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Large Time Behavior of Solution for Hartree Equation with Long Range Interaction

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§1. Introduction and theorem.

In this paper, we study the asymptotic behavior as $t\to\infty$ of the solutions of time dependent Hartree equations

$$i\partial_t u=-\frac{1}{2}\Delta_x u+(|x|^{-\gamma}*|u|^2)u \qquad (H_\gamma)$$

for $\gamma\leq 1$, where $u=u(t,x)$, $(t,x)\in \boldsymbol{R}\times\boldsymbol{R}^n$. We write $\|\cdot\|_p$ for L^p-norm, $(\cdot,\cdot)$ for L^2-coupling, $H^{l,k}=\{u\in L^2:\sum_{|\alpha|\leq l}\|\partial_x^\alpha u\|_2+\sum_{|\beta|\leq k}\|x^\beta u\|_2<\infty\}$ for $l,k=0,1,2,\cdots$ and $U(t)=\exp[(i/2)t\Delta_x]$. There is a large body of literature on the equation (H_γ). It is well-known that a unique global solution exists for any $u_0\in H^{1,0}$ if $0\leq\gamma<\min\{4,n\}$. (cf. [GV], [DF] etc.) If we assume $\gamma>1$, any above solution u behaves like free solution as t goes to infinity: that is, there exists an asymptotic state u_+ such that $\|u(t)-U(t)u_+\|_X\to 0$ as $t\to\infty$ in a suitable space X. On the other hand, if $\gamma\leq 1$, no non-trivial solution becomes asymptotically free. (See e.g. [G], [HT], [NO] etc.) But inferring on the analogy of linear long range scattering theory, the solution of this case is expected to behave *almost free*. That is, if we slightly modify the solution u by a certain phase S, then this modified solution is expected to become asymptotically free. Following result for the case $n\geq 3$ suggests above expectation.

THEOREM 1. *Let* $u_0\in H^{1,1}$, $1\geq\gamma>2/3$ *if* $n\geq 4$, $1\geq\gamma>(\sqrt{17}-1)/4$ *if* $n=3$, *and* $u(t,x)$ *be a solution of* (H_γ) *such that* $u(0,x)=u_0(x)$. *If we put*

$$S(\tau,\xi):=\int_1^\tau(|x|^{-\gamma}*|u|^2)(s,s\xi)ds=\int_1^\tau\int_{\boldsymbol{R}^n}\frac{|u(s,y)|^2}{|s\xi-y|^\gamma}dyds\,, \qquad (1)$$

then $u_+:=\text{w-lim}_{t\to\infty}M(t)U(-t)\exp[iS(t,t^{-1}\cdot)]u(t,\cdot)$ *exists in* $H^{1,0}$. *Here* $M(t)=\exp[(i/(2t))|x|^2]$.

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REMARK 1. If u_+ becomes strong limit of r.h.s., we have

$$\exp[iS(t, t^{-1}\cdot)]u(t,\cdot)\sim U(t)u_+ , \qquad |u(t,\cdot)|\sim|U(t)u_+|$$

as $t\to\infty$. This means u becomes like modified free solution.

REMARK 2. The r.h.s. of (1) is reminiscent of the Dollard's modified wave operator and first introduced by Ozawa [O].

REMARK 3. One of the important problem in non-linear scattering is to determine the range of asymptotic state u_+. This is equivalent to the existence of (modified) wave operator. For the case of $\gamma>1$, this problem is treated in [NO], [S], [HT], etc. On the other hand, Ginibre and Ozawa [GO] showed if $n\geq 2$, $\gamma=1$ and $u_+\in H^{0,2}$ is sufficiently small in some sense, then the modified wave operator exists in L^2.

REMARK 4. For power type nonlinear Schrödinger equations

$$i\partial_t u=-\tfrac{1}{2}\Delta_x u+|u|^{p-1}u\,, \tag{P}$$

the case $p-1\leq 2/n$ is long range interaction: that is, no non-trivial solution of (P) becomes asymptotic free. In this case, we obtain following similar result for (H) if $n=2$ and $p=2$.

