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On an Extension of the Ikehara Tauberian Theorem II

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Abstract. We consider a positively definite self-adjoint operator P on a separable Hilbert space H which has a compact resolvent. Then a specific example of the Ikehara Tauberian theorem is extended to the case where the zeta function of P only has simple poles. In such circumstances, we can obtain the asymptotic behavior of the counting function of eigenvalues with remainder terms. And we have their applications to some partial differential operators.

Introduction.

In this paper we shall extend the result of the previous paper Aramaki [2]. To be more precise, let P be a positively definite unbounded self-adjoint operator on a separable Hilbert space H with the domain of definition K which is dense in H. Throughout this paper, we assume that P has a compact resolvent. Then it is well known that the spectrum $\sigma(P)$ of P is discrete. Thus we can write the eigenvalues of P by $0<\lambda_1\le\lambda_2\le\cdots$, $\lim_{j\to\infty}\lambda_j=\infty$ with repetition according to multiplicity and the complete orthonormal basis in H consisting of eigenvectors by $\{e_j\}_{j=1,2,\dots}$. If f is a complex valued function defined on $\sigma(P)$, we can define an operator $f(P)$ with domain of definition $D(f(P))$ as follows:

$$D(f(P))=\left\{u\in H\ ;\ \sum_{j=1}^{\infty}|f(\lambda_j)|^2|(u,e_j)|^2<\infty\right\} \tag{0.1}$$

$$f(P)u=\sum_{j=1}^{\infty}f(\lambda_j)(u,e_j)e_j\,,\qquad u\in D(f(P))$$

where $(\cdot,\cdot)$ denotes the inner product in H.

If we choose $f(\lambda)=\lambda^{-s}$, $\lambda>0$, for $s\in\boldsymbol{C}$ where λ^{-s} for $\lambda>0$ take the principal values, we can define complex powers P^{-s}. Of course, if we denote the spectral resolution associated to P by $\{E(\lambda)\,;\,\lambda\in\boldsymbol{R}\}$, we can also write:

$$P^{-s}=\int_0^{\infty}\lambda^{-s}dE(\lambda)\,,\qquad (s\in\boldsymbol{C})\,.$$

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Let $N_P(\lambda)$ be the counting function of eigenvalues:

(0.2) $$N_P(\lambda)=\#\{j\,;\,\lambda_j\le\lambda\}\,,\qquad(\lambda>0)\,.$$

Then the purpose of this paper is to find the asymptotic behavior of $N_P(\lambda)$ as $\lambda\to\infty$ with remainder terms.

A specific example of Ikehara's Tauberian theorem says:

PROPOSITION 0.1 (Wiener [16] and Donoghue [6]). *Let the trace* $\mathrm{Tr}[P^{-s}]=\sum_{j=1}^{\infty}\lambda_j^{-s}$ *of P be holomorphic for* $\mathrm{Re}\,s>a$ (>0). *Moreover, assume that there exists a constant $A>0$ such that*

$$\mathrm{Tr}[P^{-s}]-\frac{A}{s-a}$$

is continuous for $\mathrm{Re}\,s\ge a$. *Then we have*

$$N_P(\lambda)=\frac{A}{a}\lambda^a(1+o(1))\qquad as\quad \lambda\to\infty\,.$$

For realization P in $H=L^2(\boldsymbol{R}^n)$ of global elliptic differential or pseudodifferential operators, it is well known that $\mathrm{Tr}[P^{-s}]$ has a meromorphic extension $Z_P(s)$ in $\boldsymbol{C}$ and a simple pole at the first singularity. Thus we can apply this proposition and then determine the first term in the expansion of $N_P(\lambda)$. For example, if P is the unique self-adjoint extension of the harmonic oscillator: $P^0=-\Delta+|x|^2$ starting from $C_0^\infty(\boldsymbol{R}^n)$, we see that the first singularity of $\mathrm{Tr}[P^{-s}]$ is at $s=n$ which is a simple pole with the residue $n/2^n$ and $N_P(\lambda)=2^{-n}\lambda^n(1+o(1))$ as $\lambda\to\infty$. (See also Seeley [14]). However, if we use the complex powers method in order to determine the remainder terms, we must extend Proposition 0.1 into the form of Theorem 1.1 below in §1. According to Shubin [15], if $N_P(\lambda)$ admits the following asymptotic expansion

(0.3) $$N_P(\lambda)=c_1\lambda^{a_1}+c_2\lambda^{a_2}+\cdots+c_p\lambda^{a_p}+O(\lambda^{a_{p+1}})\qquad\text{as}\quad\lambda\to\infty$$

where $a_1>a_2>\cdots>a_p>a_{p+1}$, then we have

$$Z_P(s)=\sum_{l=1}^{p}\frac{c_la_l}{s-a_l}+f_p(s)\,,\qquad \mathrm{Re}\,s>a_{p+1}$$

where $f_p(s)$ is holomorphic for $\mathrm{Re}\,s>a_{p+1}$. Then our investigation is to make a response for the converse of (0.3) under some conditions. Of course, there are some cases where the singularities of $\mathrm{Tr}[P^{-s}]$ have multiple poles, however, for such cases we only refer to Aramaki [3].

The plan of this paper is as follows. In §1, we give some notational remarks and the main theorem. Section 2 is devoted to the proof of the main theorem. The proof consists of three parts (A), (B) and (C). In part (A), we examine the asymptotic behavior of $\mathrm{Tr}[e^{-zQ}]$ as $z\to0$ with $\mathrm{Re}\,z>0$ where Q is a suitable power of P (cf. Duistermaat and Guillemin [7]). Part (B) is devoted to the study of asymptotic behavior of $I(\mu)$ where

$$I(\mu)=\int_{-\infty}^{\infty}\rho(\mu-\tau)dN_Q(\tau)$$

for some rapidly decreasing function ρ. In part (C), we get the asymptotic behavior of $N_P(\lambda)$ using the result of part (B). Finally, in §3 we give two applications of the main theorem to eigenvalue asymptotics for some partial differential operators.

§1. Statements of results.

Let H be a separable Hilbert space and P a densely defined positively definite unbounded self-adjoint operator on H with the domain of definition K. We regard K equipped with the graph norm as a Hilbert space. We assume:

(H) The canonical injection from K to H is compact.

Then it follows from (H) that the spectrum $\sigma(P)$ of P is discrete, i.e., both the following hold:

(1.1) $\lambda\in\sigma(P)$ is an isolated point of $\sigma(P)$.

(1.2) $\lambda\in\sigma(P)$ is an eigenvalue of finite multiplicity.

Thus we can denote the sequence of eigenvalues with repetition according to multiplicity by $0<\lambda_1\leq\lambda_2\leq\cdots$, $\lim_{k\to\infty}\lambda_k=+\infty$.

Since complex powers of P are defined as in introduction, we can define

$$\mathrm{Tr}[P^{-s}]=\sum_{j=1}^{\infty}\lambda_j^{-s}$$

which denotes the trace of P^{-s} if P^{-s} is of trace class.

