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An Ill-Posed Estimate of Positive Solutions of a Degenerate Nonlinear Parabolic Equation

Kazuya HAYASIDA and Takaaki YAMASHIRO

Kanazawa University

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1. Introduction.

In this paper we shall consider the non-characteristic Cauchy problem for a degenerate nonlinear parabolic equation and derive an estimate of positive solutions in terms of the bounds on the Cauchy data and the solutions.

In [10] Pucci studied the non-characteristic Cauchy problem of the linear parabolic differential equation of second order, where he showed the estimate of continuous dependence for positive solutions. Cannon [2] removed the positivity of solutions and derived an estimate, which is not of continuous dependence. His result is as follows:

Suppose $u_t=u_{xx}$, $0<x<1$, $0<t<T$, $u(x,0)=0$, $0<x<1$, $|u(0,t)|$, $|u_x(0,t)|\leq\varepsilon$, $0<t<T$, $|u(1,t)|\leq M$, $0<t<T$. Then $|u(x,t)|\leq M_1^{1-\beta(x)}\varepsilon^{\beta(x)}$, $0<x<1$, $0<t<T$, where $0<\beta(x)<1$ and M_1 depends on M.

The similar estimation was proved by Glagoleva [3] for more general linear parabolic equations. In this connection, there are referred to several contributions (see e.g., [13], [14]). From the estimate given there, we immediately see that the solution will vanish near the Cauchy surface, if the Cauchy data equals zero, particularly. Previously to these results Mizohata [8] established the uniqueness in non-characteristic Cauchy problem for linear parabolic equations.

In this way naturally arises the question: Does the above result hold too for nonlinear parabolic equations? Concerning this problem, Varin [15] has recently extended Glagoleva's work [3] to the case for a semilinear parabolic equation.

In this paper we shall consider the non-characteristic Cauchy problem for degenerate nonlinear parabolic equations and achieve an ill-posed estimate for positive solutions.

Our equation is of the form (2.1). But in the latter half we shall assume that the space dimension is one. Let $x=(x_1,\cdots,x_N)$ and $\Delta=\sum_{i=1}^N\partial_{x_i}^2$. There are many

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investigations for solutions $v(x, t)$ of the nonlinear equation

$$u_t = \Delta v^\delta , \tag{1.1}$$

which is referred to [7]. The function $v(x, t)$ normally represents the density of some substance. In the "fast" diffusion case $0<\delta<1$, (1.1) arises in plasma physics and was studied by several authors from the view-point of mathematics (see e.g., [1], [5]). Setting $u=v^\delta$ in (1.1) and changing the coordinates, we find that (1.1) becomes (2.1). From the reasoning above it seems to us that it is natural to treat only the positive solution of (2.1) or (2.4).

The ill-posed problem of the other classes may appar for nonlinear parabolic equations. Recently one of the authors [4] has obtained an ill-posed estimate of positive solutions for the backward Cauchy problem of the porous media equation. In this paper our objective consists in the estimate as described in [6] and [9].

2. Results.

For $a>0$ we define the domain

$$G_a = \{(x_1, \cdots, x_N, t) \,;\, 0<x_N<a^2-t^2-\textstyle\sum_{i=1}^{N-1} x_i^2\}$$

and the boundary portions of G_a

$$S_a = \{(x_1, \cdots, x_N, t) \,;\, 0\le x_N = a^2-t^2-\textstyle\sum_{i=1}^{N-1} x_i^2\} ,$$

$$T_a = \{(x_1, \cdots, x_{N-1}, 0, t) \,;\, t^2+\textstyle\sum_{i=1}^{N-1} x_i^2 \le a^2\}$$

(see Figure 1). We write simply $x=(x_1, \cdots, x_N)$.

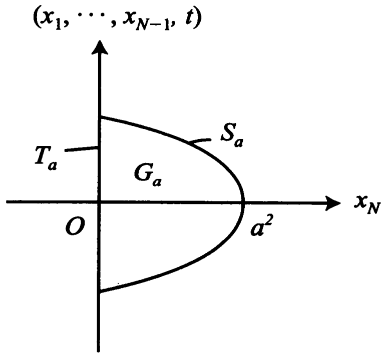

FIGURE 1

We consider the following equation in G_a:

$$u^\alpha \partial_t u = \Delta u + F(x, t, u) , \tag{2.1}$$

where $\alpha\ge 0$, $\Delta=\sum_{i=1}^N \partial_{x_i}^2$ and it is assumed that $|F(x, t, u)|\le K|u|$ for a positive constant K.

If u belongs to $C^1(\overline{G_a})$ and satisfies

$$\int_{G_a} u^\alpha \partial_t u \cdot \phi dxdt + \int_{G_a} \nabla u \cdot \nabla \phi dxdt - \int_{G_a} F(x, t, u)\phi dxdt - \int_{\partial G_a} \partial_{\mathrm{n}} u \cdot \phi d\sigma = 0 \tag{2.2}$$

for any $\phi \in C^1(\overline{G_a})$, we say as usual that u is a strong solution of (2.1) in G_a. Here n is the outer normal at ∂G_a and $\nabla = (\partial_{x_1}, \cdots, \partial_{x_N})$. If u is a classical solution of (2.1), it is naturally a strong solution of (2.1).

We use the Holmgren's transformation

$$\begin{cases} t' = t\,, \qquad x'_i = x_i \quad (i \neq N) \\ x'_N = x_N + t^2 + \sum_{i=1}^{N-1} x_i^2\,. \end{cases} \tag{2.3}$$

Obviously

$$\partial_t u = \partial_{t'} u + 2t' \partial_{x'_N} u\,, \qquad \partial^2_{x_N} u = \partial^2_{x'_N} u\,,$$

$$\partial^2_{x_i} u = \partial^2_{x'_i} u + 4x'_i \partial_{x'_i} \partial_{x'_N} u + 4x'^2_i \partial^2_{x'_N} u + 2\partial_{x'_N} u \qquad (i \neq N)\,.$$

We rewrite newly $(x'_1, \cdots, x'_N, t')$ with $(x_1, \cdots, x_N, t)$, and x'_N with y, respectively. Then (2.1) becomes

$$u^\alpha \partial_t u = \sum_{i=1}^{N-1} \partial^2_{x_i} u + g \partial^2_y u + 4 \sum_{i=1}^{N-1} x_i \partial_y \partial_{x_i} u + 2(N-1)\partial_y u - 2tu^\alpha \partial_y u + F\,, \tag{2.4}$$

where $g = 1 + 4\sum_{i=1}^{N-1} x_i^2$. By the transformation (2.3), G_a is mapped onto D_a:

$$D_a = \{(x_1, \cdots, x_{N-1}, y, t)\ ;\ t^2 + \textstyle\sum_{i=1}^{N-1} x_i^2 < y < a^2\}\,.$$

The boundary portions Γ_a and L_a of D_a are defined respectively as follows:

$$\Gamma_a = \{(x_1, \cdots, x_{N-1}, y, t)\ ;\ y = t^2 + \textstyle\sum_{i=1}^{N-1} x_i^2,\ y \leq a^2\}\,,$$

$$L_a = \{(x_1, \cdots, x_{N-1}, a^2, t)\ ;\ t^2 + \textstyle\sum_{i=1}^{N-1} x_i^2 \leq a^2\}$$

(see Figure 2).

