TOKYO J. MATH.
VOL. 19, NO. 2, 1996

On Local Conformal Hermitian-Flatness of Hermitian Manifolds

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(Communicated by Y. Maeda)

Introduction.

Let M be a $2m$-dimensional almost Hermitian manifold with the almost complex structure J and the Hermitian metric g. It is well-known (cf. [6], [10]) that there is a unique linear connection D of M such that $Dg=0$ (*metric connection*) and $DJ=0$ (*almost complex connection*) and that the torsion tensor T is pure in the following sense:

$$T(JX, Y)=T(X, JY) \qquad \text{for any vector fields } X, Y \text{ on } M .$$

This connection is called the *Hermitian connection* (or *canonical connection*) of the almost Hermitian manifold (M, J, g). In the Hermitian case, it was introduced by S.-S. Chern [4].

Balas [1] studied Hermitian manifolds of constant holomorphic sectional curvature and obtained an example of compact non-Kählerian Hermitian manifold which is Hermitian-flat. It is still unknown whether there are compact non-Kählerian Hermitian manifolds of non-zero constant holomorphic sectional curvature.

In [11], we introduced the notion of a locally conformally Hermitian-flat manifold as a complex analogue of a conformally flat Riemannian manifold. Then we derived a necessary and sufficient condition for a Hermitian manifold to be locally conformally Hermitian-flat in the case where the dimension is no less than 6, and we then constructed a family of examples of locally conformally Hermitian-flat metrics.

In Section 1 of the present paper, we recall the Hermitian connection D of an almost Hermitian manifold by the *Koszul-type formula*. In Section 2, we recall the fundamental formulas for the curvature tensor H of the Hermitian connection D of a Hermitian manifold (M, J, g).

In Section 3, we derive a necessary and sufficient condition for the product of certain normal almost contact Riemannian manifolds to be Hermitian-flat. Conse-

Received April 24, 1995
Revised June 9, 1995

quently, we obtain some examples of Hermitian-flat manifolds.

In Section 4, we recall the definition of a locally (or globally) conformally Hermitian-flat manifold and derive a necessary and sufficient condition for a Hermitian manifold to be locally conformally Hermitian-flat in the case where the dimension is no less than 4. We then introduce a conformally invariant tensor $\mathfrak{B}$ which is naturally required from the local conformal Hermitian-flatness of a Hermitian manifold. We notice that this tensor $\mathfrak{B}$ appears in the theory of Hermitian vector bundles (see [9]). On a locally conformally Hermitian-flat manifold, the Ricci-type tensors Q, R, S and the scalar curvatures s, $\hat{s}$ have the remarkable properties. We show that, if two of the three Ricci-type tensors Q, R and S of a locally conformally Hermitian-flat manifold (M, J, g) coinside, then (M, J, g) is either Hermitian-flat or of pointwise constant holomorphic sectional curvature and that there is no compact locally conformally Hermitian-flat manifold with negative scalar curvature. Closedness of the Ricci form of the Hermitian connection is closely related to the local conformal Hermitian-flatness of a Hermitian manifold. Moreover, the Ricci form represents the first Chern class of the Hermitian manifold. We hence study the Chern classes of locally conformally Hermitian-flat manifolds and show that every Chern class of a locally conformally Hermitian-flat manifold is power of the first Chern class up to the constant factor. In particular, we show that all the Chern classes of a locally conformally Kählerian-flat manifold vanish.

In the Appendix, we study compact Hermitian manifolds of constant sectional curvature and obtain some results.

PRELIMINARY REMARKS. 1) Throughout this paper, we always assume the differentiability of class C^∞ and assume manifolds to be connected and without boundary. We use the real dimensions for ones of manifolds. Since every 2-dimensional Hermitian manifold is Kählerian (cf. [10]), we always assume that the dimensions of almost Hermitian manifolds are no less than 4.

2) Given a manifold M, $C^\infty(M)$ denotes the space of all real valued differentiable functions on M and $\mathfrak{X}(M)$ denotes the Lie algebra of all vector fields on M.

ACKNOWLEDGMENT. The author would like to express his sincere thanks to Professor S. Yorozu, Professor K. Mikami and Professor F. Narita for their many valuable suggestions in the preparation of this paper.

1. Hermitian connections of almost Hermitian manifolds.

Let M be an almost complex manifold with the almost complex structure J and g a Hermitian metric on M, that is, a Riemannian metric such that $g(JX, JY) = g(X, Y)$ for any $X, Y \in \mathfrak{X}(M)$. The triple (M, J, g) is called an almost Hermitian manifolds. If the almost complex structure J is integrable, then (M, J, g) is called a Hermitian manifold. The following theorem is well-known.

THEOREM 1.1 (cf. [6], [10]). *Every almost Hermitian manifold* (M, J, g) *admits a unique linear connection* D *such that* $Dg=0$ *and* $DJ=0$ *and that the torsion tensor* T *is pure in the following sense*:

$$T(JX, Y)=T(X, JY) \qquad \text{for any } X,\ Y\in\mathfrak{X}(M)\,.$$

The connection D is called the *Hermitian connection* of (M, J, g).

In our discussion of the present paper, we want the explicit expression of the Hermitian connection D so that we prove Theorem 1.1 in the different way from [7] and [10]. We consider a mapping $\mathscr{V}: \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ defined by

$$\mathscr{V}(X, Y)=[JX, JY]+[X, Y]-J[X, JY]+J[JX, Y] \tag{1.1}$$

for any $X,\ Y\in\mathfrak{X}(M)$. The proof of the following lemma is a straightforward calculation.

LEMMA 1.1. *The mapping* $\mathscr{V}$ *has the following properties*:

$$\mathscr{V}(X\pm Y, Z)=\mathscr{V}(X, Z)\pm\mathscr{V}(Y, Z)\,, \qquad \mathscr{V}(X, Y\pm Z)=\mathscr{V}(X, Y)\pm\mathscr{V}(X, Z)\,, \tag{1.2}$$

$$\mathscr{V}(fX, Y)=f\mathscr{V}(X, Y)\,, \qquad \mathscr{V}(X, fY)=f\mathscr{V}(X, Y)+2(Xf)Y+2(JXf)JY\,, \tag{1.3}$$

$$\mathscr{V}(JX, JY)=\mathscr{V}(X, Y)\,, \quad \mathscr{V}(JX, Y)=-J\mathscr{V}(X, Y)\,, \quad \mathscr{V}(X, JY)=J\mathscr{V}(X, Y)\,, \tag{1.4}$$

$$\mathscr{V}(X, Y)+\mathscr{V}(Y, X)=-2J[X, JY]+2J[JX, Y]\,, \tag{1.5}$$

$$\mathscr{V}(X, Y)-\mathscr{V}(Y, X)=2[JX, JY]+2[X, Y] \tag{1.6}$$

for any $X,\ Y, Z\in\mathfrak{X}(M)$ *and any* $f\in C^\infty(M)$.

