2 \# f_1! \# f_2!$. In Figure 2(a) $J_i!$ denotes the knot J_i with its crossings changed; $J_i!$ upside down is obtained from the tangle J_i by reflecting in a horizontal axis. Then, connecting the two components of $l_1([f_1, f_2])$ by any band yields a slice knot. Figure 2(b) illustrates a slice knot K_J obtained in such a manner along with a basis $\{a, b, c, d\}$ for $H_1(F)$ where F is the evident Seifert surface. Our task is to deduce the claimed results in the proposition from the fact that K_J is slice for *any* knot J. We appeal to the following result of Gilmer, which combines the slicing obstructions of Levine [3] with those of

Casson-Gordon [1].

Let K be a knot with a Seifert surface F and a Seifert pairing $\varphi : H_1(F)\times H_1(F)\to \mathbf{Z}$. Define $\varepsilon : H_1(F)\to H^1(F)$ by $\varepsilon(x)(y)=\varphi(x, y)+\varphi(y, x)$. Let $A'\in H_1(F)\otimes \mathbf{Q}/\mathbf{Z}$ be the subset of elements of $\ker(\varepsilon\otimes \mathrm{id}_{\mathbf{Q}/\mathbf{Z}})$ with prime power order.

THEOREM 4 (Gilmer [2, Corollary (0.2)]). *If K is a slice knot, then there is a direct summand H of $H_1(F)$ with the properties*:

(1) $2\,\mathrm{rank}\,H=\mathrm{rank}\,H_1(F)$;

(2) $\varphi(H\times H)=0$;

(3) *Let $x\in H$ be an arbitrary primitive element such that $x\otimes s/m\in A'$ for some $0<s<m$. Then $|\sigma_{(s/m)}(J_x)|\le \mathrm{genus}(F)$ for any simple loop $J_x\subset F$ representing $x\in H_1(F)$.*

In the next section we first find all summands H satisfying conditions (1) and (2) above for the knot K_J, and then evaluate a signature of some knot by Theorem 4(3).

REMARK 5. The following facts show the difficulty of proving Θ being non-commutative.

(1) Given two theta-curves f and g, the link $L=l_i([f, g])$ has zero Conway polynomial (see [6]).

(2) Any knot obtained by a band connected sum of the components of L is algebraically slice.

2. Proof of Proposition 3.

With respect to the ordered basis $\{a, b, c, d\}$ of $H_1(F)$ in Figure 2(b), compute the Seifert matrix V for K_J : the (i, j) entry of V is the linking number of the ith base and the jth base which is pushed up off F. Then V and its inverse V^{-1} are given by:

$$V=\begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 5 & -4\\ -2 & 4 & -3 & 0\\ 0 & -5 & 0 & 6\end{pmatrix},\qquad V^{-1}=-\begin{pmatrix} 3/2 & 3/5 & 1/2 & 2/5\\ 3/2 & 3/10 & 0 & 1/5\\ 1 & 0 & 0 & 0\\ 5/4 & 1/4 & 0 & 0\end{pmatrix}.$$

Let φ be the Seifert pairing on $H_1(F)$.

Step 1. Find all 2-dimensional direct summands of $H_1(F)$ on which φ vanishes.

By [4] this is equivalent to finding 2-dimensional subspaces of $\mathbf{Q}^4$ on which the symmetric bilinear form β given by $V+V^T$ vanishes and which are invariant under the linear transformation $T=V^{-1}V^T$. In our case we have:

$$T=\begin{pmatrix} 1/2 & -9/10 & 21/10 & 3/5\\ 0 & 4/5 & 9/5 & 3/10\\ 0 & 0 & 2 & 0\\ 0 & 0 & 3/2 & 5/4\end{pmatrix},\qquad V+V^T=\begin{pmatrix} 0 & 0 & -3 & 0\\ 0 & 0 & 9 & -9\\ -3 & 9 & -6 & 0\\ 0 & -9 & 0 & 12\end{pmatrix}.$$

The transformation T has an eigenvector $x_1=(-1,0,0,0)^T$ for eigenvalue $1/2$; $x_2=(-3,1,0,0)^T$ for $4/5$; $x_3=(1,2,1,2)^T$ for 2; $x_4=(0,2,0,3)^T$ for $5/4$. Since all eigenvalues are pairwise distinct, any vector space invariant under T is spanned by eigenvectors. On the other hand, it is easy to verify that $\beta(x_i, x_j)=0$ if and only if $|i-j|\neq 2$. Thus we obtain:

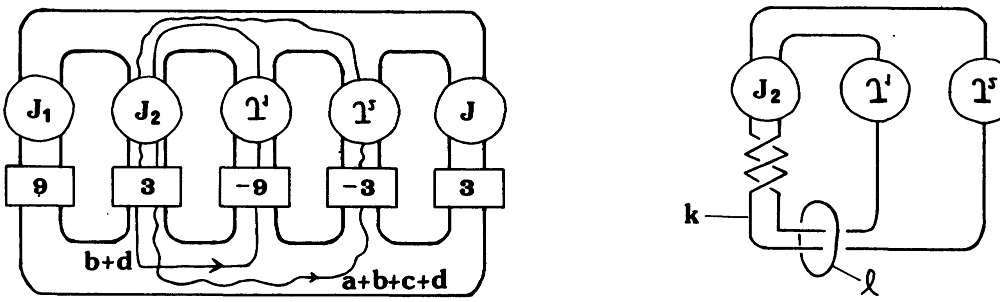
LEMMA 6. *There are exactly four 2-dimensional summands H_i, $1\leq i\leq 4$, of $H_1(F)$ on which φ vanishes*: $H_1=\langle x_1, x_2\rangle$, $H_2=\langle x_1, x_4\rangle$, $H_3=\langle x_2, x_3\rangle$, $H_4=\langle x_3, x_4\rangle$.