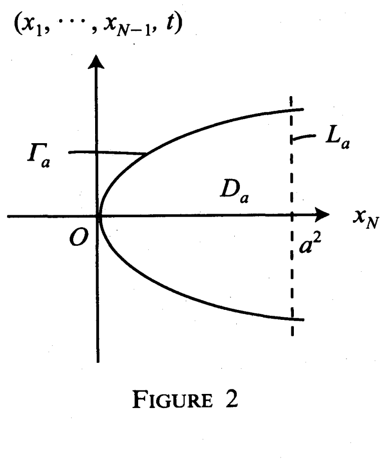

FIGURE 2

Let ν be the outer normal at ∂D_a. Let us denote by $\langle \nu, x_i\rangle$, $\langle \nu, y\rangle$, $\langle \nu, t\rangle$ the angles between ν and x_i, y, t-axis, respectively. Let $dx = dx_1 \cdots dx_{N-1}dy$. Multiplying the both sides of (2.4) with $\phi \in C^1(\overline{D_a})$ and using the integration by parts, we have

$$
\begin{aligned}
(2.5)\qquad 0 = &\int_{D_a} u^\alpha \partial_t u \cdot \phi dxdt + \sum_{i=1}^{N-1} \int_{D_a} \partial_{x_i} u \cdot \partial_{x_i}\phi dxdt \\
&+ \int_{D_a} g\partial_y u \cdot \partial_y \phi dxdt + 4\sum_{i=1}^{N-1} \int_{D_a} x_i \partial_{x_i} u \cdot \partial_y \phi dxdt \\
&-2(N-1)\int_{D_a} \partial_y u \cdot \phi dxdt + 2\int_{D_a} t u^\alpha \partial_y u \cdot \phi dxdt \\
&-\int_{D_a} F\phi dxdt - \sum_{i=1}^{N-1}\int_{\partial D_a} \partial_{x_i} u \cdot \phi \cos\langle \nu, x_i\rangle d\sigma \\
&-\int_{\partial D_a} g\partial_y u \cdot \phi \cos\langle \nu, y\rangle d\sigma - 4\sum_{i=1}^{N-1}\int_{\partial D_a} x_i \partial_{x_i} u \cdot \phi \cos\langle \nu, y\rangle d\sigma .
\end{aligned}
$$

Thus if u is in $C^1(\overline{D_a})$ and (2.5) is satisfied for any $\phi \in C^1(\overline{D_a})$, we say as usual that u is a strong solution of (2.4) in D_a. Obviously (2.1) and (2.4) are equivalent in the sense of each strong solution.

Our first theorem is the following

THEOREM 1. *Let $\alpha \geq 0$ and $0 < a < 1$. Let u be a strong solution of* (2.4) *in D_a and $u \geq 0$ there. Then, if*

$$\int_{\Gamma_a} (u + u^{1+\alpha} + |\nabla u|) d\sigma \leq \varepsilon , \qquad \int_{L_a} (u^{1+\alpha} + |\nabla u|) d\sigma \leq M ,$$

and $M \geq \varepsilon \exp[4\sqrt{2}\, a^2(N^2 + K + (\|u\|_\infty)^{2\alpha})^{1/2}]$, there is valid the inequality

$$\int_{D_{a/\sqrt{2}}} u dxdt \leq \kappa N \varepsilon^{1/2} M^{1/2} ,$$

where $\|u\|_\infty = \max_{\overline{D}_a} u$ and κ is a positive constant independent of a, α, N, K, $\|u\|_\infty$, M and ε.

REMARK 1. We consider the equation (2.1) on the original domain G_a. The condition in Theorem 1 is imposed on a and α. Then the statement of Theorem 1 is rewritten as follows:

Let u be a non-negative strong solution of (2.1) in G_a. Let

$$\int_{T_a} (u + u^{1+\alpha} + |\nabla u|) d\sigma \leq \varepsilon , \qquad \int_{S_a} (u^{1+\alpha} + |\nabla u|) d\sigma \leq M .$$

The same relation between ε and M is again assumed. Then it holds that

$$\int_{G_{a/\sqrt{2}}} udxdt \le \kappa N \varepsilon^{1/2} M^{1/2}, \tag{2.6}$$

where the constant κ is the same as in Theorem 1.

We give a brief proof for the above statement. We denote again the original (new) coordinates by $(x_1, \cdots, x_N, t)$ $((x'_1, \cdots, x'_N, t'))$, respectively. Let $\nabla_x u = (\partial_{x_1} u, \cdots, \partial_{x_N} u)$ and $\nabla_{x'} u = (\partial_{x'_1} u, \cdots, \partial_{x'_N} u)$. From (2.3), $|\nabla_{x'} u| \le 3|\nabla_x u|$. And the surface element on Γ_a is written as:

$$d\sigma = \left(1 + 4\left(t^2 + \sum_{i=1}^{N-1} x_i^2\right)\right)^{1/2} dx_1 \cdots dx_{N-1} dt .$$

So, $d\sigma \le (1+4a^2)^{1/2} dx_1 \cdots dx_{N-1} dt$ on Γ_a. Next the surface element $d\sigma$ on S_a is written as:

$$d\sigma = \left(1 + 4\left(t'^2 + \sum_{i=1}^{N-1} x_i'^2\right)\right)^{1/2} dx'_1 \cdots dx'_{N-1} dt' .$$

So, $dx'_1 \cdots dx'_{N-1} dt' \le d\sigma$ on S_a. From the above we have

$$\int_{\Gamma_a} (u + u^{1+\alpha} + |\nabla_{x'} u|) d\sigma \le 3(1+4a^2)^{1/2} \int_{T_a} (u + u^{1+\alpha} + |\nabla_x u|) dx_1 \cdots dx_{N-1} dt ,$$

$$\int_{L_a} (u^{1+\alpha} + |\nabla_{x'} u|) dx'_1 \cdots dx'_{N-1} dt' \le 3 \int_{S_a} (u^{1+\alpha} + |\nabla_x u|) d\sigma ,$$

which implies that

$$\int_{\Gamma_a} (u + u^{1+\alpha} + |\nabla_{x'} u|) d\sigma \le 3(1+4a^2)^{1/2} \varepsilon , \qquad \int_{L_a} (u^{1+\alpha} + |\nabla_{x'} u|) d\sigma \le 3M .$$

Therefore using Theorem 1, we obtain (2.6).