PROOF OF THEOREM 1.1. (*Existence*) For each $X,\ Y\in\mathfrak{X}(M)$, we define D_XY by the following equation:

$$4g(D_XY, Z)=2Xg(Y, Z)-2JXg(JY, Z)+g(\mathscr{V}(X, Y), Z)-g(\mathscr{V}(X, Z), Y) \tag{1.7}$$

for any $Z\in\mathfrak{X}(M)$. By (1.2) and (1.3), it is a straightforward verification that the mapping $(X, Y)\to D_XY$ determines a linear connection of M, denoted by D. Since the last three terms in the right hand side of (1.7) are skew-symmetric in Y and Z, it is clear that $g(D_XY, Z)+g(D_XZ, Y)=Xg(Y, Z)$, i.e., $Dg=0$. To show that $DJ=0$, it is sufficient to prove

$$D_X(JY)=JD_XY \qquad \text{for any } X,\ Y\in\mathfrak{X}(M)\,.$$

This follows immediately from (1.4) and (1.7). Moreover, by (1.7) and $DJ=0$, we have

$$D_{JX}Y=D_X(JY)-\tfrac{1}{2}J\mathscr{V}(X, Y)\,. \tag{1.8}$$

From (1.8) and (1.5), we easily obtain $T(JX, Y)=T(X, JY)$.

(*Uniqueness*) It is sufficient to prove that if a linear connection D satisfies $Dg=0$, $DJ=0$ and $T(JX, Y)=T(X, JY)$ for any $X,\ Y\in\mathfrak{X}(M)$, then it satisfies the equation (1.7). From $T(JX, Y)=T(X, JY)$, we have

$$g(T(X, Y), Z)+g(T(JX, JY), Z)=0\,, \tag{1.9}$$

$$-g(T(X, Z), Y)-g(T(JX, JZ), Y)=0\,, \tag{1.10}$$

$$g(T(X, JY), JZ)-g(T(JX, Y), JZ)=0\,, \tag{1.11}$$

$$-g(T(X, JZ), JY)+g(T(JX, Z), JY)=0 \tag{1.12}$$

for any $X, Y, Z\in\mathfrak{X}(M)$. Let us make the sum of the equations (1.9)–(1.12). Then, from the definition of T and $DJ=0$, we obtain

$$\begin{aligned}2g(D_XY, Z)-2g(D_XZ, Y)+2g(D_{JX}JY, Z)+2g(D_{JX}Z, JY)\\ -g(\mathscr{V}(X, Y), Z)+g(\mathscr{V}(X, Z), Y)=0\,.\end{aligned}$$

Furthermore, by $Dg=0$, we obtain

$$4g(D_XY, Z)-2Xg(Y, Z)+2JXg(JY, Z)-g(\mathscr{V}(X, Y), Z)+g(\mathscr{V}(X, Z), Y)=0\,.$$

Hence D satisfies (1.7). □

The expression of the Hermitian connection D given by (1.7) is our requirement.

Let ∇ be the Levi-Civita connection with respect to the same metric g, that is, ∇_XY is defined by the following *Koszul formula* (cf. [10]):

$$\begin{aligned}2g(\nabla_XY, Z)=Xg(Y, Z)+Yg(Z, X)-Zg(X, Y)\\ +g([X, Y], Z)-g([Y, Z], X)+g([Z, X], Y)\end{aligned}$$

for any $X, Y, Z\in\mathfrak{X}(M)$. By using (1.8) and $Dg=0$, we easily obtain

$$2g(\nabla_XY, Z)=2g(D_XY, Z)-g(T(X, Y), Z)+g(T(Y, Z), X)-g(T(Z, X), Y)\,. \tag{1.13}$$

Thus, if D is torsion-free, then (1.13) means that $\nabla=D$. The following fact is also well-known.

THEOREM 1.2 (cf. [10]). *Let (M, J) be an almost complex manifold. The almost complex structure J is integrable if and only if there exists a linear connection which is almost complex and torsion-free.*

These facts yield the following well-known theorem.

THEOREM 1.3 (cf. [6], [10]). *An almost Hermitian manifold (M, J, g) is a Kählerian manifold if and only if the Hermitian connection D coincides with the Levi-Civita connection ∇.*

2. Curvature tensors.

Let (M, J, g) be an almost Hermitian manifold. The curvature tensor H of the Hermitian connection D of M is defined by

$$H(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z \qquad \text{for any } X, Y, Z \in \mathfrak{X}(M).$$

Furthermore, we introduce a tensor of type (0, 4), denoted by the same symbol H, defined by

$$H(X, Y, Z, W) = g(H(Z, W)Y, Z) \qquad \text{for any } X, Y, Z, W \in \mathfrak{X}(M).$$

LEMMA 2.1 (cf. [10]). *The curvature tensor H of the Hermitian connection D of an almost Hermitian manifold (M, J, g) satisfies the following equations: For any $X, Y, Z, W \in \mathfrak{X}(M)$,*

$$H(X, Y, Z, W) = -H(Y, X, Z, W) = -H(X, Y, W, Z),$$

$$H(JX, JY, Z, W) = H(X, Y, Z, W),$$

$$\mathfrak{S}\{H(X, Y)Z\} = \mathfrak{S}\{T(T(X, Y), Z) + (D_X T)(Y, Z)\}, \qquad \textit{(Bianchi's first identity)}$$

$$\mathfrak{S}\{(D_X H)(Y, Z) + H(T(X, Y), Z)\} = 0, \qquad \textit{(Bianchi's second identity)}$$

where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z.