PROPOSITION 2. *Let $u_0\in H^{1,1}$, $n=2$, $p=2$ and $u(t,x)$ be a solution of (P) such that $u(0,x)=u_0(x)$. If we put*

$$S(\tau,\xi):=\int_1^\tau |u(s,s\xi)|\,ds\,,$$

then $u_+:=\text{w-lim}_{t\to\infty}M(t)U(-t)\exp[iS(t,t^{-1}\cdot)]u(t,\cdot)$ exists in H^1.

REMARK 5. For many other results for general nonlinear Schrödinger equations, a good summary exists in Cazenave's textbook [C].

§2. The proof of Theorem 1 and Proposition 2.

We first remark mass and energy conservation laws for the solution of (H_γ): $\|u(t,\cdot)\|_2=\|u_0\|_2$ and $E(u(t,\cdot))=E(u_0)$, where $E(u)=\|\nabla_x u\|_2^2+(|u|^2,|x|^{-\gamma}*|u|^2)$. Furthermore, for $u_0\in H^{1,1}$, we have so-called pseudo conformal conservation law:

$$\begin{aligned}\|J(t)u(t,\cdot)\|_2^2&+t^2(|u(t,\cdot)|^2,|x|^{-\gamma}*|u(t,\cdot)|^2)\\ &=\|J(t_0)u(t_0,\cdot)\|_2^2+t_0^2(|u(t_0,\cdot)|^2,|x|^{-\gamma}*|u(t_0,\cdot)|^2)\\ &\quad+(2-\gamma)\int_{t_0}^t s(|u(s,\cdot)|^2,|x|^{-\gamma}*|u(s,\cdot)|^2)ds\end{aligned}$$

where $J(t)=x+it\nabla_x=U(t)xU(-t)=M(t)(it\nabla_x)M(-t)$. Moreover, we obtain the following lemma. (See e.g. [HO1].)

LEMMA 3. *$\|J(t)u(t,\cdot)\|_2 \le Ct^{1-\gamma/2}$ for $t \ge 1$ and*

$$\int_1^\infty s^{\gamma-3}\|J(s)u(s,\cdot)\|_2^2\,ds < \infty .$$

From this lemma, we obtain following corollary.

COROLLARY 4. *If we set $A(t) := M(t)U(-t)\exp[iS(t, t^{-1}\cdot)]u(t,\cdot)$, $A(t)$ is uniformly bounded in H^1.*

Then it suffices to prove that $((d/dt)A(t), \psi) \in L^1([1,\infty))$ for any $\psi \in \mathscr{S}$. By using the relation $-i\nabla_x M(t)U(-t) = M(t)U(-t)t^{-1}x$, we calculate

$$\begin{aligned}
&i\frac{d}{dt}A(t,x)\\
&= M(t)\frac{|x|^2}{2t^2}U(-t)\exp[iS(t,t^{-1}x)]u(t) + \frac{1}{2}M(t)U(-t)\Delta_x\cdot\exp[iS(t,t^{-1}x)]u(t)\\
&\quad + M(t)U(-t)\exp[iS(t,t^{-1}x)]\left\{-(\partial_t S)(t,t^{-1}x) + \frac{x}{t^2}(\nabla_\xi S)(t,t^{-1}x)\right\}u(t)\\
&\quad + iM(t)U(-t)\exp[iS(t,t^{-1}x)]\partial_t u(t)\\
&= \frac{1}{2t^2}M(t)U(-t)J(t)^2\exp[iS(t,t^{-1}x)] + \frac{1}{2}M(t)U(-t)\Delta_x\exp[iS(t,t^{-1}x)]u(t)\\
&\quad + M(t)U(-t)\exp[iS(t,t^{-1}x)]\left\{-(\partial_t S)(t,t^{-1}x) + \frac{x}{t^2}(\nabla_\xi S)(t,t^{-1}x)\right\}u(t)\\
&\quad + M(t)U(-t)\exp[iS(t,t^{-1}x)]\left\{-\frac{1}{2}\Delta_x u(t) + (|x|^{-\gamma}*|u(t)|^2)\right\}u(t)\\
&= \frac{1}{2}M(t)U(-t)\left\{\frac{|x|^2}{t^2} + \frac{i}{t}(x\cdot\nabla_x + \nabla_x\cdot x)\right\}\exp[iS(t,t^{-1}x)]\,u(t)\\
&\quad + M(t)U(-t)\exp[iS(t,t^{-1}x)]\left\{-(\partial_t S)(t,t^{-1}x) + \frac{x}{t^2}\cdot(\nabla_x S)(t,t^{-1}x)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left. -\frac{1}{2}\Delta_x + (|x|^{-\gamma}*|u(t)|^2)\right\}u(t)\\
&= M(t)U(-t)\exp[iS(t,t^{-1}x)]\\
&\quad \times\left\{\frac{|x|^2}{2t^2} + \frac{i}{2t}(x\cdot\nabla_x + \nabla_x\cdot x) - \frac{1}{2}\Delta_x - (\partial_t S)(t,t^{-1}x) + (|x|^{-\gamma}*|u(t)|^2)\right\}u(t)\\
&= \frac{1}{2t^2}M(t)U(-t)\exp[iS(t,t^{-1}x)]J(t)^2u(t)\,.
\end{aligned}$$