REMARK 2. Let $1/2 < \delta < 1$ and $0 < a < 1$. We define

$$G_{a,\delta} = \{(x_1, \cdots, x_N, t) \,;\, 0 < x_N < a^2 - \delta^2 t^2 - \textstyle\sum_{i=1}^{N-1} x_i^2\} ,$$
$$S_{a,\delta} = \{(x_1, \cdots, x_N, t) \,;\, 0 \le x_N = a^2 - \delta^2 t^2 - \textstyle\sum_{i=1}^{N-1} x_i^2\} ,$$
$$T_{a,\delta} = \{(x_1, \cdots, x_N, t) \,;\, \delta^2 t^2 + \textstyle\sum_{i=1}^{N-1} x_i^2 \le a^2\} .$$

We consider the fast diffusion (1.1) on $G_{a,\delta}$, where v is a generalized solution. Its definition is referred to [7]. We assume that the solution v is of class C^2 and non-negative on $\overline{G_{a,\delta}}$. If we set $u = v^\delta$, then u belongs to $C^1(\overline{G_{a,\delta}})$. Indeed, denoting $\partial_{x_1}, \cdots, \partial_{x_N}, \partial_t$ simply by ∂, we see from Lemma 2 in Section 4 that $v^{-1/2}\partial v$ is bounded in $\overline{G_{a,\delta}}$. Since

$$|\partial u| \le \delta v^{\delta - 1/2} |v^{-1/2} \partial v| ,$$

it follows that u is of class C^1 on $\overline{G_{a,\delta}}$.

Taking the transformation $t'=\delta t$, we see that $G_{a,\delta}$ is mapped onto G_a and (1.1) becomes

$$u^{\alpha}\partial_{t'}u=\Delta u \quad \text{in } G_a \quad (\alpha=1/\delta-1)\,, \tag{2.7}$$

where it is in the sense of strong solutions. Let

$$\int_{T_{a,\delta}}(v^{\delta}+v^{(1+\alpha)\delta}+|\nabla v^{\delta}|)d\sigma\le\varepsilon\,, \qquad \int_{S_{a,\delta}}(v^{(1+\alpha)\delta}+|\nabla v^{\delta}|)d\sigma\le M\,,$$

and $M\ge\varepsilon\exp[4\sqrt{2}\,a^2(N^2+K+(\max_{\overline{G_{a,\delta}}}u)^{2\alpha})^{1/2}]$. Then applying Remark 1 to (2.7), we have

$$\int_{G_{a/\sqrt{2},\delta}}v^{\delta}dxdt\le\kappa N\varepsilon^{1/2}M^{1/2}\,. \tag{2.8}$$

REMARK 3. Let u be the solution in Remark 2. In particular we consider (2.7). By Sabinina [11], [12], it is known that if u vanishes at some point $(x^0, t^0)\in G_a$, then it vanishes at once on the hyperplane $t=t^0$. Since $u\ge 0$ in G_a, $u(x^0, t^0)=0$ implies that the first derivatives of u vanish at (x^0, t^0). In other words, the zero point of u propagates along the hyperplane passing it. In relation to this matter, our estimate (2.8) means that the Cauchy data of u on T_a controles the behavior of u through the whole subdomain $G_{a/\sqrt{2}}$, under the bound of u on S_a.

REMARK 4. In Remark 2 we have assumed the C^2 regularity of v. Concerning the regularity of v, the following fact is known (see e.g., [7]):

Let $(1/N)\max(N-2, 0)<\delta<1$. Let v be a generalized solution of $\partial_t v=\Delta v^{\delta}$ in $\mathbf{R}^N\times(0, T)$ with $v(x, 0)=v_0\in L^1(\mathbf{R}^n)$. Then $v\in C^{\infty}(\mathbf{R}^N\times(0, T))$.

Next letting $N=1$, we shall treat the equation (2.1). We start from the new situation after the Holmgren's transformation. So the equation is of the form

$$u^{\alpha}u_t=u_{xx}-2tu^{\alpha}u_x+F(x, t, u)\,, \tag{2.9}$$

where $|F(x, t, u)|\le K|u|$. Write again

$$\begin{aligned} D_a&=\{(x, t)\ ;\ t^2<x<a^2\}\,,\\ \Gamma_a&=\{(x, t)\ ;\ x=t^2,\ x\le a^2\}\,,\\ L_a&=\{(x, t)\ ;\ x=a^2,\ |t|\le a\}\,. \end{aligned}$$

Let $0<a<1$ and define the square

$$Q_0=\{(x, t)\ ;\ |x|, |t|<a_0\}$$

for $a<a_0$. The domain D_a is contained in Q_0 (see Figure 3).

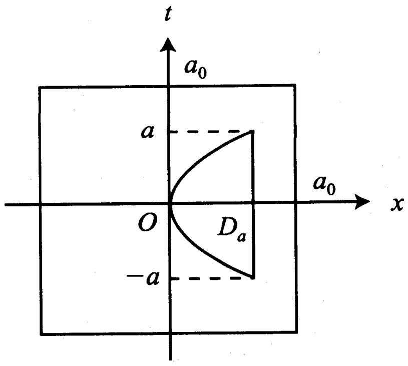

FIGURE 3

We distinguish three positive numbers α, β and γ such that

$$\begin{cases} 3<\gamma<5\,, \\ 0<\alpha<(5-\gamma)/2\,, \\ \beta=5-\gamma-2\alpha\,, \end{cases} \tag{2.10}$$

where α is the exponent in (2.4). And we define for given $M>0$

$$S=\max\left[\frac{6\gamma}{\alpha(\gamma-3)},\ \frac{2^8}{\alpha\beta}(1+K+aM^\alpha)^2,\ (a_0-a^2)^{-1}+K+M+M^\alpha\right].$$

Then we have

THEOREM 2. *Let $0<a<1$ and assume* (2.10). *Let u be a solution of* (2.4) ((2.9)) *in Q_0. Let u belong to $C^2(Q_0)$ and $u\geq 0$ in Q_0. Suppose that*

$$u\,,\quad |u_x|\,,\quad |u_t|\leq M \qquad in\quad Q_0\,,$$

$$\int_{\Gamma_a}(u^2+u_x^2+u_t^2)d\sigma\leq\varepsilon^2\,,\qquad 0\leq\varepsilon\leq 1\,,$$

$$\varepsilon\exp(2a^2S)\leq M.$$

Then there is fulfilled the inequality

$$\int_{D_{a/\sqrt{2}}}(u^2+u_x^2+\alpha\beta u^{2\alpha}u_t^2)dxdt\leq\kappa a^{-8}\varepsilon^{1/3}M^{2/3}(1+M)\,,$$

where κ is a positive constant independent of ε, M, a_0, a, α, γ and K.