By a direct computation, we have

$$\begin{aligned} &H(X, Y, JZ, JW) - H(X, Y, Z, W) \\ &\quad = -\tfrac{1}{2} J\mathcal{N}(Z, W) g(JX, Y) - \tfrac{1}{8}\mathcal{N}_1(X, Y, Z, W) + \tfrac{1}{8}\mathcal{N}_1(Y, X, Z, W) \end{aligned}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$, where $\mathcal{N}$ denotes the Nijenhuis tensor of J, i.e.,

$$\mathcal{N}(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]$$

and

$$\begin{aligned} \mathcal{N}_1(X, Y, Z, W) &= g(\mathcal{V}(W, \mathcal{N}(Z, X)), Y) - g(\mathcal{V}(Z, \mathcal{N}(W, X)), Y) \\ &\quad + 2g(\mathcal{N}(W, [Z, X]), Y) - 2g(\mathcal{N}(Z, [W, X]), Y) - 2g(\mathcal{N}(JW, [JZ, X]), Y) \\ &\quad + 2g(\mathcal{N}(JZ, [JW, X]), Y) - 2g(\mathcal{N}([W, Z], X), Y) + 2g(\mathcal{N}([JW, JZ], X), Y) \\ &\quad + 4g(J[J\mathcal{N}(W, Z), X], Y). \end{aligned}$$

In particular, we have

LEMMA 2.2 (cf. [6], [10]). *The curvature tensor H of the Hermitian connection D of a Hermitian manifold (M, J, g) satisfies*

$$H(X, Y, JZ, JW) = H(X, Y, Z, W) \qquad \textit{for any } X, Y, Z, W \in \mathfrak{X}(M).$$

Let (M, J, g) be a Hermitian manifold of dimension $2m$. We define three tensors Q, R and S which are analogous to the Ricci tensor in the Riemannian geometry. These are defined by

$$Q(X, Y)=\frac{1}{2}\sum_{\alpha=1}^{m}\{H(e_\alpha, X, e_\alpha, Y)+H(Je_\alpha, X, Je_\alpha, Y)$$
$$+H(e_\alpha, Y, e_\alpha, X)+H(Je_\alpha, Y, Je_\alpha, X)\},$$
$$R(X, Y)=\sum_{\alpha=1}^{m} H(e_\alpha, Je_\alpha, X, JY),$$
$$S(X, Y)=\sum_{\alpha=1}^{m} H(X, JY, e_\alpha, Je_\alpha)$$

for any $X, Y\in\mathfrak{X}(M)$, where $\{e_1, \cdots, e_m, Je_1, \cdots, Je_m\}$ is a local adapted orthonormal frame field of (M, J, g). Then, by Lemma 2.1 and Lemma 2.2, we have

LEMMA 2.3. *All the Ricci-type tensor Q, R and S defined above are symmetric and compatible with J, i.e.,*

$$Q(X, Y)=Q(Y, X), \qquad Q(JX, JY)=Q(X, Y),$$
$$R(X, Y)=R(Y, X), \qquad R(JX, JY)=R(X, Y),$$
$$S(X, Y)=S(Y, X), \qquad S(JX, JY)=S(X, Y)$$

for any $X, Y\in\mathfrak{X}(M)$.

We can associate 2-forms ρ_Q, ρ_R and ρ_S with the Ricci-type tensors Q, R and S respectively in the usual manner:

$$\rho_Q(X, Y)=Q(X, JY), \quad \rho_R(X, Y)=R(X, JY), \quad \rho_S(X, Y)=S(X, JY)$$

for any $X, Y\in\mathfrak{X}(M)$. In particular, we then have

$$d\rho_R(X, Y, Z)=\tfrac{1}{3}\mathfrak{S}\{(D_X\rho_R)(Y, Z)+\rho_R(T(X, Y), Z)\}$$

for any $X, Y, Z\in\mathfrak{X}(M)$. Furthermore, we have

$$\begin{aligned}
&(D_X\rho_R)(Y, Z)\\
&=X\rho_R(Y, Z)-\rho_R(D_XY, Z)-\rho_R(Y, D_XZ)\\
&=\sum_{\alpha=1}^{m}\{Xg(H(Y, JZ)Je_\alpha, e_\alpha)-g(H(D_XY, Z)Je_\alpha, e_\alpha)-g(H(Y, D_XZ)Je_\alpha, e_\alpha)\}\\
&=\sum_{\alpha=1}^{m}\{g((D_XH)(Y, JZ)Je_\alpha, e_\alpha)+2g(H(Y, Z)D_X(Je_\alpha), e_\alpha)\}.
\end{aligned}$$

However, we can show that $\sum_{\alpha=1}^{m} g(H(Y, Z)D_X(Je_\alpha), e_\alpha)=0$. In fact,

$$\sum_{\alpha=1}^{m} g(H(Y, Z)D_X(Je_\alpha), e_\alpha)$$

$$= \sum_{\alpha,\beta=1}^{m} \{g(H(Y,Z)e_\beta, e_\alpha)g(D_X(Je_\alpha), e_\beta) + g(H(Y,Z)Je_\beta, e_\alpha)g(D_X(Je_\alpha), Je_\beta)\}$$

$$= \sum_{\alpha,\beta=1}^{m} \{g(H(Y,Z)e_\beta, e_\alpha)g(e_\alpha, D_X(Je_\beta)) + g(H(Y,Z)e_\beta, Je_\alpha)g(Je_\alpha, D_X(Je_\beta))\}$$

$$= \sum_{\beta=1}^{m} g(H(Y,Z)e_\beta, D_X(Je_\beta)) = -\sum_{\alpha=1}^{m} g(H(Y,Z)D_X(Je_\alpha), e_\alpha)\,.$$

Thus we have

$$d\rho_R(X,Y,Z) = \frac{1}{3}\sum_{\alpha=1}^{m} g(\mathfrak{S}\{(D_X H)(Y,Z) + H(T(X,Y),Z)\}e_\alpha, Je_\alpha)\,.$$

Hence the Bianchi's second identity gives us the following

LEMMA 2.4 (cf. [9], [10]). *The 2-form ρ_R is closed.*

The 2-form ρ_R is called the *Ricci form* of the Hermitian connection D.