Then, for any $\psi \in \mathscr{S}$, we calculate

$$\begin{aligned}
&\left(i\frac{d}{dt}A(t), \psi\right) \\
&=\frac{1}{2t^2}(M(t)U(-t)\exp[iS(t, t^{-1}x)]J(t)^2u(t), \psi) \\
&=\frac{1}{2t^2}(J(t)u(t), J(t)\exp[-iS(t, t^{-1}x)]U(t)M(-t)\psi) \\
&=\frac{1}{2t^2}(J(t)u(t), U(t)M(-t)x\cdot\mathscr{F}^{-1}\exp[-iS(t, \cdot)]\mathscr{F}\psi) \\
&=\frac{1}{2t^2}(J(t)u(t), U(t)M(-t)\mathscr{F}^{-1}\cdot i\nabla_\xi\cdot\exp[-iS(t, \cdot)]\mathscr{F}\psi) \\
&=\frac{1}{2t^2}(J(t)u(t), U(t)M(-t)\mathscr{F}^{-1}\exp[-iS(t, \cdot)]\{i\nabla_\xi+(\nabla_\xi S)(t, \cdot)\}\mathscr{F}\psi), \qquad (2)
\end{aligned}$$

where $\mathscr{F}$ is Fourier transform from $\boldsymbol{R}_x^n$ to $\boldsymbol{R}_\xi^n$ and then,

$$\left|\left(\frac{d}{dt}A(t), \psi\right)\right| \le \frac{1}{2}t^{-2}\|J(t)u(t)\|_2(\|\nabla_\xi\mathscr{F}\psi\|_2+\|(\nabla_\xi S)(t, \cdot)\mathscr{F}\psi\|_2). \qquad (3)$$

By Hölder's and generalized Young's inequalities, we have

$$\begin{aligned}
\|(\nabla_\xi S)(t, \cdot)\mathscr{F}\psi\|_2 &\le \|(\nabla_\xi S)(t, \cdot)\|_{p_1}\|\mathscr{F}\psi\|_{p_2} \\
&\le C\|\mathscr{F}\psi\|_{p_2}\int_1^t s\|(|x|^{-\gamma-1}*|u|^2)(s, s\cdot)\|_{p_1}ds \\
&= C\|\mathscr{F}\psi\|_{p_2}\int_1^t s^{1-n/p_1}\|(|x|^{-\gamma-1}*|u|^2)(s)\|_{p_1}ds \\
&\le C\|\mathscr{F}\psi\|_{p_2}\int_1^t s^{1-n/p_1}\|u(s)\|_{p_3}^2 ds,
\end{aligned}$$

where $1/p_1+1/p_2=1/2$, $1+1/p_1=(\gamma+1)/n+2/p_3$, $2\le p_1<\infty$ and $2<p_3<\infty$. Here, using Gagliardo-Nirenberg's inequality, we have

$$\begin{aligned}
\|u(s)\|_{p_3} &= \|M(-s)u(s)\|_{p_3} \\
&\le C\|\nabla_x M(-s)u(s)\|_2^\alpha\|M(-s)u(s)\|_2^{1-\alpha} \\
&= Cs^{-\alpha}\|J(s)u(s)\|_2^\alpha\|u_0\|_2^{1-\alpha},
\end{aligned}$$

where $1/p_3=1/2-\alpha/n$, and we obtain

$$\|(\nabla_\xi S)(t, \cdot)\mathscr{F}\psi\|_2 \le C\int_1^t s^{1-n/p_1-2\alpha}\|J(s)u(s)\|_2^{2\alpha}ds\cdot\|\mathscr{F}\psi\|_{p_2}. \qquad (4)$$