On the other hand, for $z\in \boldsymbol{C}$, $\mathrm{Re}\,z>0$, choosing $f(\lambda)=e^{-z\lambda}$, $\lambda>0$, as a function in (0.1), we also define an operator e^{-zP}. If we can assume that there exist constants $c, C>0$ such that $\lambda_j\sim Cj^c$ as $j\to\infty$, it is easily seen that e^{-zP} is of trace class and $\mathrm{Tr}[e^{-zP}]$ is holomorphic for $\mathrm{Re}\,z>0$. For $z=it$ ($t\in\boldsymbol{R}$), we can define an tempered distribution

$$\mathrm{Tr}[e^{-itP}]:\phi\in\mathscr{S}(\boldsymbol{R})\mapsto\mathrm{Tr}[\hat{\phi}(P)]=\sum_{j=1}^{\infty}\hat{\phi}(\lambda_j).$$

Here $\mathscr{S}(\boldsymbol{R})$ denotes the set of all rapidly decreasing smooth functions on $\boldsymbol{R}$ and $\hat{\phi}$ the Fourier transformation of ϕ. We note that

$$\mathrm{Tr}[e^{-(\tau+it)P}]\to\mathrm{Tr}[e^{-itP}]\qquad\text{as}\quad\tau\downarrow 0\ \text{in}\ \mathscr{S}'(\boldsymbol{R})$$

where $\mathscr{S}'(\boldsymbol{R})$ denotes the totality of tempered distributions on $\boldsymbol{R}$. Thus we may assume that $\mathrm{Tr}[e^{-zP}]$ is a holomorphic function in z for $\mathrm{Re}\,z>0$ and a tempered distribution with respect to $t=\operatorname{Im} z\in \boldsymbol{R}$.

Now we state the main theorem.

THEOREM 1.1. *Let P be a positively definite self-adjoint operator on H satisfying the condition* (H). *Moreover assume that*

(i) *P^{-s} is of trace class for large $\operatorname{Re} s>0$ and $\operatorname{Tr}[P^{-s}]$ has a meromorphic extension $Z_P(s)$ in $\boldsymbol{C}$ whose singularities $\{a_0, a_1, a_2, \cdots\}$ are all simple poles distributed on the real line with the residues $\{A_0, A_1, A_2, \cdots\}$ respectively and satisfy: $A_0>0$, $a_0>0$ and*

$$(1.3) \qquad a_0>a_1>a_2>\cdots .$$

(ii) *There exists an integer $n\geq 1$ such that the tempered distribution on $\boldsymbol{R}$:*

$$(1.4) \qquad \operatorname{Tr}[e^{-itP^{a_0/n}}]=\sum_{j=1}^{\infty} e^{-it\lambda_j^{a_0/n}}$$

has $t=0$ as an isolated singularity.

In the particular case where we can choose the above n as $n=1$, we moreover suppose that $Z_P(s)$ is holomorphic at $s=0$.

(iii) *$Z_P(s)$ is at most of polynomial order uniformly with respect to $\operatorname{Im} s$ in all vertical strips, excluding neighborhoods of poles, i.e., we can find a constant $\tilde{N}>0$ such that for any $d_1<d_2$ there exists a constant $C=C_{d_1,d_2}>0$ such that*

$$(1.5) \qquad |Z_P(s)|\leq C(1+|\operatorname{Im} s|)^{\tilde{N}}$$

for all $s\in \Pi_{d_1,d_2}=\{s\in \boldsymbol{C}; d_1\leq \operatorname{Re} s\leq d_2, |\operatorname{Im} s|\geq 1\}$.

Then we have the asymptotic formula of $N_P(\lambda)$:

$$(1.6) \qquad N_P(\lambda)=\sum_{j=0}^{p} \frac{A_j}{a_j}\lambda^{a_j}+O(\lambda^{(n-1)a_0/n}) \qquad as \quad \lambda\to\infty$$

where $p=\max\{j\,;\, a_j>(n-1)a_0/n\}$.

Before the proof, we have some remarks. If we replace P with $Q=P^{a_0/n}$, we see that the eigenvalues μ_j of Q satisfy $\mu_j=(\lambda_j)^{a_0/n}$, $Z_Q(s)=Z_P(a_0s/n)$ and $N_Q(\lambda)=N_P(\lambda^{n/a_0})$. Since $Z_Q(s)$ has the simple poles $b_j=a_jn/a_0$ with residues $B_j=nA_j/a_0$, thus the proof is reduced to the case $a_0=n$. Therefore, from now we concentrate on the case $a_0=n$.

If we note that $Z_P(s)$ has the first singularity at $s=n$ which is a simple pole with residue A_0, we obtain from Proposition 0.1 that

$$(1.7) \qquad N_P(\lambda)=\frac{A_0}{n}\lambda^n(1+o(1)) \qquad \text{as} \quad \lambda\to\infty .$$

Therefore, since it follows from (1.7) that $\lambda_j\sim c^{-1/n}j^{1/n}$ where $c=A_0/a_0$ (cf. [15]), we can define a tempered distribution $\operatorname{Tr}[e^{-itP}]$ on $\boldsymbol{R}$.

§2. Proof of Theorem 1.1.

In this section, we shall prove Theorem 1.1. The proof consists of three parts (A), (B) and (C). Let P be the operator satisfying the hypotheses of Theorem 1.1 with $a_0 = n$.

(A) If we put

$$\theta_P(z) = \mathrm{Tr}[e^{-zP}] = \sum_{j=1}^{\infty} e^{-z\lambda_j}, \tag{2.1}$$

it follows that $\theta_P(z)$ exists and holomorphic in $z \in \Pi_0 = \{z \in \boldsymbol{C};\ \mathrm{Re}\, z > 0\}$.

In fact, by (1.7), we see that $\lambda_j \sim c^{-1/n} j^{1/n}$. Therefore, for any $a > 0$, if $\mathrm{Re}\, z \geq a$, there exists a constant $C_a > 0$ such that

$$\sum_{j=1}^{\infty} |e^{-z\lambda_j}| \leq C_a \sum_{j=1}^{\infty} 1/j^2 < \infty .$$

Thus $\sum_{j=1}^{\infty} |e^{-z\lambda_j}|$ is absolutely and uniformly convergent in the wider sense in Π_0.

On the other hand, by the inverse Mellin transformation, we can rewrite $\theta_P(z)$ in the form:

$$\theta_P(z) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s = d} z^{-s} Z_P(s) \Gamma(s)\, ds \tag{2.2}$$

where $d > 0$ is large enough. Here $\Gamma(s)$ is the Gamma function:

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt , \qquad \mathrm{Re}\, s > 0$$

and for $\mathrm{Re}\, s \leq 0$, $\Gamma(s)$ is defined by the analytic continuation:

$$\Gamma(s) = \frac{\Gamma(s+k)}{(s+k-1)(s+k-2)\cdots s} \qquad \text{if} \quad \mathrm{Re}\, s > -k,\ k > 0 .$$

Since $t = 0$ is an isolated singularity of $\mathrm{Tr}[e^{-itP}]$ by the hypothesis of Theorem, we can choose $T_0 > 0$ such that

$$[-T_0, T_0] \cap \mathrm{sing\,supp}\, \mathrm{Tr}[e^{-itP}] = \{0\} . \tag{2.3}$$

In this part (A), we shall search for the asymptotic behavior of $\theta_P(z)$ as $z \downarrow 0$, i.e., $z \to 0$ with $\mathrm{Re}\, z > 0$ ($|z| < T_0$). In order to do so, we must list up all the poles of $Z_P(s)\Gamma(s)$ in the integral (2.2). Since $Z_P(s)$ and $\Gamma(s)$ have the sets of the singularities $\{a_0, a_1, \cdots\}$ and $\boldsymbol{Z}_- = \{0, -1, -2, \cdots\}$, respectively which are all simple poles and $a_j > 0$ for $0 \leq j \leq p$, we denote the rearrangement of $\{a_{p+1}, a_{p+2}, \cdots\} \cup \boldsymbol{Z}_-$ by $\{a'_{p+1}, a'_{p+2}, \cdots\}$ so that $a'_{p+1} > a'_{p+2} > \cdots$. Here we note that if $a'_{p+k} = a_{p+k'} = -l \in \boldsymbol{Z}_-$ for some $k' \geq 1$, $Z_P(s)\Gamma(s)$ has a double pole at $s = -l$ and if $a'_{p+k} \notin \boldsymbol{Z}_-$ for $k \geq 1$, $Z_P(s)\Gamma(s)$ has a simple pole at $s = a'_{p+k}$. Now we shall prove:

PROPOSITION 2.1 (cf. Helffer and Robert [9; Theorem 6.6]). *Under the reduced hypotheses of Theorem* 1.1, *for every large integer N, we can write*

(2.4) $$\theta_P(z)=\sum_{j=0}^{p}A_j\Gamma(a_j)z^{-a_j}+\sum_{j=p+1}^{N-1}\sum_{k=0}^{l_j}A'_{j,k}z^{-a'_j}(\log z)^k+\theta_P^{b_N}(z),$$

for $\mathrm{Re}\,z\geq 0$ *and* $0<|z|<T_0$.