Moreover, we define two *scalar* curvatures s and $\hat{s}$ which are analogous to the scalar curvature in the Riemannian geometry:

$$s = 2\sum_{\alpha=1}^{m} R(e_\alpha, e_\alpha) = 2\sum_{\alpha=1}^{m} S(e_\alpha, e_\alpha)\,, \qquad \hat{s} = 2\sum_{\alpha=1}^{m} Q(e_\alpha, e_\alpha)\,.$$

3. Hermitian-flat manifolds.

DEFINITION 3.1 ([11]). We call a Hermitian manifold (M, J, g) *Hermitian-flat* and g a *Hermitian-flat metric* if the curvature tensor of the Hermitian connection with respect to g vanishes everywhere.

Balas [1] gave an example of compact non-Kählerian Hermitian-flat manifolds.

EXAMPLE 3.1 ([1], [11]). The *Iwasawa manifold M* is defined by $M = G/\Gamma$, where

$$G = \left\{ \begin{pmatrix} 1 & z^1 & z^2 \\ 0 & 1 & z^3 \\ 0 & 0 & 1 \end{pmatrix} : z^i \in \mathbf{C} \right\}, \qquad \Gamma = \left\{ \begin{pmatrix} 1 & \alpha^1 & \alpha^2 \\ 0 & 1 & \alpha^3 \\ 0 & 0 & 1 \end{pmatrix} : \alpha^i \in \mathbf{Z} + \sqrt{-1}\,\mathbf{Z} \right\}.$$

Since M is the quotient space of a complex Lie group G by a discrete subgroup Γ, it is a complex manifold. And it is also well-known that this manifold is compact. The holomorphic 1-form $\varphi = dz^2 - z^3 dz^1$ defined on G is invariant under the action of Γ. Thus φ is identified with a 1-form on M. Moreover we have that $d\varphi = -dz^3 \wedge dz^1 \neq 0$. Hence M does not admit any Kählerian metrics. The following Hermitian metric ds^2 on G is a Hermitian-flat metric invariant under the action of Γ. Hence ds^2 induces a Hermitian-flat metric on the Iwasawa manifold M.

$$\begin{aligned}ds^2 = {} & A_1 dz^1 d\bar{z}^1 + A_2(dz^2 - z^3 dz^1)(d\bar{z}^2 - \bar{z}^3 d\bar{z}^1) + A_3 dz^3 d\bar{z}^3 \\ & + A_4 dz^1(d\bar{z}^2 - \bar{z}^3 d\bar{z}^1) + A_4(dz^2 - z^3 dz^1)d\bar{z}^1 \\ & + A_5(dz^2 - z^3 dz^1)d\bar{z}^3 + A_5 dz^3(d\bar{z}^2 - \bar{z}^3 d\bar{z}^1) \\ & + A_6 dz^3 d\bar{z}^1 + A_6 dz^1 d\bar{z}^3 ,\end{aligned}$$

where A_i ($i=1, \cdots, 6$) are real numbers satisfying

$$A_2>0\,, \quad A_1A_2-A_4^2>0\,, \quad (A_1A_2-A_4^2)A_3-A_1A_5^2-A_2A_6^2+2A_4A_5A_6>0\,.$$

Next we shall give examples of non-compact Hermitian-flat manifolds which are the products of certain normal almost contact Riemannian manifolds. Let M be an almost contact Riemannian manifold with the structure tensors (ϕ, ξ, η, g). Then we have the following relations:

$$\phi\xi=0\,, \quad \eta(\phi X)=0\,, \quad \eta(\xi)=1\,, \quad \phi^2 X=-X+\eta(X)\xi\,,$$
$$g(X, \xi)=\eta(X)\,, \qquad g(\phi X, \phi Y)=g(X, Y)-\eta(X)\eta(Y)$$

for any $X, Y \in \mathfrak{X}(M)$. Moreover, it is well-known that an almost contact structure (ϕ, ξ, η) is normal if and only if $[\phi, \phi]+2d\eta\otimes\xi=0$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ, i.e.,

$$[\phi, \phi](X, Y)=[\phi X, \phi Y]+\phi^2[X, Y]-\phi[X, \phi Y]-\phi[\phi X, Y]$$

for any $X, Y \in \mathfrak{X}(M)$.

Let M and N be two normal almost contact Riemannian manifolds with the structure tensors $(\phi_M, \xi_M, \eta_M, g_M)$ and $(\phi_N, \xi_N, \eta_N, g_N)$ respectively. On the product manifold $M\times N$, we define an almost complex structure J as follows: Identifying $T_{(x,y)}(M\times N)$ with $T_xM\oplus T_yN$, we may express $X\in\mathfrak{X}(M\times N)$ as $X=X_1+X_2$, where $X_1\in\mathfrak{X}(M)$ and $X_2\in\mathfrak{X}(N)$. We also consider a function on M (resp. N) as a function on $M\times N$ as usual. Then an almost complex structure J is defined by Morimoto [12] as

$$JX=\phi_M X_1-\eta_N(X_2)\xi_M+\phi_N X_2+\eta_M(X_1)\xi_N \tag{3.1}$$

for any $X\in\mathfrak{X}(M\times N)$. This almost complex structure J is integrable because both M and N are normal. Next we consider the product metric $g=g_M+g_N$ on the complex manifold $(M\times N, J)$. Then g is compatible with J, that is, $(M\times N, J, g)$ is a Hermitian manifold.

Kenmotsu [8] studied a class of almost contact Riemannian manifolds which satisfy the following two conditions:

$$(\nabla_X\phi)Y=g(\phi X, Y)\xi-\eta(Y)\phi X\,, \qquad \nabla_X\xi=X-\eta(X)\xi\,.$$

Such manifolds are called *Kenmotsu manifolds*. Kenmotsu manifolds are normal because of $[\phi, \phi]=0$ and $d\eta=0$. Moreover, Kenmotsu manifolds are non-compact because of $\operatorname{div}\xi=\dim M-1$. A warped product space $\mathbf{R}\times_f \mathbf{C}^p$ is an example of Kenmotsu manifolds, where $f(t)=ce^t$ (c: positive constant) is a function on $\mathbf{R}$. Here the metrics on $\mathbf{R}$ and $\mathbf{C}^p$ are the standard flat metrics. We notice that this Kenmotsu manifold $\mathbf{R}\times_f \mathbf{C}^p$ has constant sectional curvature -1.