REMARK. Surprisingly, Theorem 2 can not be proved for the case of $\alpha=0$, as will be seen later.

3. Proof of Theorem 1.

Let us denote simply by (,) the $L^2(D_a)$-inner product. Let ϕ belong to $C^2(\overline{D_a})$. By integration by parts we see that

$$(u^\alpha \partial_t u, \phi) = -\frac{1}{1+\alpha}(u^{1+\alpha}, \partial_t \phi) + \frac{1}{1+\alpha}\int_{\Gamma_a} u^{1+\alpha}\phi \cos\langle \nu, t\rangle d\sigma\,,$$

$$(\partial_{x_i} u, \partial_{x_i}\phi) = -(u, \partial_{x_i}^2 \phi) + \int_{\Gamma_a} u\partial_{x_i}\phi \cdot \cos\langle \nu, x_i\rangle d\sigma \quad (i \neq N)\,,$$

$$(g\partial_y u, \partial_y \phi) = -(gu, \partial_y^2 \phi) + \int_{\partial D_a} gu\partial_y \phi \cdot \cos\langle \nu, y\rangle d\sigma\,,$$

$$(x_i \partial_{x_i} u, \partial_y \phi) = -(x_i u, \partial_{x_i}\partial_y \phi) - (u, \partial_y \phi) + \int_{\Gamma_a} x_i u \partial_y \phi \cdot \cos\langle \nu, x_i\rangle d\sigma \quad (i \neq N)\,,$$

$$(\partial_y u, \phi) = -(u, \partial_y \phi) + \int_{\partial D_a} u\phi \cos\langle \nu, y\rangle d\sigma\,,$$

$$(tu^\alpha \partial_y u, \phi) = -\frac{1}{1+\alpha}(tu^{1+\alpha}, \partial_y \phi) + \frac{1}{1+\alpha}\int_{\partial D_a} tu^{1+\alpha}\phi \cos\langle \nu, y\rangle d\sigma\,.$$

Here we have used the fact that $\cos\langle \nu, t\rangle = \cos\langle \nu, x_i\rangle = 0$, $i = 1, \cdots, N-1$, on L_a. Thus we obtain from (2.5)

$$\begin{aligned}
0 = {} & \frac{1}{1+\alpha}(u^{1+\alpha}, \partial_t \phi) + \sum_{i=1}^{N-1}(u, \partial_{x_i}^2 \phi) + (gu, \partial_y^2 \phi) \\
& + 4\sum_{i=1}^{N-1}(x_i u, \partial_{x_i}\partial_y \phi) + 2(N-1)(u, \partial_y \phi) \\
& + \frac{2}{1+\alpha}(tu^{1+\alpha}, \partial_y \phi) + (F, \phi) - \frac{1}{1+\alpha}\int_{\Gamma_a} u^{1+\alpha}\phi \cos\langle \nu, t\rangle d\sigma \\
& - \sum_{i=1}^{N-1}\int_{\Gamma_a} u\partial_{x_i}\phi \cdot \cos\langle \nu, x_i\rangle d\sigma - \int_{\partial D_a} gu\partial_y \phi \cdot \cos\langle \nu, y\rangle d\sigma \\
& - 4\sum_{i=1}^{N-1}\int_{\Gamma_a} x_i u\partial_y \phi \cdot \cos\langle \nu, x_i\rangle d\sigma + \sum_{i=1}^{N-1}\int_{\Gamma_a} \partial_{x_i} u \cdot \phi \cos\langle \nu, x_i\rangle d\sigma \\
& + \int_{\partial D_a} g\partial_y u \cdot \phi \cos\langle \nu, y\rangle d\sigma + 4\sum_{i=1}^{N-1}\int_{\partial D_a} x_i \partial_{x_i} u \cdot \phi \cos\langle \nu, y\rangle d\sigma \\
& + 2(N-1)\int_{\partial D_a} u\phi \cos\langle \nu, y\rangle d\sigma - \frac{2}{1+\alpha}\int_{\partial D_a} tu^{1+\alpha}\phi \cos\langle \nu, y\rangle d\sigma\,.
\end{aligned} \tag{3.1}$$

We set $\phi=e^{\lambda y}$, where $\lambda\leq -1$ for the time being. Then from (3.1) we have

$$0=\lambda^2(gu, \exp[\lambda y])+2\lambda(N-1)(u, \exp[\lambda y])+\frac{2}{1+\alpha}\lambda(tu^{1+\alpha}, \exp[\lambda y])+(F, \exp[\lambda y])$$

$$+\sum_{i=1}^{N-1}\int_{\partial D_a}\partial_{x_i}u\cdot\exp[\lambda y]\cos\langle v, x_i\rangle d\sigma-\lambda\int_{\partial D_a}gu\exp[\lambda y]\cos\langle v, y\rangle d\sigma$$

$$+\int_{\partial D_a}g\partial_y u\cdot\exp[\lambda y]\cos\langle v, y\rangle d\sigma-4\lambda\sum_{i=1}^{N-1}\int_{\partial D_a}x_i u\exp[\lambda y]\cos\langle v, x_i\rangle d\sigma$$

$$+4\sum_{i=1}^{N-1}\int_{\partial D_a}x_i\partial_{x_i}u\cdot\exp[\lambda y]\cos\langle v, y\rangle d\sigma+2(N-1)\int_{\partial D_a}u\exp[\lambda y]\cos\langle v, y\rangle d\sigma$$

$$-\frac{2}{1+\alpha}\int_{\partial D_a}tu^{1+\alpha}\exp[\lambda y]\cos\langle v, y\rangle d\sigma-\frac{1}{1+\alpha}\int_{\partial D_a}u^{1+\alpha}\exp[\lambda y]\cos\langle v, t\rangle d\sigma\,.$$

We note that $\cos\langle v, t\rangle=\cos\langle v, x_i\rangle=0$ $(i\neq N)$ on L_a. Setting $A=\|u\|_\infty$, we have from the above

$$[\lambda^2-2\left(N-1+\frac{a}{1+\alpha}A^\alpha\right)|\lambda|-K](u, \exp[\lambda y]) \tag{3.2}$$

$$\leq\sum_{i=1}^{N-1}\int_{\Gamma_a}|\partial_{x_i}u|\exp[\lambda y]d\sigma+(1+4a^2)|\lambda|\int_{\Gamma_a}u\exp[\lambda y]d\sigma$$

$$+(1+4a^2)\left(\int_{\Gamma_a}|\partial_y u|\exp[\lambda y]d\sigma+\int_{L_a}|\partial_y u|\exp[\lambda y]d\sigma\right)$$

$$+4a(N-1)|\lambda|\int_{\Gamma_a}u\exp[\lambda y]d\sigma$$

$$+4a\left(\sum_{i=1}^{N-1}\int_{\Gamma_a}|\partial_{x_i}u|\exp[\lambda y]d\sigma+\sum_{i=1}^{N-1}\int_{L_a}|\partial_{x_i}u|\exp[\lambda y]d\sigma\right)$$