Here $A'_{j,k}$ $(j\geq p+1)$ *are some constants,* $l_j=0$ *if* a'_j *is a simple pole of* $Z_P(s)\Gamma(s)$ *and* $l_j=1$ *if* $a'_j=-l$ *is a double pole of* $Z_P(s)\Gamma(s)$, $a'_N<b_N<a'_{N-1}$ *and we have*

$$\theta_P^{b_N}(z)=\frac{1}{2\pi i}\int_{\mathrm{Re}\,s=b_N}z^{-s}Z_P(s)\Gamma(s)ds=O(z^{-b_N}),$$

as $z\downarrow 0$, *i.e.,* $\mathrm{Re}\,z>0$, $z\to 0$.

REMARK. If P is the unique self-adjoint extension in $L^2(\boldsymbol{R}^n)$ of the harmonic oscillator: $P_0=-\Delta+|x|^2$ starting from $C_0^\infty(\boldsymbol{R}^n)$, the Weyl symbol of e^{-zP} is equal to

$$c(z\,;\,x,p)=(\cosh z)^{-n}\exp\{-(\tanh z)(|x|^2+|p|^2)\}$$

for $\mathrm{Re}\,z\geq 0$ and $z\neq i(2k+1)\pi/2$, $(k\in\boldsymbol{Z})$. Therefore we have $\theta_P(z)=(2\sinh z)^{-n}$ for $\mathrm{Re}\,z\geq 0$ and $z\neq ik\pi$, $(k\in\boldsymbol{Z})$. Thus by a simple calculation, we have, for $0<|z|<\pi$,

$$\theta_P(z)=(2\sinh z)^{-n}=\sum_{j=0}^{\infty}B_jz^{-n+2j}$$

where $B_0=2^{-n}$, $B_1=-n/(3\cdot 2^{(n+1)})$, $B_2=n(5n+2)/(2^{(n+3)}\cdot 3^2\cdot 5)$, etc. In this case, since P is a differential operator, the logarithmic terms disappear (cf. Robert [13]).

For the proof of Proposition 2.1, we firstly need the following Lemma.

LEMMA 2.2. *Let* $\mathrm{Re}\,z>0$. *If we put* $s=\tau+i\sigma$ $(\tau,\sigma\in\boldsymbol{R})$, *then for any* $d_1<d_2$, *there exists a constant* $C=C_{z,d_1,d_2}>0$ *such that*

$$|z^{-s}Z_P(s)\Gamma(s)|\leq Ce^{|\sigma|(|\arg z|-|\arg s|)}(1+|\sigma|)^{d_2+\tilde{N}-1/2}$$

for all $s\in\Pi_{d_1,d_2}$ *where* $\tilde{N}$ *is as in* (1.5).

PROOF. By the Stirling formula, there exists a constant C_1 such that

(2.5) $$|\Gamma(s)|\leq C_1e^{-\tau}|s|^{\tau-1/2}e^{-|\sigma||\arg s|}.$$

Here for any $\delta>0$, the estimate (2.5) is uniformly in $D_\delta=\{s\in\boldsymbol{C}\,;\,|\arg s|\leq\pi-\delta\}$. On the other hand, by the hypothesis (iii) of Theorem 1.1, there exists a constant $C_2>0$ such that

$$|Z_P(s)|\leq C_2(1+|\sigma|)^{\tilde{N}}$$

for $s\in\Pi_{d_1,d_2}$. Therefore there exists a constant $C_3>0$ such that

$$|z^{-s}Z_P(s)\Gamma(s)|\leq C_3|z|^{-\tau}(1+|\sigma|)^{\tau+\tilde{N}-1/2}e^{\sigma\arg z-|\sigma||\arg s|}$$

for all $s\in\Pi_{d_1,d_2}$. This completes the proof.

PROOF OF PROPOSITION 2.1. For fixed z with $\mathrm{Re}\,z>0$, there exist $M>0$ and $\varepsilon_0>0$ such that $|\arg z|-|\arg s|\leq -\varepsilon_0$ for all $s=\tau+i\sigma\in\Pi_{d_1,d_2}$ and $|\sigma|\geq M$. Then it follows from Lemma 2.2 that

$$\int_{d_1}^{d_2}|z^{-(\tau\pm iM)}Z_P(\tau\pm iM)\Gamma(\tau\pm iM)|\,d\tau$$
$$\leq C(\mathrm{Re}\,z)^{-d_1}e^{-\varepsilon_0 M}(1+M)^{d_2+\tilde{N}-1/2}(d_1-d_2)\to 0 \qquad \text{as} \quad M\to\infty\,.$$

If we choose d_1 and d_2 so that $a_{j+1}<d_1<a_j<d_2<a_{j-1}$ $(1\leq j\leq p)$, $Z_P(s)$ has only one simple pole at $s=a_j$ with residue A_j in $\{s\in\boldsymbol{C};\ a_{j+1}<\mathrm{Re}\,s<a_{j-1}\}$. Therefore by the residue theorem and the above argument, we have

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\,s=d_2}z^{-s}Z_P(s)\Gamma(s)ds=\frac{1}{2\pi i}\int_{\mathrm{Re}\,s=d_1}z^{-s}Z_P(s)\Gamma(s)ds+A_j\Gamma(a_j)z^{-a_j}\,.$$

Similarly, if $a'_{j+1}<d_1<a'_j<d_2<a'_{j-1}$ and a'_j is a double pole of $Z_P(s)\Gamma(s)$, we see that

$$Z_P(s)\Gamma(s)-\frac{A'_{j,0}}{s-a'_j}-\frac{A'_{j,1}}{(s-a'_j)^2}$$

is holomorphic for $d_1\leq\mathrm{Re}\,s\leq d_2$. Then we have

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\,s=d_2}z^{-s}Z_P(s)\Gamma(s)ds$$
$$=\frac{1}{2\pi i}\int_{\mathrm{Re}\,s=d_1}z^{-s}Z_P(s)\Gamma(s)ds+A'_{j,0}z^{-a'_j}-A'_{j,1}z^{-a'_j}\log z\,.$$

If a'_j is a simple pole of $Z_P(s)\Gamma(s)$, it is easily seen that we may put $A'_{j,1}=0$ in the last equality. In order to complete the proof, it suffices to show that the following Lemma on the remainder term $\theta_P^{b_N}(z)$ holds.

LEMMA 2.3. *Let $b<-\tilde{N}-1/2$ and $b\notin\{a'_{p+1},a'_{p+2},\cdots\}$. Then for every non-negative integer l such that $l<-b-\tilde{N}-1/2$, there exists a constant $C_l>0$ such that*

$$\left|z^{b+l}\left(\frac{d}{dz}\right)^l\theta_P^b(z)\right|\leq C_l \tag{2.6}$$

for all z with $\mathrm{Re}\,z>0$ *and* $|z|\leq T_0$.