Janssens and Vanhecke [7] introduced the notion of an α-Kenmotsu manifold, which is a generalization of a Kenmotsu manifold. An α-Kenmotsu manifold M is an almost contact Riemannian manifold whose structure tensors (ϕ, ξ, η, g) satisfy the following two conditions:

$$(\nabla_X\phi)Y=\alpha\{g(\phi X, Y)\xi-\eta(Y)\phi X\}, \qquad \nabla_X\xi=\alpha\{X-\eta(X)\xi\},$$

where α is non-zero constant. α-Kenmotsu manifolds are also normal and non-compact. Using the $\mathscr{D}$-homothetic deformation defined by

$$g'(X, Y)=\alpha^{-1}g(X, Y)+\alpha^{-1}(\alpha^{-1}-1)\eta(X)\eta(Y), \qquad \alpha>0,$$

on a Kenmotsu manifold (M, ϕ, ξ, η, g), we obtain an α-Kenmotsu manifold with $\alpha>0$. By starting with $(M, -\phi, -\xi, -\eta, g)$ we obtain a $(-\alpha)$-Kenmotsu manifold with $-\alpha<0$. We notice that, using the $\mathscr{D}$-homothetic deformation above for the Kenmotsu manifold $\mathbf{R}\times_f \mathbf{C}^p$, we obtain an α-Kenmotsu manifold with constant sectional curvature $-\alpha^2$.

Let M (resp. N) be an α-Kenmotsu (resp. a β-Kenmotsu) manifold with the structure tensors $(\phi_M, \xi_M, \eta_M, g_M)$ (resp. $(\phi_N, \xi_N, \eta_N, g_N)$). On the product manifold $M\times N$, we consider the almost complex structure J defined by (3.1) and the product metric $g=g_M+g_N$. Then $(M\times N, J, g)$ is a Hermitian manifold because both $(M, \phi_M, \xi_M, \eta_M, g_M)$ and $(N, \phi_N, \xi_N, \eta_N, g_N)$ are normal. The Hermitian connection D of $M\times N$ relates to the Levi-Civita connections of M and N as follows:

$$\begin{aligned} D_XY={}^M\nabla_{X_1}Y_1+{}^N\nabla_{X_2}Y_2&-\alpha\{\eta_M(Y_1)X_1-\eta_N(X_2)\phi_MY_1-g_M(X_1, Y_1)\xi_M\}\\ &-\beta\{\eta_N(Y_2)X_2+\eta_M(X_1)\phi_NY_2-g_N(X_2, Y_2)\xi_N\}, \end{aligned}$$

where ${}^M\nabla$ (resp. ${}^N\nabla$) denotes the Levi-Civita connection of M (resp. N). The curvature tensor H of D satisfies

$$\begin{aligned} H(X, Y, Z, W)=K_M(X_1, Y_1, Z_1, W_1)&+K_N(X_2, Y_2, Z_2, W_2)\\ &+\alpha^2\{g_M(X_1, Z_1)g_M(Y_1, W_1)-g_M(X_1, W_1)g_M(Y_1, Z_1)\}\\ &+\beta^2\{g_N(X_2, Z_2)g_N(Y_2, W_2)-g_N(X_2, W_2)g_N(Y_2, Z_2)\}, \end{aligned}$$

where K_M (resp. K_N) denotes the curvature tensor of ${}^M\nabla$ (resp. ${}^N\nabla$). Evidently we obtain the following

THEOREM 3.1. *The product of an α-Kenmotsu manifold M and a β-Kenmotsu manifold N is Hermitian-flat if and only if M and N have constant sectional curvature $-\alpha^2$ and $-\beta^2$ respectively.*

Since Kenmotsu manifolds are 1-Kenmotsu manifolds, we obtain the following

COROLLARY 3.1. *The product of two Kenmotsu manifolds M and N is Hermitian-flat if and only if both M and N have constant sectional curvature* -1.

Let $N=\mathbf{R}$ be the real line and let $(\phi_N, \xi_N, \eta_N, g_N)=(0, d/dt, dt, dt^2)$, where t is the coordinate in $\mathbf{R}$. Then we may consider that $(N, \phi_N, \xi_N, \eta_N, g_N)$ is a normal almost contact Riemannian manifold. Furthermore, we easily see that ${}^N\nabla\phi_N={}^N\nabla\xi_N=0$. Thus, for the sake of the convenience, let us consider that $(N, \phi_N, \xi_N, \eta_N, g_N)$ is a 0-Kenmotsu manifold. Then, by Theorem 3.1, we have

COROLLARY 3.2. *The product of an α-Kenmotsu manifold M and the real line* $\mathbf{R}$ *is Hermitian-flat if and only if M has constant sectional curvature* $-\alpha^2$.

REMARK. In [15], Vaisman showed that a compact locally conformally Kählerian manifold with $H=0$ is Kählerian. This fact does not hold in the non-compact case. This follows from Corollary 3.2 and the fact [13] that the product of an α-Kenmotsu manifold and the real line is locally conformally Kählerian and not Kählerian.

4. Locally conformally Hermitian-flat manifolds.

In this section, we study Hermitian manifolds which are locally conformal to Hermitian-flat manifolds.

DEFINITION 4.1 ([11]). A Hermitian manifold (M, J, g) is *locally conformally Hermitian-flat* if every $x\in M$ has an open neighborhood U with a differentiable function $\sigma: U\to\mathbf{R}$ such that $g'=e^{-\sigma}g_{|U}$ is a Hermitian-flat metric on U. In particular, we call (M, J, g) a *globally conformally Hermitian-flat* manifold if we can take $U=M$.

EXAMPLE 4.1. Every Hermitian-flat manifold is locally conformally Hermitian-flat.

Vaisman [16] studied a class of locally conformally Kählerian manifolds whose local Kählerian metrics are flat, which are called *locally conformally Kählerian-flat* manifolds. They are automatically examples of locally conformally Hermitian-flat manifolds.