$$+2(N-1)\int_{\Gamma_a}u\exp[\lambda y]d\sigma$$

$$+\frac{2a}{1+\alpha}\left(\int_{\Gamma_a}u^{1+\alpha}\exp[\lambda y]d\sigma+\int_{\Gamma_a}u^{1+\alpha}\exp[\lambda y]d\sigma\right)$$

$$+\frac{1}{1+\alpha}\int_{\Gamma_a}u^{1+\alpha}\exp[\lambda y]d\sigma$$

$$\leq\sum_{i=1}^{N-1}\int_{\Gamma_a}|\partial_{x_i}u|d\sigma+5|\lambda|\int_{\Gamma_a}ud\sigma+5\int_{\Gamma_a}|\partial_y u|d\sigma+4(N-1)|\lambda|\int_{\Gamma_a}ud\sigma$$

$$+4\sum_{i=1}^{N-1}\int_{\Gamma_a}|\partial_{x_i}u|d\sigma+2(N-1)\int_{\Gamma_a}ud\sigma+3\int_{\Gamma_a}u^{1+\alpha}d\sigma$$

$$+\exp[\lambda a^2]\cdot\left(5\int_{L_a}|\partial_y u|d\sigma+4\sum_{i=1}^{N-1}\int_{L_a}|\partial_{x_i}u|d\sigma+2\int_{L_a}u^{1+\alpha}d\sigma\right)$$

$$\leq 5\sum_{i=1}^{N-1}\int_{\Gamma_a}|\partial_{x_i}u|\,d\sigma+6N|\lambda|\int_{\Gamma_a}ud\sigma+3\int_{\Gamma_a}u^{1+\alpha}d\sigma$$

$$+\exp[\lambda a^2]\cdot\left(5\sum_{i=1}^{N-1}\int_{L_a}|\partial_{x_i}u|d\sigma+2\int_{L_a}u^{1+\alpha}d\sigma\right).$$

Setting $B=N-1+(a/(1+\alpha))A^\alpha$, we write $\lambda^2-2B|\lambda|-K=\frac{1}{2}\lambda^2+\frac{1}{2}(\lambda^2-4B|\lambda|-2K)$. If $|\lambda|\geq 2B+\sqrt{4B^2+2K}$, then $\lambda^2-4B|\lambda|-2K\geq 0$. Since $B\leq N+A^\alpha$, we have $4B^2+2K\leq 8(N^2+K+A^{2\alpha})$. Hence we see that

$$\lambda^2-2B|\lambda|-K\geq\tfrac{1}{2}\lambda^2\,,$$

if $|\lambda|\geq 4\sqrt{2(N^2+K+A^{2\alpha})}$. Let ε and M be the same quantities as in the statement of Theorem 1. We replace the integral domain D_a with $D_{a/\sqrt{2}}$ on the left-hand side of (3.2). Then

$$|\lambda|\int_{D_{a/\sqrt{2}}}u\cdot\exp[\lambda y]dxdt\leq\kappa\Bigg[N|\lambda|\int_{\Gamma_a}(u+u^{1+\alpha}+|\nabla u|)d\sigma$$

$$+\exp[\lambda a^2]\int_{L_a}(u^{1+\alpha}+|\nabla u|)d\sigma\Bigg].$$

From the above calculation we conclude that

$$\int_{D_{a/\sqrt{2}}}udxdt\leq\kappa N\exp[-\lambda a^2/2](\varepsilon+M\exp[\lambda a^2])$$

for $\lambda\leq -4\sqrt{2(N^2+K+A^{2\alpha})}$. Setting $\lambda=-a^{-2}\log(M/\varepsilon)$, we complete the proof of Theorem 1.

4. Lemmas.

We prepare a lemma (Lemma 2) for the proof of Theorem 2. First we prepare

LEMMA 1. *Let $f(x)$ belong to $C^2([0,c])$, $c>0$. Let $f>0$ and $f'<0$ in $[0,c)$. Then it holds that*

$$f'(0)^2/f(0)\leq 2\left(\max_{0\leq x\leq c}|f''(x)|+2c^{-1}\max_{0\leq x\leq c}|f'(x)|\right).$$

PROOF. We define the function g in $[1, \infty)$:

$$g(x)=f(c-c/x)\,.$$

Then $g(x)>0$ and $g'(x)=cx^{-2}f'(c-c/x)<0$ in $[1, \infty)$. And

$$g''(x)=-2cx^{-3}f'(c-c/x)+c^2x^{-4}f''(c-c/x)\,.$$

Further, $g(1)=f(0)$, $g'(1)=cf'(0)$ and $g'(x)\to -0$ as $x\to\infty$.

For $x>1$ we have

$$\frac{g'(1)^2}{g(1)}\le\frac{g'(1)^2}{g(1)-g(x)}=\frac{g'(1)^2-g'(x)^2}{g(1)-g(x)}+\frac{g'(x)^2}{g(1)-g(x)}\,.$$

By Cauchy's theorem

$$\frac{g'(1)^2-g'(x)^2}{g(1)-g(x)}=2g''(\xi)$$

for some ξ with $1<\xi<x$. Therefore

$$\frac{g'(1)^2}{g(1)}\le 2\max_{x\ge 1}|g''(x)|+\lim_{x\to\infty}\frac{g'(x)^2}{g(1)-g(x)}=2\max_{x\ge 1}|g''(x)|\,.$$

This completes the proof.

LEMMA 2. *Let $0<c<d$. Let f belong to $C^2([-d, d])$. Let $f\ge 0$ in $[-d, d]$ and $f\ne 0$ in $[-c, c]$. Then it holds that, in $[-c, c]$,*

$$f'(x)^2/f(x)\le 2\Big(\max_{|x|\le d}|f''(x)|+2(d-c)^{-1}\max_{|x|\le d}|f'(x)|\Big)\,. \tag{4.1}$$

PROOF. First we assume that $f>0$ in $[-d, d]$. Let $E=\{|x|\le d\colon f'(x)=0\}$. We note that E is closed. If $E=\varnothing$, then $f'(x)<0$ or $f'(x)>0$ in $[-d, d]$. In the former case this lemma is obvious from Lemma 1. In the latter case it is reduced to the former case. In fact, it is enough to reverse the coordinate. Hence we may assume that $E\ne\varnothing$, without loss of generality.