PROOF. Put $s=b+i\sigma$. First of all, we must consider the integral

$$I_l(z)=z^{b+l}\left(\frac{d}{dz}\right)^l\frac{1}{2\pi i}\int_{\mathrm{Re}\,s=b}z^{-s}Z_P(s)\Gamma(s)\,ds$$

$$=z^{b+l}\frac{(-1)^l}{2\pi i}\int_{\mathrm{Re}\,s=b} s(s+1)\cdots(s+l-1)z^{-s-l}Z_P(s)\Gamma(s)ds$$

$$=\frac{(-1)^l}{2\pi}\int_{-\infty}^{\infty}\prod_{j=0}^{l-1}(b+i\sigma-j)z^{-i\sigma}Z_P(b+i\sigma)\Gamma(b+i\sigma)d\sigma\,.$$

Then using the Stirling formula (2.5) again, it follows that the absolute value of the integrand in the last integral is estimated by

$$C(1+|\sigma|)^{\tilde{N}+l+b-1/2}e^{|\sigma|(|\arg z|-|\arg(b+i\sigma)|)}$$

for some constant C. Since $|\arg z|<\pi/2\le|\arg(b+i\sigma)|$ and $\tilde{N}+l+b-1/2<-1$, the integrand is estimated by an integrable function $C(1+|\sigma|)^{\tilde{N}+l+b-1/2}$ which is independent of z. Thus we see that $|I_l(z)|\le C_l$ for some constant C_l. This completes the proof.

(B) In this part, we shall examine the asymptotic behavior of

$$I(\mu)=\int_{-\infty}^{\infty}\rho(\mu-\tau)dN_P(\tau) \tag{2.7}$$

where the function $\rho\in\mathscr{S}(\boldsymbol{R})$ and satisfies

$$\begin{cases}\rho\ge 0\,, & \rho(0)>0\,,\\ \hat{\rho}\in C_0^\infty(\boldsymbol{R})\,, & \hat{\rho}(0)=1\,,\quad \hat{\rho}\text{ is an even function and}\\ \operatorname{supp}\hat{\rho}\subset(-T_0,T_0)\,. & \end{cases}$$

For the existence of such a function, see Helffer [8]. Since by (1.7), $N_P(\tau)$ is of at most polynomial order as $\tau\to\infty$, it follows from the Lebesgue theorem that

$$\begin{aligned}I(\mu)&=\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}e^{-\varepsilon\tau}\rho(\mu-\tau)dN_P(\tau)\\ &=\lim_{\varepsilon\downarrow 0}\sum_{j=1}^{\infty}e^{-\varepsilon\lambda_j}\rho(\mu-\lambda_j)\\ &=\lim_{\varepsilon\downarrow 0}(2\pi)^{-1}\int_{-\infty}^{\infty}\theta_P(\varepsilon+it)\hat{\rho}(t)e^{i\mu t}dt\\ &=\sum_{j=0}^{p}A_jI_j(\mu)+\sum_{j=p+1}^{N-1}\sum_{k=0}^{l_j}A'_{j,k}I_{j,k}(\mu)+R_{b_N}(\mu)\end{aligned}$$

where

$$I_j(\mu)=\lim_{\varepsilon\downarrow 0}(2\pi)^{-1}\int_{-\infty}^{\infty}(\varepsilon+it)^{-a_j}\Gamma(a_j)\hat{\rho}(t)e^{i\mu t}dt\,, \tag{2.8}$$

$$(2.9) \qquad I_{j,k}(\mu)=\lim_{\varepsilon\downarrow 0}(2\pi)^{-1}\int_{-\infty}^{\infty}(\varepsilon+it)^{-a'_j}(\log(\varepsilon+it))^k\hat{\rho}(t)e^{i\mu t}\,dt\,,$$

$$R_{b_N}(\mu)=\lim_{\varepsilon\downarrow 0}(2\pi)^{-1}\int_{-\infty}^{\infty}\theta_P^{b_N}(\varepsilon+it)\hat{\rho}(t)e^{i\mu t}\,dt\,.$$

Now we want to get the asymptotic behavior of $I(\mu)$ as $\mu\to\infty$ modulo $O(\mu^{n-2})$ if $n\geq 2$ and $O(\mu^{-1-\delta})$ for some $\delta>0$ if $n=1$. For this purpose, we prove the following:

PROPOSITION 2.4. *Under the preceding situations, we have*

$$(2.10) \qquad I(\mu)=\sum_{j=0}^{p}A_jI_j(\mu)+R(\mu)=\sum_{j=0}^{p}A_j\mu^{a_j-1}+R(\mu)$$

where

$$R(\mu)=\begin{cases}O(\mu^{n-2}) & \text{if } n\geq 2\,,\\ O(\mu^{-1-\delta}) & \text{for some } \delta>0 \text{ if } n=1\,.\end{cases}$$

In order to prove this proposition, we need the following three lemmas.

LEMMA 2.5 (cf. Aramaki [1]). *Let* $0<\sigma<1$ *and* ρ *be as above. Then we have*

$$\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}(\varepsilon+it)^{-\sigma}e^{i\mu t}\,dt=\int_{-\infty}^{\infty}(it)^{-\sigma}e^{i\mu t}\,dt\,.$$

PROOF. The mean value theorem leads to

$$(\varepsilon+it)^{-\sigma}=(it)^{-\sigma}-s\varepsilon\int_0^1(\varepsilon\theta+it)^{-\sigma-1}\,d\theta\,.$$

Here we have, with a constant C independent of ε,

$$\varepsilon\int_{|t|\geq 1}\left|\int_0^1(\varepsilon\theta+it)^{-\sigma-1}\,d\theta\right|dt\leq C\varepsilon\int_{|t|\geq 1}|t|^{-\sigma-1}\,dt=\frac{2C\varepsilon}{\sigma}\to 0 \qquad \text{as} \quad \varepsilon\to 0\,.$$

For any δ with $0<\delta<1$, we have

$$\varepsilon\int_{|t|\leq 1}\left|\int_0^1(\varepsilon\theta+it)^{-\sigma-1}\,d\theta\right|dt\leq\varepsilon C\int_{|t|\leq 1}\int_0^1(\varepsilon\theta)^{\delta-1}\,d\theta\,|t|^{-\sigma-\delta}\,dt\,.$$

Thus if we choose $\delta>0$ such that $0<\sigma+\delta<1$, the last integral is equal to

$$\frac{\varepsilon^\delta C}{\delta}\int_{|t|\leq 1}|t|^{-\sigma-\delta}\,dt=\frac{2\varepsilon^\delta C}{\delta(-\sigma-\delta+1)}\to 0 \qquad \text{as} \quad \varepsilon\to 0\,.$$

This completes the proof.

LEMMA 2.6 (cf. [1]). *Let* $0<\operatorname{Re}s<1$. *Then we have*

$$J(s)\overset{\text{def}}{=}(2\pi)^{-1}\Gamma(s)\int_{-\infty}^{\infty}(it)^{-s}e^{i\mu t}dt=\mu^{s-1}. \tag{2.11}$$

PROOF. First of all, it is well known that for $0<\operatorname{Re}s<1$, $\operatorname{Re}z\geq 0$, $z\neq 0$,

$$\Gamma(1-s)=z^{1-s}\int_0^{\infty}e^{-zt}t^{-s}dt.$$

Putting $z=\pm i\mu$, $\mu>0$ in this formula, we have

$$\int_0^{\infty}t^{-s}e^{\mp i\mu t}dt=(\pm i\mu)^{s-1}\Gamma(1-s).$$

Now, if we decompose $J(s)=J^+(s)+J^-(s)$ where

$$J^+(s)=(2\pi)^{-1}\Gamma(s)\int_0^{\infty}(it)^{-s}e^{i\mu t}dt,$$

$$J^-(s)=(2\pi)^{-1}\Gamma(s)\int_{-\infty}^{0}(it)^{-s}e^{i\mu t}dt$$

$$=(2\pi)^{-1}\Gamma(s)\int_0^{\infty}(-it)^{-s}e^{i\mu t}dt,$$

we have

$$\begin{aligned}J(s)&=J^+(s)+J^-(s)\\&=(2\pi)^{-1}\{i^{-s}(-i\mu)^{s-1}+(-i)^{-s}(i\mu)^{s-1}\}\Gamma(s)\Gamma(1-s)\\&=(2\pi)^{-1}2\sin(\pi s)\Gamma(s)\Gamma(1-s)\mu^{s-1}=\mu^{s-1}.\end{aligned}$$

This completes the proof.