Then, combining (3), (4) and Lemma 3, we deduce that

$$\left|\left(\frac{d}{dt}A(t), \psi\right)\right| \leq C(\|\nabla_\xi \mathscr{F}\psi\|_2 + \|\mathscr{F}\psi\|_{p_2} t^{2-n/p_1-\alpha\gamma}) t^{-2+1-\gamma/2} .$$

Above estimate shows if we can take α and p_1 such that

$$2 < \frac{\gamma}{2} + \alpha\gamma + \frac{n}{p_1} , \tag{5}$$

$$1 + \frac{1}{p_1} = \frac{\gamma+1}{n} + 2\left(\frac{1}{2} - \frac{\alpha}{n}\right) \quad \text{i.e.} \quad \frac{2\alpha}{n} = \frac{\gamma+1}{n} - \frac{1}{p_1} , \tag{6}$$

$$2 \leq p_1 < \infty , \tag{7}$$

$$0 < \alpha \leq 1 , \tag{8}$$

then $((d/dt)A(t), \psi) \in L^1([1, \infty))$ and desired weak limit exists.

We claim that if p_1 satisfies

$$\frac{4-2\gamma-\gamma^2}{n(2-\gamma)} < \frac{1}{p_1} \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{p_1} < \frac{\gamma+1}{n} , \tag{9}$$

then (5) $\sim$ (8) hold.

In fact, (9) means (7) obviously and

$$\frac{1}{p_1} < \frac{\gamma+1}{n} , \tag{10}$$

$$\begin{aligned} &(4-2\gamma-\gamma^2)p_1 < n(2-\gamma) \\ &\quad \Leftrightarrow 4p_1 < p_1\gamma(\gamma-2) - n\gamma + 2n \\ &\quad \Leftrightarrow 2 < \frac{\gamma}{2} + \frac{n\gamma}{2}\left(\frac{\gamma+1}{n} - \frac{1}{p_1}\right) + \frac{n}{p_1} . \end{aligned} \tag{11}$$

By defining α as (6), (10) and (11) mean (8) and (5) respectively.

Now, we can take p_1 satisfying (9) if we have

$$\frac{4-2\gamma-\gamma^2}{n(2-\gamma)} \leq \frac{1}{2} \quad \text{and} \quad \frac{4-2\gamma-\gamma^2}{n(2-\gamma)} < \frac{\gamma+1}{n} .$$

Solving these inequalities, we obtain our result and complete the proof of Theorem 1.

Next, we consider Proposition 2. In this case, similar calculation as Lemma 3 shows

$$\int_1^\infty t^{-2} \|J(t)u(t)\|_2^2 \, dt < \infty ,$$

and

$$\begin{aligned}
&\|(\nabla_\xi S)(t)\mathscr{F}\psi\|_2 \\
&\quad \le \|(\nabla_\xi S)(t)\|_2 \|\mathscr{F}\psi\|_\infty \\
&\quad = \left(\int_{\boldsymbol{R}^n_\xi} \left| \int_1^t \nabla_\xi \left| \exp\left[-\frac{i}{2}s|\xi|^2\right] \cdot u(s, s\xi) \right| ds \right|^2 d\xi \right)^{1/2} \cdot \|\mathscr{F}\psi\|_\infty \\
&\quad \le \left(\int_{\boldsymbol{R}^n_\xi} \left(\int_1^t \left| i\nabla_\xi \left\{ \exp\left[-\frac{i}{2}s|\xi|^2\right] \cdot u(s, s\xi) \right\} \right| ds \right)^2 d\xi \right)^{1/2} \cdot \|\mathscr{F}\psi\|_\infty \\
&\quad \le \int_1^t s^{-1}\|J(s)u(s)\|_2 ds \cdot \|\mathscr{F}\psi\|_\infty .
\end{aligned}$$