There is a sequence of disjoint open intervals $\{(a_{i-1}, a_i)\}_{i=1}^{\infty}$ such that

$$(-d, d)-E=\bigcup_{i=1}^{\infty}(a_{i-1}, a_i)\,.$$

We handle the case with $-d<a_{i-1}<a_i<d$. Then, obviously $f'(a_{i-1})=f'(a_i)=0$ and $f'(x)\ne 0$ in (a_{i-1}, a_i). If $f'(x)>0$ in (a_{i-1}, a_i), for any x with $a_{i-1}<x\le a_i$ there exists ξ_1 such that $a_{i-1}<\xi_1<a_i$ and

$$\frac{f'(x)^2}{f(x)}\le\frac{f'(x)^2-f'(a_{i-1})^2}{f(x)-f(a_{i-1})}=2f''(\xi_1)\,. \tag{4.2}$$

If $f'(x)<0$ in (a_{i-1}, a_i), there exists ξ_2 for any x with $a_{i-1}\le x<a_i$ such that $a_{i-1}<\xi_2<a_i$ and

$$\frac{f'(x)^2}{f(x)}\le\frac{f'(x)^2-f'(a_i)^2}{f(x)-f(a_i)}=2f''(\xi_2)\,.$$

From the above, (4.1) is correct for $x\in(a_{i-1}, a_i)$, if $-d<a_{i-1}<a_i<d$.

Next we consider the case when $a_i=d$, $a_{i-1}\neq -d$ or $a_i\neq d$, $a_{i-1}=-d$. Without loss of generality we may assume that $a_i=d$ and $a_{i-1}\neq -d$. If $f'>0$ in (a_{i-1}, a_i), the inequality (4.2) holds. If $f'<0$ in (a_{i-1}, a_i) and $a_{i-1}<c$, (4.1) holds for x with $\max(a_{i-1}, -c)\le x\le c$ in virtue of Lemma 1.

If $f\ge 0$ in $[-d, d]$, we replace f with $f+\varepsilon$ for $\varepsilon>0$ and we take $\varepsilon\to +0$. Thus we complete the proof.

5. Proof of Theorem 2.

We shall prove Theorem 2. From now on let κ be the constant in the statement of Theorem 2. Let $\lambda<-1$ for the time being. We set $v(x, t)=\exp[\lambda x]u(x, t)$ and $G(x, t, u, u_x)=F(x, t, u)-2tu^\alpha u_x$. Then (2.4) ((2.9)) becomes

$$v_{xx}-2\lambda v_x+\lambda^2 v-\exp[-\lambda\alpha x]v^\alpha v_t=-\exp[\lambda x]G\,. \tag{5.1}$$

As previously we denote by (,) the $L^2(D_a)$-inner product. Multiplying (5.1) with v,

$$(v_{xx}, v)-2\lambda(v_x, v)+\lambda^2(1, v^2)-(\exp[-\lambda\alpha x]v^{1+\alpha}, v_t)=-(\exp[\lambda x]G, v)\,.$$

By integration by parts,

$$(v_x, v)=\frac{1}{2}\int_{\partial D_a} v^2\cos\langle\nu, x\rangle d\sigma\,,$$

$$(v_{xx}, v)=-(1, v_x^2)+\int_{\partial D_a} vv_x\cos\langle\nu, x\rangle d\sigma\,,$$

$$(\exp[-\lambda\alpha x]v^{1+\alpha}, v_t)=\frac{1}{2+\alpha}\int_{\partial D_a}\exp[-\lambda\alpha x]v^{2+\alpha}\cos\langle\nu, t\rangle d\sigma\,.$$

Hence we gain

$$(\exp[\lambda x]G, v)=(1, v_x^2)-\lambda^2(1, v^2)+I_1\,, \tag{5.2}$$

where

$$I_1=\lambda\int_{\partial D_a} v^2\cos\langle\nu, x\rangle d\sigma-\int_{\partial D_a} vv_x\cos\langle\nu, x\rangle d\sigma$$
$$+\frac{1}{2+\alpha}\int_{\partial D_a}\exp[-\lambda\alpha x]v^{2+\alpha}\cos\langle\nu, t\rangle d\sigma\,.$$

Next multiply (5.1) with v_x. Then

$$-(\exp[\lambda x]G, v_x)=(v_{xx}, v_x)-2\lambda(1, v_x^2)+\lambda^2(v, v_x)-(\exp[-\lambda\alpha x]v^\alpha v_t, v_x)\,.$$

Hence we have

$$(\exp[\lambda x]G, v_x)=(\exp[-\lambda\alpha x]v^\alpha v_t, v_x)-2|\lambda|(1, v_x^2)-\tfrac{1}{2}I_2\,, \tag{5.3}$$

where

$$I_2=\lambda^2\int_{\partial D_a} v^2\cos\langle\nu, x\rangle d\sigma+\int_{\partial D_a} v_x^2\cos\langle\nu, x\rangle d\sigma\,.$$

Hereafter let $\delta>0$ be sufficiently small. We multiply (5.1) with $-\exp[-\lambda\alpha x](v+\delta)^\alpha v_t$. Then

$$\begin{aligned}(\exp[\lambda(1-\alpha)x]G, (v+\delta)^\alpha v_t)&\\ =-(\exp[-\lambda\alpha x](v+\delta)^\alpha, v_tv_{xx})&+2\lambda(\exp[-\lambda\alpha x](v+\delta)^\alpha, v_tv_x)\\ -\lambda^2(\exp[-\lambda\alpha x](v+\delta)^\alpha, vv_t)&+(\exp[-2\lambda\alpha x](v+\delta)^\alpha, v^\alpha v_t^2)\,.\end{aligned} \tag{5.4}$$

Obviously

$$\begin{aligned}&-(\exp[-\lambda\alpha x](v+\delta)^\alpha, v_tv_{xx})\\ &=-\lambda\alpha(\exp[-\lambda\alpha x](v+\delta)^\alpha, v_tv_x)+\alpha(\exp[-\lambda\alpha x](v+\delta)^{\alpha-1}, v_tv_x^2)\\ &\quad+(\exp[-\lambda\alpha x](v+\delta)^\alpha, v_{tx}v_x)-\int_{\partial D_a}\exp[-\lambda\alpha x](v+\delta)^\alpha v_tv_x\cos\langle\nu, x\rangle d\sigma\,,\end{aligned}$$

$$\begin{aligned}(\exp[-\lambda\alpha x](v+\delta)^\alpha, v_{tx}v_x)&=-\frac{\alpha}{2}(\exp[-\lambda\alpha x](v+\delta)^{\alpha-1}, v_tv_x^2)\\ &\quad+\frac{1}{2}\int_{\partial D_a}\exp[-\lambda\alpha x](v+\delta)^\alpha v_x^2\cos\langle\nu, t\rangle d\sigma\,.\end{aligned}$$