LEMMA 2.7. *Let* $0<\sigma\leq 1$ *and* $j\geq 0$ *integer. Then for every* $\phi\in C_0^{\infty}(\boldsymbol{R})$, *we have*

$$\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}(\varepsilon+it)^{-\sigma\pm j}\phi(t)e^{i\mu t}dt=O(\mu^{\sigma\mp j-1}) \tag{2.12}$$

as $\mu\to\infty$.

PROOF. Since $\phi\in C_0^{\infty}(\boldsymbol{R})$, we may assume that $\operatorname{supp}\phi\subset(-a,a)$ for some $a>0$. At first, we shall prove (2.12) in the case $j=0$ and $0<\sigma<1$. For this case, we have

$$\int_{-\infty}^{\infty}(\varepsilon+it)^{-\sigma}\phi(t)e^{i\mu t}dt=K_1(\mu;\varepsilon)+K_2(\mu;\varepsilon)+K_3(\mu;\varepsilon)$$

where

(2.13) $$K_1(\mu\,;\,\varepsilon)=\phi(0)\int_{-\infty}^{\infty}(\varepsilon+it)^{-\sigma}e^{i\mu t}dt\,,$$

(2.14) $$K_2(\mu\,;\,\varepsilon)=\int_{-a}^{a}(\varepsilon+it)^{-\sigma}(\phi(t)-\phi(0))e^{i\mu t}dt\,,$$

(2.15) $$K_3(\mu\,;\,\varepsilon)=\phi(0)\int_{|t|\geq a}(\varepsilon+it)^{-\sigma}e^{i\mu t}dt\,.$$

By Lemma 2.5, it follows that $\lim_{\varepsilon\downarrow 0}K_1(\mu\,;\,\varepsilon)=O(\mu^{\sigma-1})$ as $\mu\to\infty$. In the integral $K_3(\mu\,;\,\varepsilon)$, the integration by parts leads to

$$\begin{aligned}K_3(\mu\,;\,\varepsilon)=&\frac{\phi(0)}{i\mu}\{(\varepsilon-ia)^{-\sigma}e^{-i\mu a}-(\varepsilon+ia)^{-\sigma}e^{i\mu a}\}\\&+\frac{\sigma}{\mu}\phi(0)\int_{|t|\geq a}(\varepsilon+it)^{-\sigma-1}e^{i\mu t}dt\,.\end{aligned}$$

Since $|(\varepsilon+it)^{-\sigma-1}|\leq C|t|^{-\sigma-1}$ and $0<\sigma<1$, the integrand of (2.15) is estimated by an integrable function independent of ε for $|t|\geq a$. Thus it follows from the Lebesgue theorem that $\lim_{\varepsilon\downarrow 0}K_3(\mu\,;\,\varepsilon)=O(\mu^{-1})$ as $\mu\to\infty$. Also in the integral $K_2(\mu\,;\,\varepsilon)$, the integration by parts leads to

$$\begin{aligned}K_2(\mu\,;\,\varepsilon)=&\frac{1}{i\mu}[(\varepsilon+it)^{-\sigma}(\phi(t)-\phi(0))e^{-i\mu t}]_{-a}^{a}\\&+\frac{\sigma}{i\mu}\int_{-a}^{a}(\varepsilon+it)^{-\sigma-1}(\phi(t)-\phi(0))e^{i\mu t}dt\\&-\frac{1}{i\mu}\int_{-a}^{a}(\varepsilon+it)^{-\sigma}\phi'(t)e^{i\mu t}dt\,.\end{aligned}$$

The first and the third terms are of order $O(\mu^{-1})$ uniformly in ε. In the second term, since $\phi(t)-\phi(0)=t\phi'(\theta t)$ for some $\theta\in(0,1)$, we have

$$|(\varepsilon+it)^{-\sigma-1}(\phi(t)-\phi(0))|\leq C|t|^{-\sigma}$$

where C is a constant independent of ε. Therefore the second term is integrable on $[-a,a]$ and estimated by

$$C\mu^{-1}\int_{-a}^{a}|t|^{-\sigma}dt=2C\mu^{-1}a^{1-\sigma}/(1-\sigma)\,.$$

Thus using the Lebesgue theorem again, we see that $\lim_{\varepsilon\downarrow 0}K_2(\mu\,;\,\varepsilon)=O(\mu^{-1})$ as $\mu\to\infty$.

Next we consider the case $j=0$ and $\sigma=1$:

$$\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} (\varepsilon + it)^{-1} \phi(t) e^{i\mu t} dt .$$

Define a function:

$$a_\varepsilon(\mu) = \begin{cases} e^{-\varepsilon\mu} & \text{if } \mu > 0 , \\ 0 & \text{if } \mu < 0 . \end{cases}$$

Since the convolution of a_ε and $\psi \in \mathscr{S}(\boldsymbol{R})$ becomes

$$(a_\varepsilon * \psi)(\mu) = \int_{-\infty}^{\infty} a_\varepsilon(\mu - \tau)\psi(\tau)\, d\tau = \int_{-\infty}^{\mu} e^{-\varepsilon(\mu-\tau)}\psi(\tau)\, d\tau$$

for any $\varepsilon > 0$ and

$$\hat{a}_\varepsilon(t) = \int_0^{\infty} e^{-(\varepsilon + it)\mu} d\mu = \frac{1}{\varepsilon + it} ,$$

we have

$$(2\pi)^{-1} \int_{-\infty}^{\infty} (\varepsilon + it)^{-1} \hat{\psi}(t) e^{i\mu t} dt = [\hat{a}_\varepsilon \cdot \hat{\psi}]^{\check{}}(\mu) = (a_\varepsilon * \psi)(\mu)$$

where $\check{}$ denotes the inverse Fourier transformation. Thus putting $\psi = \tilde{\phi}$, we have

$$\begin{aligned} \text{(2.16)} \qquad & \lim_{\varepsilon \downarrow 0} (2\pi)^{-1} \int_{-\infty}^{\infty} (\varepsilon + it)^{-1} \phi(t) e^{i\mu t} dt \\ & = \int_{-\infty}^{\mu} \tilde{\phi}(\tau) d\tau = \phi(0) - \int_{-\mu}^{\infty} \tilde{\phi}(\tau) d\tau = \phi(0) + O(\mu^{-N}) \end{aligned}$$

for any integer $N > 0$. Thus (2.12) holds for the case $j = 0$.

Let $j \geq 1$. Repeating the integration by parts leads to

$$\begin{aligned} I_\varepsilon(\sigma, j, \mu, \phi) & \overset{\text{def}}{=} \int_{-\infty}^{\infty} (\varepsilon + it)^{-\sigma - j} \phi(t) e^{i\mu t} dt \\ & = \frac{\mu}{\sigma + j - 1} I_\varepsilon(\sigma, j-1, \mu, \phi) + \frac{1}{\sigma + j - 1} I_\varepsilon(\sigma, j-1, \mu, \phi') \\ & \cdots\cdots \\ & = \sum_{l=0}^{j} \frac{\mu^{j-1}}{(\sigma + j - 1) \cdots (\sigma + 1)} I_\varepsilon(\sigma, 0, \mu, \phi^{(l)}) . \end{aligned}$$

Therefore it follows that $I_\varepsilon(\sigma, j, \mu, \phi)$ is a linear combination of

$$\frac{\mu^{j-1}}{(\sigma + j - 1) \cdots (\sigma + 1)} I_\varepsilon(\sigma, 0, \mu, \phi^{(l)}) , \qquad (l = 0, 1, \cdots, j) .$$

By the result of the case $j=0$, we see that

$$\lim_{\varepsilon\downarrow 0} I_\varepsilon(\sigma, j, \mu, \phi)=O(\mu^{\sigma+j-1}) \qquad \text{as} \quad \mu\to\infty .$$

Similarly, for $j\geq 1$, by integration by parts, we have

$$I_\varepsilon(\sigma, -j, \mu, \phi)=\mu^{-1}(\sigma-j-1)I_\varepsilon(\sigma, -(j-1), \mu, \phi)+i\mu^{-1}I_\varepsilon(\sigma, -j, \mu, \phi') .$$

Thus by the similar arguments as above we see that (2.12) holds. This completes the proof.