EXAMPLE 4.2 ([16]). Let α be any non-zero complex number with $|\alpha|\neq 1$ and fixed. The quotient space $H_\alpha^m=(\mathbf{C}^m-\{0\})/G_\alpha$, $m\geq 2$, is a complex manifold of dimension $2m$, where G_α is the infinite cyclic group generated by the transformation $(z^1, \cdots, z^m)\to(\alpha z^1, \cdots, \alpha z^m)$ of $\mathbf{C}^m-\{0\}$. This manifold H_α^m is called a (*homogeneous*) *Hopf manifold.* Because H_α^m is diffeomorphic to $S^1\times S^{2m-1}$, H_α^m is compact. Here S^1 (resp. S^{2m-1}) denotes the standard 1 (resp. $(2m-1)$)-dimensional sphere. On $\mathbf{C}^m-\{0\}$, we consider a Hermitian metric

$$ds^2 = \frac{2}{\|z\|^2} \sum_{i=1}^{m} dz^i d\bar{z}^i \,, \tag{4.1}$$

where $\|z\|^2 = \sum_{i=1}^{m} z^i \bar{z}^i$. Since this metric is invariant under the action of G_α, it induces a Hermitian metric on H_α^m. The Hopf manifold H_α^m with the metric (4.1) is a locally conformally Kählerian-flat manifold.

In [11], we gave an example of the family of locally conformally Hermitian-flat metrics on the non-compact complex manifold $\mathbf{R}^{m-1} \times T^{m+1}$, where T^{m+1} denotes the $(m+1)$-dimensional torus.

EXAMPLE 4.3 ([11]). Let W be the m-times $(m \geq 2)$ direct product of $\mathbf{C}^* = \mathbf{C} - \{0\}$, i.e., $W = \mathbf{C}^* \times \cdots \times \mathbf{C}^*$ and α any fixed non-zero complex number with $|\alpha| \neq 1$. Consider the transformation g_α of W defined by $g_\alpha(z^1, \cdots, z^m) = (\alpha z^1, \cdots, \alpha z^m)$. The infinite cyclic group G_α generated by g_α acts on W freely and properly discontinuously. Thus $M_\alpha = W/G_\alpha$ is a complex manifold of dimension $2m$. Moreover we can easily show that M_α is diffeomorphic to $\mathbf{R}^{m-1} \times T^{m+1}$. Then we can construct the family $\{ds_q^2\}$ of locally conformally Hermitian-flat metrics on M_α which are neither Kählerian nor locally conformally Kählerian:

$$ds_q^2 = \frac{2}{\|z\|^{2q}} \sum_i A_i |z^{\mu(i)}|^{2(q-1)} dz^i d\bar{z}^i \tag{4.2}$$

for each $(A_1, \cdots, A_m) \in (^+\mathbf{R})^m$, $\mu \in \mathfrak{S}_m - \{Id\}$ and $q \in \mathbf{R} - \{0, 1\}$, where $\mathfrak{S}_m$ is the permutation group of $\{1, \cdots, m\}$.

Let (M, J, g) be a Hermitian manifold of dimension $2m$. Consider a conformal change $g' = e^{-\sigma} g$ of metric g where $\sigma \in C^\infty(M)$. Denoting by D', H' and $\rho_{R'}$ the Hermitian connection, the curvature tensor and the Ricci form of g' respectively, we have

$$D'_X Y = D_X Y - \frac{1}{2} d\sigma(X) Y - \frac{1}{2} d^c\sigma(X) JY \,, \tag{4.3}$$

$$H' = e^{-\sigma}(H - \Omega \otimes dd^c\sigma) \,, \tag{4.4}$$

$$\rho_{R'} = \rho_R - m dd^c\sigma \,, \tag{4.5}$$

where Ω denotes the fundamental form of (M, J, g), i.e., $\Omega(X, Y) = g(X, JY)$ for any $X, Y \in \mathfrak{X}(M)$ and d^c is a differential operator (see [3]) as follows: For any $r(\geq 0)$-form φ on M, we define

$$\begin{aligned} J\varphi &= \varphi && \text{for } r = 0 \,, \\ J\varphi(X_1, \cdots, X_r) &= (-1)^r \varphi(JX_1, \cdots, JX_r) && \text{for } r > 0 \,, \end{aligned}$$

where $X_1, \cdots, X_r \in \mathfrak{X}(M)$. The operator d^c is defined by

$$d^c\phi = -J^{-1} dJ\varphi = (-1)^r JdJ\varphi \qquad \text{for any } r\text{-form } \varphi \text{ on } M \,.$$

From (4.4) and (4.5), we have

$$H'-\frac{1}{m}\Omega'\otimes\rho_{R'}=e^{-\sigma}\left(H-\frac{1}{m}\Omega\otimes\rho_R\right). \tag{4.6}$$

Thus we naturally obtain a tensor field $\mathfrak{B}$ defined by

$$\mathfrak{B}=H-\frac{1}{m}\Omega\otimes\rho_R\,. \tag{4.7}$$

Furthermore, we introduce a tensor of type (1, 3), denoted by the same symbol $\mathfrak{B}$, defined by $g(\mathfrak{B}(X, Y)Z, W)=\mathfrak{B}(W, Z, X, Y)$. Then we have

THEOREM 4.1. *The tensor $\mathfrak{B}$ of type* (1, 3) *is conformally invariant.*

REMARK. This tensor $\mathfrak{B}$ coincides with the tensor introduced by Kobayashi [9] which concerns projective flatness of Hermitian vector bundle.

We are interested in locally conformally Hermitian-flat manifolds. Since the equations (4.3)–(4.6) are valid for a local conformal change $g'=e^{-\sigma}g_{|U}$, we have that, if $H'=0$, then $\mathfrak{B}=0$. We claim that the converse is also true. That is, we can prove the following

THEOREM 4.2. *A Hermitian manifold (M, J, g) is locally conformally Hermitian-flat if and only if the tensor $\mathfrak{B}$ vanishes everywhere on M.*

PROOF. Since the Ricci form ρ_R is a closed 2-form of type (1, 1) (see Lemma 2.3 and Lemma 2.4), it is well-known (cf. [3]) that, on a neighborhood U of every point of M, there exists a differentiable function σ such that $\rho_R=mdd^c\sigma$. By means of this function σ, we consider a local conformal change $g'=e^{-\sigma}g_{|U}$. Assume that $\mathfrak{B}$ vanishes everywhere on M. Then we have $\mathfrak{B}'=0$ on U. Thus we obtain

$$H'=\frac{1}{m}\Omega'\otimes\rho_{R'}=\frac{e^{-\sigma}}{m}\Omega\otimes(\rho_R-mdd^c\sigma)=0 \qquad \text{on } U\,.$$

Hence (M, J, g) is locally conformally Hermitian-flat. The converse was claimed. □