Hence

$$\begin{aligned}&-(\exp[-\lambda\alpha x](v+\delta)^\alpha, v_tv_{xx})\\ &=-\lambda\alpha(\exp[-\lambda\alpha x](v+\delta)^\alpha, v_tv_x)+\frac{\alpha}{2}(\exp[-\lambda\alpha x](v+\delta)^{\alpha-1}, v_tv_x^2)\\ &\quad+\frac{1}{2}\int_{\partial D_a}\exp[-\lambda\alpha x](v+\delta)^\alpha v_x^2\cos\langle\nu, t\rangle d\sigma\\ &\quad-\int_{\partial D_a}\exp[-\lambda\alpha x](v+\delta)^\alpha v_tv_x\cos\langle\nu, x\rangle d\sigma\,.\end{aligned} \tag{5.5}$$

Multiplying (5.1) with $(\alpha/2)(v+\delta)^{-1}v_x^2$, it turns out that

$$\frac{\alpha}{2}(\exp[-\lambda\alpha x]v^{\alpha}(v+\delta)^{-1}, v_t v_x^2)=\frac{\alpha}{2}((v+\delta)^{-1}, v_x^2 v_{xx})-\lambda\alpha((v+\delta)^{-1}, v_x^3)$$

$$+\frac{\alpha}{2}\lambda^2(v(v+\delta)^{-1}, v_x^2)+\frac{\alpha}{2}(\exp[\lambda x]G, (v+\delta)^{-1}v_x^2)\,.$$

Here

$$((v+\delta)^{-1}, v_x^2 v_{xx})=\frac{1}{3}((v+\delta)^{-2}, v_x^4)+\frac{1}{3}\int_{\partial D_a}(v+\delta)^{-1}v_x^3\cos\langle\nu, x\rangle d\sigma\,.$$

So we have

$$\tag{5.6} \frac{\alpha}{2}(\exp[-\lambda\alpha x]v^{\alpha}(v+\delta)^{-1}, v_t v_x^2)$$

$$=\frac{\alpha}{6}((v+\delta)^{-2}, v_x^4)-\lambda\alpha((v+\delta)^{-1}, v_x^3)+\frac{\alpha}{2}\lambda^2(v(v+\delta)^{-1}, v_x^2)$$

$$+\frac{\alpha}{2}(\exp[\lambda x]G, (v+\delta)^{-1}v_x^2)+\frac{\alpha}{6}\int_{\partial D_a}(v+\delta)^{-1}v_x^3\cos\langle\nu, x\rangle d\sigma\,.$$

Let us set $E=\{(x, t)\in D_a\,;\, u(x, t)=0\}$. Then $v_x=v_t=0$ on E, since $v\geq 0$ there. We write

$$(\exp[-\lambda\alpha x]v^{\alpha}(v+\delta)^{-1}, v_t v_x^2)=\int_{D_a-E}\exp[-\lambda\alpha x]v^{\alpha}(v+\delta)^{-1}v_t v_x^2 dxdt\,.$$

From our assumptions it follows that

$$|u_{xx}|\leq u^{\alpha}|u_t|+2|t|u^{\alpha}|u_x|+|F|\leq\kappa M(K+M^{\alpha})\qquad\text{in}\quad Q_0\,.$$

Combining this inequality with Lemma 2, we obtain

$$\tag{5.7} u_x(x, t)^2/u(x, t)\leq\kappa M(K+M^{\alpha}+(a_0-a^2)^{-1})$$

for $(x, t)\in D_a-E$. Hence $(v+\delta)^{-1}v_x^2$ is uniformly bounded in D_a-E with respect to δ. This implies that

$$(\exp[-\lambda\alpha x]v^{\alpha}(v+\delta)^{-1}, v_t v_x^2)\longrightarrow(\exp[-\lambda\alpha x]v^{\alpha-1}, v_t v_x^2)$$

as $\delta\to+0$, where we define $v^{-1}v_x^2=0$ on E. Similarly, as $\delta\to+0$,

$$((v+\delta)^{-2}, v_x^4)\longrightarrow(v^{-2}, v_x^4)\,,$$

$$((v+\delta)^{-1}, v_x^3)\longrightarrow(v^{-1}, v_x^3)\,,$$

$$(v(v+\delta)^{-1}, v_x^3)\longrightarrow(1, v_x^3)\,,$$

$$\int_{\partial D_a} (v+\delta)^{-1} v_x^3 \cos\langle v, x\rangle d\sigma \longrightarrow \int_{\partial D_a} v^{-1} v_x^3 \cos\langle v, x\rangle d\sigma .$$

From these relations and (5.6) we deduce

$$\frac{\alpha}{2}(\exp[-\lambda\alpha x]v^{\alpha-1}, v_t v_x^2) = \frac{\alpha}{6}(v^{-2}, v_x^4) - \lambda\alpha(v^{-1}, v_x^3)$$

$$+\frac{\alpha}{2}\lambda^2(1, v_x^2) + \frac{\alpha}{2}(\exp[\lambda x]G, v^{-1}v_x^2) + \frac{\alpha}{6}\int_{\partial D_a} v^{-1}v_x^3\cos\langle v, x\rangle d\sigma .$$

Taking $\delta \to +0$ in (5.5), we combine these two equalities to have

$$-(\exp[-\lambda\alpha x]v^{\alpha}, v_t v_{xx})$$

$$= -\lambda\alpha(\exp[-\lambda\alpha x]v^{\alpha}, v_t v_x) + \frac{\alpha}{6}(v^{-2}, v_x^4) - \lambda\alpha(v^{-1}, v_x^3)$$

$$+\frac{\alpha}{2}\lambda^2(1, v_x^2) + \frac{\alpha}{2}(\exp[\lambda x]G, v^{-1}v_x^2) + \frac{\alpha}{6}\int_{\partial D_a} v^{-1}v_x^3\cos\langle v, x\rangle d\sigma$$

$$+\frac{1}{2}\int_{\partial D_a}\exp[-\lambda\alpha x]v^{\alpha}v_x^2\cos\langle v, t\rangle d\sigma - \int_{\partial D_a}\exp[-\lambda\alpha x]v^{\alpha}v_x v_t\cos\langle v, x\rangle d\sigma .$$

Taking $\delta \to +0$ in (5.4), we note that

$$(\exp[-\lambda\alpha x]v^{1+\alpha}, v_t) = \frac{1}{2+\alpha}\int_{\partial D_a}\exp[-\lambda\alpha x]v^{2+\alpha}\cos\langle v, t\rangle d\sigma .$$