REMARK 2.8. If we choose $\phi=\hat{\rho}$ where ρ is as in (2.7), we see that

$$\lim_{\varepsilon\downarrow 0}(2\pi)^{-1}\int_{-\infty}^{\infty}(\varepsilon+it)^{-1}\hat{\rho}(t)e^{i\mu t}dt=1+O(\mu^{-N}) \tag{2.17}$$

for any integer $N>0$. In fact, it suffices to note that $\hat{\rho}(0)=1$ in (2.16).

PROOF OF PROPOSITION 2.4. For brevity of notations, we put $a_j=n-b_j$ $(j=0, 1, \cdots, p)$. Then $b_0=0$ and $0<b_j<1$ if $1\leq j\leq p$. Repeating the integration by parts and applying the Leibniz formula, we have

$$\begin{aligned} I_j(\mu\,;\varepsilon)&=\int_{-\infty}^{\infty}(\varepsilon+it)^{-n+b_j}\hat{\rho}(t)e^{i\mu t}dt \\ &=\frac{(-i)^{n-1}}{(n-1-b_j)\cdots(1-b_j)}\int_{-\infty}^{\infty}(\varepsilon+it)^{-1+b_j}\left(\frac{d}{dt}\right)^{n-1}(\hat{\rho}(t)e^{i\mu t})dt \\ &=I_j^{(1)}(\mu\,;\varepsilon)+I_j^{(2)}(\mu\,;\varepsilon) \end{aligned}$$

where

$$I_j^{(1)}(\mu\,;\varepsilon)=\frac{\mu^{n-1}}{(n-1-b_j)\cdots(1-b_j)}\int_{-\infty}^{\infty}(\varepsilon+it)^{-1+b_j}\hat{\rho}(t)e^{i\mu t}dt ,$$

$$I_j^{(2)}(\mu\,;\varepsilon)=\sum_{l=1}^{n-1}\binom{n-1}{l}\frac{i^{-l}\mu^{n-1-l}}{(n-1-b_j)\cdots(1-b_j)}\int_{-\infty}^{\infty}(\varepsilon+it)^{-1+b_j}\hat{\rho}^{(l)}(t)e^{i\mu t}dt .$$

Applying Lemma 2.7, it follows that $I_j^{(2)}(\mu\,;\varepsilon)=O(\mu^{n-2-b_j})$ uniformly in ε. Noting that $I_j^{(2)}(\mu\,;\varepsilon)=0$ for $n=1$, we see that

$$I_j^{(2)}(\mu\,;\varepsilon)=\begin{cases} O(\mu^{n-2}) & \text{if} \quad n\geq 2 , \\ O(\mu^{-1-\delta}) \quad \text{for some } \delta>0 & \text{if} \quad n=1 \end{cases}$$

as $\mu\to\infty$ uniformly in $\varepsilon\in(0, 1]$. Applying Lemma 2.6, Lemma 2.7 and Remark 2.8 for $0\leq j\leq p$, we see

$$\lim_{\varepsilon\downarrow 0}(2\pi)^{-1}I_j^{(1)}(\mu\,;\varepsilon)=\frac{\mu^{n-1-b_j}}{(n-1-b_j)\cdots(1-b_j)\Gamma(1-b_j)}=\frac{\mu^{a_j-1}}{\Gamma(a_j)} .$$

Thus we have

$$I_j(\mu)=\lim_{\varepsilon\downarrow 0}(2\pi)^{-1}\Gamma(a_j)I_j(\mu\,;\,\varepsilon)=\mu^{a_j-1}+R_j^{(n)}(\mu)$$

where

$$R_j^{(n)}(\mu)=\begin{cases}O(\mu^{n-2}), & \text{if } n\geq 2,\\ O(\mu^{-1-\delta}) \quad \text{for some } \delta>0 & \text{if } n=1\end{cases}$$

as $\mu\to\infty$.

Next we shall show that $I_{j,k}(\mu)$ $(p+1\leq j\leq N-1, 0\leq k\leq 1)$ are negligible, i.e., $I_{j,k}(\mu)$ satisfies the same estimate as $R_j^{(n)}(\mu)$ above.

When $k=0$, if we replace n in the above arguments with $n-1$, we easily see that $I_{j,0}(\mu)$ are negligible. When $k=1$ and $a_j'\neq 1$, the integration by parts leads to

$$\begin{aligned}I_{j,1}(\mu)=&\lim_{\varepsilon\downarrow 0}\frac{i}{1-a_j'}\int_{-\infty}^{\infty}(\varepsilon+it)^{-a_j'}\hat{\rho}(t)e^{i\mu t}dt\\ &+\lim_{\varepsilon\downarrow 0}\frac{i}{1-a_j'}\int_{-\infty}^{\infty}(\varepsilon+it)^{-a_j'+1}\log(\varepsilon+it)\hat{\rho}'(t)e^{i\mu t}dt\\ &+\lim_{\varepsilon\downarrow 0}\frac{-\mu}{1-a_j'}\int_{-\infty}^{\infty}(\varepsilon+it)^{-a_j'+1}\log(\varepsilon+it)\hat{\rho}(t)e^{i\mu t}dt\,.\end{aligned}$$

Since for any $\delta_0>0$ there exists a constant $C>0$ such that

$$|(\varepsilon+it)^{\delta_0}\log(\varepsilon+it)|\leq C$$

in supp $\hat{\rho}(t)$, we see from Lemma 2.7 that $I_{j,1}(\mu)=O(\mu^{a_j'-1+\delta_0})$. If $n\geq 2$, it follows from the hypothesis of Theorem 1.1 that $n-1\geq a_j'$. However if $n-1=a_j'$, a_j' is a simple pole, so $k=0$. If $n-1>a_j'$, we choose δ_0 so that $a_j'-1+\delta_0\leq n-2$. Thus we see that $I_{j,1}(\mu)=O(\mu^{n-2})$. When $n=1$, if $a_j'=0$, a_j' is a simple pole of $Z_P(s)\Gamma(s)$ by the hypothesis of Theorem 1.1. Therefore we reduce to the case $k=0$. If $a_j'<0$, we choose δ_0 so that $a_j'-1+\delta_0<-1$. Thus we see that $I_{j,1}(\mu)=O(\mu^{-1-\delta})$ for some $\delta>0$.

Finally we shall estimate the remainder term $R_{b_N}(\mu)$.

LEMMA 2.9. *Let $-b_N>\tilde{N}+5/2$. Then we have*

$$\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\theta_P^{b_N}(\varepsilon+it)\hat{\rho}(t)e^{i\mu t}dt=O(\mu^{-2}) \qquad \textit{as} \quad \mu\to\infty\,. \tag{2.18}$$

PROOF. By the integration by parts, we have

$$\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\theta_P^{b_N}(\varepsilon+it)\hat{\rho}(t)e^{i\mu t}dt$$

$$=\frac{-1}{i\mu}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}(\theta_P^{b_N}(\varepsilon+it)\hat{\rho}(t))'e^{i\mu t}dt$$
$$=\frac{-1}{\mu^2}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\left(\frac{d}{dt}\right)^2(\theta_P^{b_N}(\varepsilon+it)\hat{\rho}(t))e^{i\mu t}dt\,.$$

Then it follows from Proposition 2.3 that there exists a constant $C>0$ independent of $\varepsilon\in(0, 1]$ such that

$$\left|\left(\frac{d}{dt}\right)^2(\theta_P^{b_N}(\varepsilon+it)\hat{\rho}(t))\right|\le C\,.$$

This completes the proof.