Here we use Kato's inequality $\nabla|v| \le |\nabla v|$ at forth line. Thus, we have

$$\begin{aligned}
&\int_1^\infty t^{-2}\|J(t)u(t)\|_2 \|\{i\nabla_\xi + (\nabla_\xi S)(t, \cdot)\}\mathscr{F}\psi\|_2 dt \\
&\qquad \le \int_1^\infty t^{-2}\|J(t)u(t)\|_2 dt \cdot \|\nabla_\xi \mathscr{F}\psi\|_2 \\
&\qquad\quad + \int_1^\infty \left(t^{-1}\|J(t)u(t)\|_2 \cdot t^{-1} \int_1^t s^{-1}\|J(s)u(s)\|_2 ds \right) dt .
\end{aligned}$$

Now, using Hardy's inequality:

$$\left\| t^{-1} \int_0^t f(s) ds \right\|_{L^2(0,\infty)} \le 2\|f\|_{L^2(0,\infty)}$$

for

$$f(s) = \begin{cases} 0 & \text{if } 0 \le s \le 1 , \\ s^{-1}\|J(s)u(s)\|_2 & \text{if } \ 1 < s , \end{cases}$$

we have

$$\left\| t^{-1} \int_1^t s^{-1}\|J(s)u(s)\|_2 ds \right\|_{L^2(0,\infty)} \le 2\left(\int_1^\infty s^{-2}\|J(s)u(s)\|_2^2 ds \right)^{1/2} .$$

Thus,

$$\begin{aligned}
&\int_1^\infty (t^{-1}\|J(t)u(t)\|_2) \cdot \left(t^{-1} \int_1^t s^{-1}\|J(s)u(s)\|_2 ds \right) dt \\
&\qquad \le 2 \int_1^\infty t^{-2}\|J(t)u(t)\|_2^2 dt < \infty .
\end{aligned}$$

From this estimate, we obtain

$$\int_1^\infty t^{-2}\|J(t)u(t)\|_2\,\|\{i\nabla_\xi+(\nabla_\xi S)(t,\cdot)\}\mathscr{F}\psi\|_2\,dt$$
$$\le C\int_1^\infty t^{-1}\cdot t^{-1}\|J(t)u(t)\|_2\,dt+2\int_1^\infty t^{-2}\|J(t)u(t)\|_2^2\,dt$$
$$\le C\left(\int_1^\infty t^{-2}\|J(t)u(t)\|_2^2\,dt\right)^{1/2}+2\int_1^\infty t^{-2}\|J(t)u(t)\|_2^2\,dt<\infty\,.$$

This means our desired result. □

§3. Remark of Theorem 1: the strong limit.

As mentioned in Lemma 3, $t^{-1}\|J(t)u(t)\|_2=\|M(t)\cdot i\nabla_x\cdot M(-t)u(t)\|_2\le Ct^{-\gamma/2}$ even though $\|\nabla_x u(t)\|_2\sim C$ as $t\to\infty$. This decay estimate means the expectation value of momentum $p=-i\nabla_x$ behaves like $t^{-1}x$, similarly as the free particle. In fact, we have $t^{-1}\|J(t)U(t)u_0\|_2=t^{-1}\|xu_0\|_2$ in free case, and this observation suggests $\|J(t)u(t)\|_2$ decays faster than what Lemma 3 shows. If this expectation is true, u_+ becomes strong limit.

PROPOSITION 5. *If $\|J((t)u(t)\|_2\le Ct^{1-\mu}$ for some $\mu>2/(\gamma+2)$, then u_+ in Theorem 1 is strong limit of $A(t)$ in L^2.*

PROOF. By virtue of Theorem 1 and

$$\|u_+-A(t)\|_2^2=(u_+-A(t),u_+)-(u_+-A(t),A(t))\,,$$

it suffices to show that

$$(u_+-A(t),A(t))\to 0 \qquad \text{as} \quad t\to\infty\,.$$

Now, by putting $A(t)$ in ψ of (2), we get

$$i(u_+-A(t),A(t))=\int_t^\infty\left(i\frac{d}{ds}A(s),A(t)\right)ds=\frac{1}{2}\int_t^\infty s^{-2}(J(s)u(s),$$
$$U(s)M(-s)\mathscr{F}^{-1}\exp[-iS(s,\cdot)]\{i\nabla_\xi+(\nabla_\xi S)(s,\cdot)\}\mathscr{F}A(t))\,ds\,. \qquad (12)$$