Applying Gilmer's theorem to K_J, we look for the summands H of $H_1(F)$ satisfying conditions (1), (2), (3) of the theorem. Then $H=H_i$ for some i.

Step 2. For each H_i choose $x\otimes(s/m)\in A'\cap(H_i\otimes \boldsymbol{Q}/\boldsymbol{Z})$ and a simple loop $J_x\subset F$ representing x. Then evaluate $\sigma_{(s/m)}(J_x)$.

First note that $x\otimes(1/3)\in A'$ for any primitive element $x\in H_1(F)$ because $V+V^T$ is divisible by 3.

Case 1. $H=H_i$ where $i=1,2$. Choose $x_1\otimes(1/3)\in A'\cap(H_i\otimes \boldsymbol{Q}/\boldsymbol{Z})$ and the simple loop $a\subset F$ representing x_1, where $i=1,2$. As knots in the 3-sphere $a=J_2!\,\#\,J$, so that $\sigma_{(1/3)}(a)=\sigma_{(1/3)}(J_2!)+\sigma_{(1/3)}(J)$. By Theorem 4(3) we get $|\sigma_{(1/3)}(J_2!)+\sigma_{(1/3)}(J)|\leq 2$.



(a) (b)

FIGURE 3

Case 2. $H=H_i$ where $i=3,4$. In this case choose $x_3\otimes(1/3)\in H_i\cap A'$. Note that $x_3=a+2b+c+2d=(a+b+c+d)+(b+d)$ (cf. Figure 3(a)). Then the knot $k\subset F$ given in Figure 3(b) represents x_3. Let k_1 be the knot k with $J_1!$ and $J_2!$ in the presentation of k replaced by trivial arcs. Since $k=k_1\,\#\,J_1!\,\#\,J_2!$, it follows:

$$\sigma_{(1/3)}(k)=\sigma_{(1/3)}(J_1!)+\sigma_{(1/3)}(J_2!)+\sigma_{(1/3)}(k_1)\,. \tag{1}$$

Let k_2 be the knot k_1 with J_2 replaced by a trivial arc; k_2 is a right handed trefoil knot. Note that k_1 is the satellite knot with pattern $k_2\subset\overline{S^3-N(l)}$ and companion J_2. The winding number of the pattern in the solid torus is 2. Using the formula of the signatures of satellite knots by Litherland [5, Theorem 2], we obtain :

$$\sigma_{(1/3)}(k_1)=\sigma_{(2/3)}(J_2)+\sigma_{(1/3)}(k_2)\,. \tag{2}$$

Note that $\sigma_{(1/3)}(k_2)=-2$, and $-\sigma_{(1/3)}(K!)=\sigma_{(1/3)}(K)=\sigma_{(2/3)}(K)$ for any knot K. It then follows from (1) and (2) that $\sigma_{(1/3)}(k)=-\sigma_{(1/3)}(J_1)-2$. By Theorem 4(3) we get $|\sigma_{(1/3)}(J_1)+2|\leq 2$, so that $\sigma_{(1/3)}(J_1)=0$, -2 or -4 as claimed in Proposition 3.

If we take J to be a knot satisfying $|-\sigma_{(1/3)}(J_2)+\sigma_{(1/3)}(J)|>2$, then Case 2 is the only possible case. Hence Proposition 3 is proved.

References

[1] A. Casson and C. McA. Gordon, Cobordism of classical knots, *A la Recherche de la Topologie Perdue* (ed. by A. Marin and L. Guillou), Progress in Math. **62** (1986), Birkhäuser, 181–199.

[2] P. M. Gilmer, Slice knots in S^3, Quart. J. Math. Oxford (2), **34** (1983), 305–322.

[3] J. Levine, Knot cobordism groups in codimension two, Comment. Math. Helv. **44** (1969), 229–244.

[4] J. Levine, Invariants of knot cobordism, Invent. Math. **8** (1969), 98–110.

[5] R. A. Litherland, Signatures of iterated torus knots, *Topology of Low-Dimensional Manifolds*, Lecture Notes in Math. **722** (1979), Springer, 71–84.

[6] K. Taniyama, Cobordism of theta curves in S^3, to appear in Math. Proc. Cambridge Philos. Soc. (1993).

[7] K. Wolcott, The knotting of theta curves and other graphs in S^3, *Geometry and Topology* (ed. by C. McCrory and T. Shifrin), Marcel Dekker (1987), 325–346.

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