(C) In this part, we shall examine the asymptotic behavior of $N_P(\lambda)$. Here we only give an outline of the proof (cf. [8], [9] and [1]).

First of all, we note that it follows from (2.10) that

$$I(\mu)=\int_{-\infty}^{\infty}\rho(\mu-\tau)dN_P(\tau)=O(\mu^{n-1})\qquad\text{as}\quad \mu\to\infty\,.$$

LEMMA 2.10 (cf. [8], [1]). *There exists a constant $\gamma>0$ such that for all $K>0$ and λ,*

$$\int_{|\lambda-\mu|\le K}dN_P(\mu)\le\gamma(1+K)^n(1+|\lambda|)^{n-1}\,. \tag{2.19}$$

LEMMA 2.11 (cf. [8], [1]). *For any $\varepsilon>0$, there exists a constant $K>0$ such that for all $\lambda\ge1$,*

$$\int_{|\lambda-\mu|\ge K}dN_P(\mu)\le\varepsilon\lambda^{n-1}\,, \tag{2.20}$$

$$\int_{\mu>\lambda-K}\left\{\int_{-\infty}^{\lambda}\rho(\tau-\mu)d\tau\right\}dN_P(\mu)\le\varepsilon\lambda^{n-1}\,, \tag{2.21}$$

$$\int_{\mu<\lambda-K}\left\{\int_{\lambda}^{\infty}\rho(\tau-\mu)d\tau\right\}dN_P(\mu)\le\varepsilon\lambda^{n-1}\,. \tag{2.22}$$

For the proof of the above lemmas, see [8].

Using the above Lemmas, we have

$$\int_1^{\tau}I(\mu)d\mu=\sum_{j=0}^{p}\frac{A_j}{a_j}\tau^{a_j}+O(\tau^{r_n})\qquad\text{as}\quad\tau\to\infty \tag{2.23}$$

where $r_n=n-1$ if $n\ge2$ and $r_1=-\delta$ for some $\delta>0$. On the other hand, the integral $\int_{-\infty}^{1}I(\mu)d\mu$ is bounded from above. Therefore we have

$$\int_{-\infty}^{\tau}\left\{\int_{-\infty}^{\infty}\rho(\tau-\mu)d\tau\right\}dN_P(\mu)=\sum_{j=0}^{p}\frac{A_j}{a_j}\tau^{a_j}+O(\tau^{r_n}) \qquad \text{as} \quad \tau\to\infty\,.$$

Now we can decompose the left hand side into $A+B+C$ where

$$A=\int_{\mu>\tau+K}\left\{\int_{-\infty}^{\tau}\rho(\lambda-\mu)d\lambda\right\}dN_P(\mu)\,,$$

$$B=\int_{\mu<\tau-K}\left\{\int_{-\infty}^{\tau}\rho(\lambda-\mu)d\lambda\right\}dN_P(\mu)\,,$$

$$C=\int_{|\mu-\tau|\le K}\left\{\int_{-\infty}^{\tau}\rho(\lambda-\mu)d\lambda\right\}dN_P(\mu)\,.$$

Here we can write $B=N_P(\tau)-B_1-B_2$ where

$$B_1=\int_{\mu<\tau-K}\left\{\int_{\tau}^{\infty}\rho(\lambda-\mu)d\lambda\right\}dN_P(\mu)\,,$$

$$B_2=\int_{\tau-K\le\mu\le\tau}dN_P(\mu)\,.$$

Therefore taking Lemma 2.10 and Lemma 2.11 into consideration, we have

$$N_P(\tau)=\sum_{j=0}^{p}\frac{A_j}{a_j}\tau^{a_j}+O(\tau^{n-1}) \qquad \text{as} \quad \tau\to\infty\,. \tag{2.24}$$

This completes the proof of Theorem 1.1.

§3. Examples.

In this section we shall apply the results given in the Theorem 1.1 to some partial differential operators.

(i) Let $P^0=-\Delta_S+I$ on S_n where Δ_S denotes the Laplace-Beltrami operator on the unit sphere S_n in R^{n+1}. It is well known that P^0 has a self-adjoint extension P in $L^2(S_n)$ with compact resolvent and the eigenvalues of P are $\nu_l=l(l+n-1)$ ($l=0, 1, 2, \cdots$) with their multiplicities

$$d_l=\binom{n+l}{n}-\binom{n+l-2}{n}\sim 2l^{n-1}/(n-1)!\,.$$

Since

$$Z_P(s)=\sum_{l=0}^{\infty}d_l\{l(l+n-1)+1\}^{-s}$$

has the first singularity at $s=a_0=n/2$ which is a simple pole with residue $A_0=1/(n-1)!$ and $Z_P(s)-A_0/(s-a_0)$ is holomorphic for $\mathrm{Re}\,s>(n-1)a_0/n=(n-1)/2$. It follows from [7] that $e^{-itP^{1/2}}$ has $t=0$ as the isolated singularity. The other hypotheses of Theorem 1.1 hold (cf. [5]). Applying this theorem, we have

$$N_P(\lambda)=\frac{2}{n!}\lambda^{n/2}+O(\lambda^{(n-1)/2}) \qquad \text{as} \quad \lambda\to\infty .$$

(ii) Let P^0 be the Schrödinger operator:

$$P^0=-\Delta+q(x) \qquad \text{on} \quad \boldsymbol{R}^n \tag{3.1}$$

where Δ denotes the Laplacian:

$$\Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_n^2}$$

and the potential $q(x)$ is of the form:

$$q(x)=(1+|x|^2)^\delta , \qquad \delta>0 \text{ an integer} .$$

Such operators of this type were considered by many authors (cf. Helffer and Robert [10] and Levendorskiĭ [11]). If we regard P^0 as an operator on $L^2(\boldsymbol{R}^n)$ with the domain $C_0^\infty(\boldsymbol{R}^n)$, it follows that P^0 is essentially self-adjoint and has the unique self-adjoint extension P which is positively definite. Moreover, it is well known that P has a compact resolvent. Thus we can write the eigenvalues with repetition according to multiplicity of P by

$$0<\lambda_1\le\lambda_2\le\cdots, \quad \lim_{j\to\infty}\lambda_j=\infty .$$

We denote the counting function of eigenvalues of P by (0.2) and consider the asymptotic behavior of $N_P(\lambda)$ as $\lambda\to\infty$.