Remarking that

$$(\mathscr{F}M(t)U(-t)\phi)(\xi)=(it)^{n/2}\exp\left[-\frac{i}{2}t|\xi|^2\right]\phi(t\xi)$$

for any ϕ, we can write $\mathscr{F}A(t)$ as following:

$$\{i\nabla_\xi+(\nabla_\xi S)(s,\cdot)\}\mathscr{F}A(t)=(it)^{n/2}\exp\left[-\frac{i}{2}t|\xi|^2+iS(t,\xi)\right]$$
$$\times\{t\xi+i\nabla_\xi+(\nabla_\xi S)(s,\xi)-(\nabla_\xi S)(t,\xi)\}u(t,t\xi)\,. \qquad (13)$$

From (12) and (13), we obtain

$$|(u_+ - A(t), A(t))| \le \frac{1}{2}\int_t^\infty s^{-2}\|J(s)u(s)\|_2 \cdot t^{n/2}\left\|\left\{(t\xi + i\nabla_\xi) + \int_t^s \frac{d}{d\tau}(\nabla_\xi S)(\tau, \xi)d\tau\right\}u(t, t\xi)\right\|_{L^2(\boldsymbol{R}^n_\xi)} ds\,.$$

Since

$$\frac{d}{d\tau}(\nabla_\xi S)(\tau, \xi) = \tau(\nabla|x|^{-\gamma} * |u|^2)(\tau, \tau\xi)\,,$$

we get

$$\begin{aligned} t^{n/2}&\left\|\int_t^s \frac{d}{d\tau}(\nabla_\xi S)(\tau, \xi)d\tau \cdot u(t, t\xi)\right\|_{L^2(\boldsymbol{R}^n_\xi)} \\ &\le \left\|\int_t^s \frac{d}{d\tau}(\nabla_\xi S)(\tau, \xi)d\tau\right\|_{L^\infty(\boldsymbol{R}^n_\xi)} \cdot \|u_0\|_2 \\ &\le \|u_0\|_2 \cdot \int_t^s \tau\|(\nabla|x|^{-\gamma} * |u|^2)(\tau, \tau\xi)\|_{L^\infty(\boldsymbol{R}^n_\xi)} d\tau\,. \end{aligned}$$

For the estimation of this term, we use following lemma.

LEMMA 6. *Let $p \in [1, \infty)$, $q < n$ and $0 \le q \le p$. Then*

$$\int_{\boldsymbol{R}^n} \frac{|\phi(x)|^p}{|x|^q} dx \le \left(\frac{p}{n-q}\right)^q \|\phi\|_p^{p-q}\|\nabla_x\phi\|_p^q$$

for any $\phi \in W^{1,p} := \{\phi \in L^p : \nabla_x\phi \in L^p\}$.

For the proof of this lemma, see [C] p. 131 Lemma 7.4.1.
In virtue of this lemma, we have

$$\begin{aligned} \|(\nabla|x|^{-\gamma} * |u|^2)(\tau, \tau\xi)\|_{L^\infty(\boldsymbol{R}^n_\xi)} &= C \sup_{\xi \in \boldsymbol{R}^n} \int_{\boldsymbol{R}^n} \frac{|u(\tau, \tau\xi - \eta)|^2}{|\eta|^{\gamma+1}} d\eta \\ &= C \sup_{\xi \in \boldsymbol{R}^n} \int_{\boldsymbol{R}^n} \frac{|\exp[(i/(2\tau))|\tau\xi - \eta|^2] \cdot u(\tau, \tau\xi - \eta)|^2}{|\eta|^{\gamma+1}} d\eta \\ &\le C \sup_{\xi \in \boldsymbol{R}^n} \|\nabla_\eta \cdot \exp[(i/(2\tau))|\tau\xi - \eta|^2] \cdot u(\tau, \tau\xi - \eta)\|_{L^2(\boldsymbol{R}^n_\eta)}^{\gamma+1} \cdot \|u(\tau, \tau\xi - \eta)\|_{L^2(\boldsymbol{R}^n_\eta)}^{1-\gamma} \\ &= C\|u_0\|_2^{1-\gamma} \cdot \sup_{\xi \in \boldsymbol{R}^n} \|-i\tau^{-1}(\tau\xi - \eta) \cdot u(\tau, \tau\xi - \eta) - (\nabla u)(\tau, \tau\xi - \eta)\|_{L^2(\boldsymbol{R}^n_\eta)}^{\gamma+1} \\ &\le C\|u_0\|_2^{1-\gamma} \cdot \tau^{-1-\gamma}\|J(\tau)u(\tau)\|_2^{\gamma+1}\,. \end{aligned}$$