COROLLARY 4.1. *A locally conformally Hermitian-flat manifold is globally conformally Hermitian-flat if and only if the Ricci form is dd^c-exact.*

On a Kählerian manifold or a Hermitian-flat manifold, the Ricci-type tensors Q, R and S coincide. We shall study the Ricci-type tensors Q, R, S and the scalar curvatures s, $\hat{s}$ of a locally conformally Hermitian-flat manifold. We first prove

THEOREM 4.3. *Let (M, J, g) be a locally conformally Hermitian-flat manifold. If two of the three Ricci-type tensors Q, R and S coincide, then (M, J, g) is either Hermitian-flat or of pointwise constant holomorphic sectional curvature.*

PROOF. Since (M, J, g) is locally conformally Hermitian-flat, the curvature tensor H has, by Theorem 4.2, the form $H=(1/m)\Omega\otimes\rho_R$, where $m=\dim M/2$. Then we have $R=mQ$, $S=(s/2m)g$, $s=m\hat{s}$. If $R=Q$, then we obtain $R=0$, i.e., $H=0$ because of $m\geq 2$. If $S=Q$, then we have $s=\hat{s}$. Since $m\geq 2$, we obtain $s=0$, i.e., $0=S=Q=R$. Hence $H=0$. If $R=S$, then $R=(s/2m)g$, i.e., $\rho_R=(s/2m)\Omega$. Thus H satisfies $H=(s/2m^2)\Omega\otimes\Omega$. Then the holomorphic sectional curvature $\kappa(X)$ at a unit vector X satisfies

$$\kappa(X)=H(X, JX, X, JX)=\frac{s}{2m^2}g(X, J^2X)g(X, J^2X)=\frac{s}{2m^2},$$

that is, (M, J, g) is of pointwise constant holomorphic sectional curvature. □

By the proof of Theorem 4.3, we have

COROLLARY 4.2. *A locally conformally Hermitian-flat manifold with $Q=R=S$ is Hermitian-flat.*

COROLLARY 4.3. *Every locally conformally Hermitian-flat Kählerian manifold is flat.*

Moreover, if (M, J, g) is of pointwise constant sectional curvature, then, by (A.12) of Appendix, we have $s=-((m-1)/(2m-1))\hat{s}$. Since $s=m\hat{s}$ and $m\geq 2$, we obtain $\hat{s}=0$, and also $s=0$. Thus, by $R=mQ$, (A.11) and (A.12), we have $Q=R=S=0$. Hence, by Corollary 5.2, we obtain

COROLLARY 4.4. *A locally conformally Hermitian-flat manifold of pointwise constant sectional curvature is Hermitian-flat.*

Next we shall prove non-existence of compact locally conformally Hermitian-flat manifolds with negative scalar curvature. On a locally conformally Hermitian-flat manifold (M, J, g), $s=m\hat{s}$ holds. If s (resp. $\hat{s}$) is negative, then $\hat{s}$ (resp. s) is also negative. Thus we have $s-\hat{s}=(m-1)\hat{s}<0$. Assume that M is compact. Then by integrating this inequality, we have

$$\int_M (s-\hat{s})dV<0,$$

where dV is the volume form of M. However, on a compact Hermitian manifold (M, J, g), the following Gauduchon's inequality holds ([5], see also Proposition A.1 of Appendix):

$$\int_M (s-\hat{s})dV\geq 0.$$

Hence we obtain

THEOREM 4.4. *There is no compact locally conformally Hermitian-flat manifold with negative scalar curvature s (or $\hat{s}$).*

Finally, since the Ricci form of the Hermitian connection represents the first Chern class of the Hermitian manifold, we shall study the Chern classes of locally conformally Hermitian-flat manifolds. The closed $2k$-forms γ_k representing the k-th Chern class $c_k(M)$ $(1 \leq k \leq m)$ of a Hermitian manifold (M, J, g), called the k-th Chern forms, can be written as follows:

$$\gamma_k = \frac{(\sqrt{-1})^k}{(2\pi)^k \cdot k!} \sum \delta^{j_1 \cdots j_k}_{i_1 \cdots i_k} \Psi^{i_1}_{j_1} \wedge \cdots \wedge \Psi^{i_k}_{j_k} ,$$

where $\Psi = (\Psi^i_j)$ denotes the curvature form of the Hermitian connection of (M, J, g) (cf. [10]). In terms of a complex local coordinate system $z^1, \cdots, z^m$, the components Ψ^i_j are given by

$$\Psi^i_j = \sum_{k,l=1}^{m} H^i_{jk\bar{l}} dz^k \wedge d\bar{z}^l ,$$

where $\sum_{i=1}^m H^i_{jk\bar{l}} \partial/\partial z^i = H(\partial/\partial z^k, \partial/\partial \bar{z}^l)\partial/\partial z^j$. The computation of the Chern forms is essentially that of the $2k$-forms ψ_k $(1 \leq k \leq m)$ defined by

$$\psi_k = \operatorname{trace} \Psi^k , \qquad \Psi^k = \Psi \wedge \cdots \wedge \Psi \ (k\text{-times}) .$$

The curvature form Ψ of a locally conformally Hermitian-flat manifold is, by Theorem 4.2, of the form $\Psi = (\sqrt{-1}/m) I_m \otimes \rho_R$, where I_m denotes the identity matrix of degree m. Thus we have

$$\psi_k = \operatorname{trace}\left(\frac{\sqrt{-1}}{m} I_m \otimes \rho_R\right)^k = \operatorname{trace}\left(I_m \otimes \left(\frac{\sqrt{-1}}{m} \rho_R\right)^k\right) = \frac{1}{m^{k-1}} \psi_1^k$$

because of $\psi_1 = \sqrt{-1} \rho_R$. Therefore we have

THEOREM 4.5. *Every Chern class of a locally conformally Hermitian-flat manifold is power of the first Chern class up to the constant factor.*

In particular, on a locally conformally Kählerian-flat manifold (M, J, g), we have $\rho_R = m d\theta$, where θ is a global 1-form on M defined by $\theta = J\omega$, where ω is the *Lee form* of (M, J, g) (cf. [14]) which is a globally defined d-closed 1-form such that $d\Omega = \omega \wedge \Omega$. Thus, every locally conformally Kählerian-flat manifold has the vanishing first Chern class. Thereby, as a corollary of Theorem 4.5, we obtain

COROLLARY 4.5. *All the Chern classes of a locally conformally Kählerian-flat manifold vanish.*

Appendix. Hermitian manifolds of pointwise constant sectional curvature.