According to Hörmander [12], we can regard $W=T^*(\boldsymbol{R}^n)=\boldsymbol{R}^n\oplus\boldsymbol{R}^n$ as the symplectic vector space with the symplectic form σ:

$$\sigma((x,\xi),(y,\eta))=\langle\xi,y\rangle-\langle\eta,x\rangle , \qquad \text{for} \quad (x,\xi),(y,\eta)\in W .$$

For every $(x,\xi)\in W$, we define a σ-temperate metric on W:

$$g_{x,\xi}(y,\eta)=\frac{1}{|\xi|^2+\langle x\rangle^{2\delta}}|\eta|^2+\frac{1}{\langle x\rangle^2}|y|^2$$

where $\langle x\rangle=(1+|x|^2)^{1/2}$. The dual to $g_{x,\xi}$ with respect to σ is also defined by:

$$g^\sigma_{x,\xi}(y,\eta)=\sup_{(z,\zeta)\neq 0}\frac{|\sigma((y,\eta),(z,\zeta))|^2}{g_{x,\xi}(z,\zeta)} .$$

Then we have

$$h(x,\xi)\overset{\text{def}}{=}\left[\sup_{(y,\eta)\neq 0}\frac{g_{x,\xi}(y,\eta)}{g^{\sigma}_{x,\xi}(y,\eta)}\right]^{1/2}=(|\xi|^2+\langle x\rangle^{2\delta})^{-1/2}\langle x\rangle^{-1}\,.$$

Here we note that there exists a constant C such that

$$h(x,\xi)\leq C\min\{\langle x,\xi\rangle^{-1},\langle x\rangle^{-(1+\delta)}\}$$

and the function $p(x,\xi)=|\xi|^2+\langle x\rangle^{2\delta}$ is σ, g-temperate. Then by [11] we see that $p\in S(g,p)$, i.e., for every multi-indices α, β, there exists a constant $C_{\alpha,\beta}$ such that

$$|D^{\alpha}_x D^{\beta}_{\xi} p(x,\xi)|\leq C_{\alpha,\beta}\,p(x,\xi)\langle x\rangle^{-|\alpha|}(|\xi|^2+\langle x\rangle^{2\delta})^{-|\beta|/2} \tag{3.2}$$

for all $(x,\xi)\in W$. In general, the operator with the Weyl symbol $a\in S(g,p)$ is defined by:

$$a^w(x,D)u(x)=(2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}a\left(\frac{x+y}{2},\xi\right)u(y)dyd\xi\,,\qquad u\in\mathscr{S}(\boldsymbol{R}^n)\,.$$

For such theory of pseudodifferential operators, see [11], [12] and also [15]. If we denote the spectral resolution associated to P by $\{E(\lambda);\lambda\in\boldsymbol{R}\}$, we can also define complex powers P^{-s} of P by (0.1). According to [14] and Aramaki [4], we see that P^{-s} are pseudodifferential operators, i.e., $P^{-s}\in S(g,p^{-\mathrm{Re}s})$ with the Weyl symbol:

$$p_{-s}\sim\sum_{j=0}^{\infty}p_{-s,j} \tag{3.3}$$

where

$$p_{-s,j}=\begin{cases}p^{-s} & \text{if } j=0\\ \displaystyle\sum_{k=1}^{2j-1}\frac{s(s+1)\cdots(s+k-1)}{k!}d_{j,k}p^{-s-k} & \text{if } j\geq 1\,.\end{cases}$$

Here we note that $d_{j,k}\in S(g,h^jp^k)$ are independent of s. In particular, $d_{1,1}=0$. Therefore we can rewrite (3.3) in the form:

$$p_{-s}=p^{-s}+r_s$$

where r_s belongs to $S(g,h^2p^{-\mathrm{Re}s})$ and is holomorphic with respect to s. For s with large real part, P^{-s} is of trace class and the trace $\mathrm{Tr}[\mathrm{P}^{-s}]=I(s)+R(s)$ where

$$I(s)=(2\pi)^{-n}\iint p(x,\xi)^{-s}dxd\xi\,,\qquad R(s)=(2\pi)^{-n}\iint r_s(x,\xi)dxd\xi\,.$$

By the change of variables $\xi\mapsto\langle x\rangle^{\delta}\xi$ and then the change of x, ξ into the polar coordinates respectively, we have

$$I(s)=(2\pi)^{-n}|S_{n-1}|^2\int_0^{\infty}(1+r^2)^{-\delta(s-n/2)}r^{n-1}dr\int_0^{\infty}(1+R^2)^{-s}R^{n-1}dR$$

where $|S_{n-1}|=2\pi^{n/2}/\Gamma(n/2)$ denotes the surface area of the unit sphere $S_{n-1}=\{x\in\boldsymbol{R}^n;\ |x|=1\}$. By the well known formula:

$$\int_0^\infty \frac{r^a}{(1+r^2)^{1+b}}dr=\frac{1}{2}\frac{\Gamma((a+1)/2)\Gamma(b+1-(a+1)/2)}{\Gamma(1+b)}$$

for $\mathrm{Re}b>-1$, $\mathrm{Re}a>-1$ and $\mathrm{Re}(b-(a+1)/2)>-1$, we see that

$$I(s)=2^{-n}\frac{\Gamma(\delta(s-n/2)-n/2)\Gamma(s-n/2)}{\Gamma(\delta(s-n/2))\Gamma(s)}$$

is holomorphic for $\mathrm{Re}s>a_0=n(1+\delta)/(2\delta)$ and has a meromorphic extension in $\boldsymbol{C}$ which is also denoted by $I(s)$. By the theory of the Gamma functions, it is not difficult to see that the singularities of $I(s)$ are all simple poles: $s=a_j=((1+\delta)n-2j)/(2\delta)$ $(j=0, 1, \cdots)$. In particular, the residues A_j at $s=a_j$ for $j=0, 1, \cdots, [(1+\delta)/2]$ where $[(1+\delta)/2]=\max\{j\in\boldsymbol{Z}: j<(1+\delta)/2\}$ satisfy:

$$A_j=\frac{(-1)^j\Gamma((n-2j+2\delta)/2\delta))}{j!2^n\Gamma((n-2j+2)/2)\Gamma(((\delta+1)n-2j)/(2\delta))}.$$

For the case: $n=1$, it is clear that if δ is an odd integer and $j=(1+\delta)/2$, $Z_P(s)$ is holomorphic at $s=a_j=0$.

Since $r_s\in S(g, h^2p^{\mathrm{Re}s})$, it follows from [4] that by the same change of variables, $R(s)$ is holomorphic for $\mathrm{Re}s>(n-2)(1+\delta)/(2\delta)$ and has a meromorphic extension in $\boldsymbol{C}$. If we put $Q=P^{a_0/n}=P^{(\delta+1)/(2\delta)}$, then the principal symbol of Q is equal to $q(x,\xi)=p(x,\xi)^{(\delta+1)/(2\delta)}$. It follows that

$$\operatorname{sing\,supp}\mathrm{Tr}[e^{-itQ}]\subset\mathscr{L}$$

where $\mathscr{L}$ denotes the set of all periods of periodic Hamilton flows associated to $q_0(x,\xi)=(|\xi|^2+|x|^{2\delta})^{(\delta+1)/(2\delta)}$ with energy 1. Here we note that $q_0(x,\xi)$ is the principal symbol regarding Q as a quasi-elliptic pseudodifferential operator. Therefore we see that $t=0$ is an isolated singularity of the tempered distribution $\mathrm{Tr}[e^{-itQ}]$ (cf. [7]). Thus all the conditions of Theorem 1.1 hold with $a_j=((\delta+1)n-2j)/(2\delta), j=0, 1, \cdots$ (cf. [4] and Dauge and Robert [5]). Therefore applying Theorem 1.1, we have

$$N_P(\lambda)=\sum_{j=0}^{[(\delta+1)/2]}B_j\lambda^{((\delta+1)n-2j)/(2\delta)}+O(\lambda^{(\delta+1)(n-1)/(2\delta)})\qquad\text{as}\quad\lambda\to\infty$$

where

$$B_j=\frac{(-1)^j\delta\Gamma((n-2j+2\delta)/(2\delta))}{((\delta+1)n-2j)j!2^{n-1}\Gamma((n-2j+2)/2)\Gamma(((\delta+1)n-2j)/(2\delta))}.$$

In the particular case: $\delta=1$, i.e., P^0 is the harmonic oscillator:

$$P^0=-\Delta+|x|^2+1,$$

we have

$$N_P(\lambda)=\frac{1}{2^n n!}\lambda^n+O(\lambda^{n-1}) \qquad \text{as} \quad \lambda\to\infty .$$

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