Then, by using the assumption for $J(t)u(t)$, we obtain

$$t^{n/2}\left\| \int_t^s \frac{d}{d\tau}(\nabla_\xi S)(\tau,\xi)\,d\tau\cdot u(t,t\xi)\right\|_{L^2(\mathbf{R}^n_\xi)}$$
$$\le C\int_t^s \tau^{-\gamma}\|J(\tau)u(\tau)\|_2^{\gamma+1}d\tau \le C\int_t^s \tau^{(\gamma+1)(1-\mu)-\gamma}d\tau$$
$$=C(t^{1-\gamma+(\gamma+1)(1-\mu)}-s^{1-\gamma+(\gamma+1)(1-\mu)})\,.$$

Thus, remarking that

$$t^{n/2}\|(t\xi+i\nabla_\xi)u(t,t\xi)\|_{L^2(\mathbf{R}^n_\eta)}=\|J(t)u(t)\|_2\,,$$

we have

$$|(u_+-A(t),A(t))|$$
$$\le C\int_t^\infty s^{-2}\|J(s)u(s)\|_2(\|J(t)u(t)\|_2+t^{1-\gamma+(\gamma+1)(1-\mu)}-s^{1-\gamma+(\gamma+1)(1-\mu)})ds$$
$$\le C\int_t^\infty s^{-2+1-\mu}(t^{1-\mu}+t^{1-\gamma+(\gamma+1)(1-\mu)})ds=C(t^{1-2\mu}+t^{(\gamma+2)(1-\mu)-\gamma})\,.$$

By the assumption of μ,

$$(\gamma+2)(1-\mu)-\gamma<0 \quad\text{and}\quad 1-2\mu<0\,.$$

This means our desired result. □

§4. Proof of Lemma 3 and Corollary 4.

PROOF OF LEMMA 3. We put

$$\int_{t_0}^t s(|u(s,\cdot)|^2,|x|^{-\gamma}*|u(s,\cdot)|^2\,ds=:G(t)\,,$$

$$\|J(t_0)u(t_0,\cdot)\|_2^2+t_0^2(|u(t_0,\cdot)^2,|x|^{-\gamma}*|u(t_0,\cdot)|^2)=:B$$

for suitable $t_0>0$. Then, we can rewrite the pseudo-conformal conservation law as following:

$$\|J(t)u(t)\|_2^2+tG'(t)=B+(2-\gamma)G(t)\,.$$

Since this means

$$\frac{d}{dt}(t^{\gamma-2}G(t))=t^{\gamma-3}(B-\|J(t)u(t)\|_2^2)\,, \tag{14}$$

integrating with respect to t, we obtain

$$t^{\gamma-2}G(t)=\int_{t_0}^{t} s^{\gamma-3}(B-\|J(s)u(s)\|_2^2)ds\,.$$

Remarking that the l.h.s. ≥ 0, we have

$$\int_{t_0}^{t} s^{\gamma-3}\|J(s)\|_2^2 ds=(2-\gamma)^{-1}B(t_0^{\gamma-2}-t^{\gamma-2})-t^{\gamma-2}G(t)<\infty$$

for any $t>t_0$. This means the latter of desired estimates. The former one obeys (14) and Gronwall's inequality. $\square$

PROOF OF COROLLARY 4. Obviously, $\|A(t)\|_2=\|u(t)\|_2=\|u_0\|_2$, it suffices to show $\|\nabla_x A(t)\|_2\leq C$. Since $-i\nabla_x A(t)=M(t)U(-t)\exp[iS(t, t^{-1}x)]\cdot t^{-1}x\cdot u(t)$,

$$\|\nabla_x A(t)\|_2=t^{-1}\|xu(t)\|_2\leq t^{-1}\|J(t)u(t)\|_2+\|\nabla_x u(t)\|_2\,.$$

Now, remarking that $\|\nabla_x u(t)\|_2^2\leq E(u)=$const., we obtain the corollary from Lemma 3. $\square$

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