In [1], Balas studied Hermitian manifolds (M, J, g) of constant (Hermitian) holomorphic sectional curvature. Then he introduced the tensor field K of type $(0, 4)$ on M, called the *Kählerian symmetric part* of the curvature tensor H of Hermitian connection. K is given by

$$K(X, Y, Z, W)=\tfrac{1}{4}\{H(X, Y, Z, W)+H(Z, W, X, Y) \\ +H(X, W, Z, Y)+H(Z, Y, X, W)\} \tag{A.1}$$

for any $X, Y, Z, W\in \mathfrak{X}(M)$. Then K satisfies the following equations:

$$K(X, Y, Z, W)=K(Y, X, W, Z)\,, \qquad K(X, Y, Z, W)=K(Z, W, X, Y)\,,$$
$$K(X, Y, Z, W)+K(X, Z, W, Y)+K(X, W, Y, Z)=0\,.$$

Using this tensor K, Balas [1] derived a necessary and sufficient condition for (M, J, g) to be of constant holomorphic sectional curvature.

Now, let us consider the (*Hermitian*) *sectional curvature* of (M, J, g). For each plane p in the tangent space T_xM, the (Hermitian) sectional curvature $\kappa(p)$ for p is defined by $\kappa(p)=H(X, Y, X, Y)$, where X, Y is an orthonormal basis for p. It is easy to see that the sectional curvature $\kappa(p)$ is independent of the choice of an orthonormal basis X, Y for p. If $\kappa(p)$ is a constant for all planes p in T_xM at every point $x\in M$, then (M, J, g) is said to be *of pointwise constant sectional curvature*. Moreover, if $\kappa(p)$ is a constant for all points $x\in M$, then (M, J, g) is said to be *of constant sectional curvature*.

We introduce a tensor field L of type $(0, 4)$ on M defined by

$$L(X, Y, Z, W)=\tfrac{2}{3}\{K(X, Y, Z, W)-K(Y, X, Z, W)\} \tag{A.2}$$

for any $X, Y, Z, W\in \mathfrak{X}(M)$. Then L satisfies the following equations:

$$L(X, Y, Z, W)=-L(Y, X, Z, W)=-L(X, Y, W, Z)\,, \tag{A.3}$$

$$L(X, Y, Z, W)=L(Z, W, X, Y)\,, \tag{A.4}$$

$$L(X, Y, Z, W)+L(X, Z, W, Y)+L(X, W, Y, Z)=0\,. \tag{A.5}$$

Moreover, we have

$$L(X, Y, X, Y)=H(X, Y, X, Y) \tag{A.6}$$

for any $X, Y\in \mathfrak{X}(M)$. We set

$$H_0(X, Y, Z, W)=g(X, Z)g(Y, W)-g(X, W)g(Y, Z)\,. \tag{A.7}$$

Then H_0 satisfies all the equations (A.3)–(A.5).

Assume that (M, J, g) is of pointwise constant sectional curvature k. Then, for any orthonormal vectors X and Y of T_xM,

$$kH_0(X, Y, X, Y)=k\{g(X, X)g(Y, Y)-g(X, Y)g(Y, X)\}=k=H(X, Y, X, Y)\,. \qquad \text{(A.8)}$$

From (A.6), (A.8) and the well-known fact (Proposition 1.2 of Chapter V in [10]), we obtain that $L=kH_0$. Thus we conclude

THEOREM A.1. *A Hermitian manifold (M, J, g) is of pointwise constant sectional curvature k if and only if*

$$L(X, Y, Z, W)=k\{g(X, Z)g(Y, W)-g(X, W)g(Y, Z)\}$$

for any $X, Y, Z, W\in\mathfrak{X}(M)$.

We shall consider the Ricci-type contractions L_1, L_2 of L defined by

$$L_1(X, Y)=\sum_{\alpha=1}^{m}\{L(e_\alpha, X, e_\alpha, Y)+L(Je_\alpha, X, Je_\alpha, Y)\}\,, \qquad \text{(A.9)}$$

$$L_2(X, Y)=\sum_{\alpha=1}^{m}L(e_\alpha, Je_\alpha, X, JY) \qquad \text{(A.10)}$$

for any $X, Y\in\mathfrak{X}(M)$, where $m=\dim M/2$ and $\{e_1, \cdots, e_m, Je_1, \cdots, Je_m\}$ a local adapted orthonormal frame field of (M, J, g). Then we have $L_1=Q$, $L_2=\frac{1}{3}(Q+R+S)$. Assume again that (M, J, g) is of pointwise constant sectional curvature k. Then we have $L_1=(2m-1)kg$, $L_2=kg$. Thus we have

$$Q=(2m-1)kg\,, \qquad \tfrac{1}{3}(Q+R+S)=kg\,. \qquad \text{(A.11)}$$

Therefore we obtain

$$\hat{s}=2m(2m-1)k\,, \qquad \text{(A.12)}$$
$$\tfrac{1}{3}(\hat{s}+2s)=2mk\,.$$

From these equations, we obtain

$$s=-\frac{m-2}{2m-1}\hat{s}\,. \qquad \text{(A.13)}$$

On a compact Hermitian manifold (M, J, g), it is known [5] that

$$s-\hat{s}=\delta\tau+\|\tau\|^2\,, \qquad \text{(A.14)}$$

where τ denotes the *torsion* 1-*form* defined by $\tau(X)=\mathrm{trace}(Y\to T(X, Y))$ and δ denotes the codifferential, i.e., $\delta=-*d*$. This equation gives us the following

PROPOSITION A.1 ([5]). *On a compact Hermitian manifold (M, J, g),*

$$\int_M (s-\hat{s})dV\geq 0\,,$$

with equality holding if and only if $\tau=0$.

Let (M, J, g) be a compact Hermitian manifold of pointwise constant positive sectional curvature k. Then, by (A.12) and (A.13), we have $s-\hat{s}=-(3(m-1)/(2m-1))\hat{s}<0$. Hence we have

THEOREM A.2. *There exists no compact Hermitian manifold of pointwise constant positive sectional curvature.*

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