Then it follows that

$$(\exp[\lambda(1-\alpha)x]G, v^{\alpha}v_t)$$

$$=(2-\alpha)\lambda(\exp[-\lambda\alpha x]v^{\alpha}, v_x v_t) + \frac{\alpha}{6}(v^{-2}, v_x^4) - \lambda\alpha(v^{-1}, v_x^3) + \frac{\alpha}{2}\lambda^2(1, v_x^2)$$

$$+(\exp[-2\lambda\alpha x]v^{2\alpha}, v_t^2) + \frac{\alpha}{2}(\exp[\lambda x]G, v^{-1}v_x^2) + I_3 ,$$

where

$$I_3 = \frac{\alpha}{6}\int_{\partial D_a} v^{-1}v_x^3\cos\langle v, x\rangle d\sigma - \int_{\partial D_a}\exp[-\lambda\alpha x]v^{\alpha}v_t v_x\cos\langle v, x\rangle d\sigma$$

$$+\frac{1}{2}\int_{\partial D_a}\exp[-\lambda\alpha x]v^{\alpha}v_x^2\cos\langle v, t\rangle d\sigma$$

$$-\frac{1}{2+\alpha}\lambda^2\int_{\partial D_a}\exp[-\lambda\alpha x]v^{2+\alpha}\cos\langle v, t\rangle d\sigma .$$

We rewrite this equality as follows:

$$(5.8)\qquad \frac{\alpha}{6(2-\alpha)}\frac{1}{|\lambda|}(v^{-2}, v_x^4)+\frac{\alpha}{2(2-\alpha)}|\lambda|(1, v_x^2)+\frac{1}{2-\alpha}\frac{1}{|\lambda|}(\exp[-2\lambda\alpha x]v^{2\alpha}, v_t^2)$$

$$=(\exp[-\lambda\alpha x]v^{\alpha}, v_x v_t)-\frac{\alpha}{2-\alpha}(v^{-1}, v_x^3)+\frac{1}{2-\alpha}\frac{1}{|\lambda|}(\exp[\lambda(1-\alpha)x]G, v^{\alpha}v_t)$$

$$-\frac{\alpha}{2(2-\alpha)}\frac{1}{|\lambda|}(\exp[\lambda x]G, v^{-1}v_x^2)-\frac{1}{2-\alpha}\frac{1}{|\lambda|}I_3 .$$

Summing up each side of (5.3) and (5.8), and using the inequality

$$|(v^{-1}, v_x^3)|\le\frac{\eta}{2}(v^{-2}, v_x^4)+\frac{1}{2\eta}(1, v_x^2), \qquad \eta>0,$$

we finally conclude that

$$(5.9)\qquad \frac{\alpha}{2(2-\alpha)}\left(\frac{1}{3|\lambda|}-\eta\right)(v^{-2}, v_x^4)+\left[\left(2+\frac{\alpha}{2(2-\alpha)}\right)|\lambda|-\frac{\alpha}{2(2-\alpha)}\frac{1}{\eta}\right](1, v_x^2)$$

$$+\frac{1}{2-\alpha}\frac{1}{|\lambda|}(\exp[-2\lambda\alpha x]v^{2\alpha}, v_t^2)$$

$$\le 2(\exp[-\lambda\alpha x]v^{\alpha}, v_x v_t)-(\exp[\lambda x]G, v_x)+\frac{1}{2-\alpha}\frac{1}{|\lambda|}(\exp[\lambda(1-\alpha)x]G, v^{\alpha}v_t)$$

$$-\frac{\alpha}{2(2-\alpha)}\frac{1}{|\lambda|}(\exp[\lambda x]G, v^{-1}v_x^2)+\frac{1}{2}|I_2|+\frac{1}{2-\alpha}\frac{1}{|\lambda|}|I_3| .$$

We set $\eta=1/(\gamma|\lambda|)$, where γ is the positive number in the statement of Theorem 2. Since $3<\gamma<5$, $\frac{1}{2}(5-\gamma)<8/(3+\gamma)$. Hence $\alpha<8/(3+\gamma)$. We write

$$2+\frac{\alpha}{2(2-\alpha)}-\frac{\alpha\gamma}{2(2-\alpha)}=\frac{8-(3+\gamma)\alpha}{2(2-\alpha)} .$$

And we set

$$A=\frac{8-(3+\gamma)\alpha}{2(2-\alpha)}|\lambda|, \qquad C=\frac{1}{(2-\alpha)|\lambda|},$$

$$H=AX^2-2XY+CY^2 .$$

Then we have

$$H\ge\frac{AC-1}{A}Y^2, \qquad \frac{AC-1}{C}X^2,$$

in virtue of A, $C>0$. Since

$$AC-1=\frac{\alpha(5-\gamma-2\alpha)}{2(2-\alpha)^2},$$

we have

$$2H\geq\frac{\alpha(5-\gamma-2\alpha)}{2(2-\alpha)}|\lambda|X^2+\frac{1}{2-\alpha}\frac{\alpha(5-\gamma-2\alpha)}{8-(3+\gamma)\alpha}\frac{1}{|\lambda|}Y^2.$$

Setting $X=v_x$, $Y=\exp[-\lambda\alpha x]v^\alpha v_t$, we combine this inequality with (5.9). Then it follows that

$$\frac{\alpha}{2(2-\alpha)}\left(\frac{1}{3}-\frac{1}{\gamma}\right)\frac{1}{|\lambda|}(v^{-2},v_x^4)+\frac{\alpha(5-\gamma-2\alpha)}{4(2-\alpha)}|\lambda|(1,v_x^2)$$

$$+\frac{1}{2(2-\alpha)}\frac{\alpha(5-\gamma-2\alpha)}{8-(3+\gamma)\alpha}\frac{1}{|\lambda|}(\exp[-2\lambda\alpha x]v^{2\alpha},v_t^2)$$

$$\leq-(\exp[\lambda x]G,v_x)+\frac{1}{2-\alpha}\frac{1}{|\lambda|}(\exp[\lambda(1-\alpha)x]G,v^\alpha v_t)$$

$$-\frac{\alpha}{2(2-\alpha)}\frac{1}{|\lambda|}(\exp[\lambda x]G,v^{-1}v_x^2)+\frac{1}{2}|I_2|+\frac{1}{2-\alpha}\frac{1}{|\lambda|}|I_3|.$$

Because of (5.2) we have

$$\frac{1}{4(2-\alpha)}|\lambda|(1,v_x^2)=\frac{1}{8(2-\alpha)}|\lambda|(1,v_x^2)+\frac{1}{8(2-\alpha)}|\lambda|^3(1,v^2)$$

$$+\frac{1}{8(2-\alpha)}|\lambda|(\exp[\lambda x]G,v)-\frac{1}{8(2-\alpha)}|\lambda|I_1$$

$$\geq\frac{1}{16}|\lambda|(1,v_x^2)+\frac{1}{16}|\lambda|^3(1,v^2)-\frac{1}{8}|\lambda||(\exp[\lambda x]G,v)|-\frac{1}{8}|\lambda||I_1|.$$

From the above inequalities we conclude that

$$\frac{\alpha}{4}\left(\frac{1}{3}-\frac{1}{\gamma}\right)\frac{1}{|\lambda|}(v^{-2},v_x^4)+\frac{\alpha}{16}(5-\gamma-2\alpha)|\lambda|(1,v_x^2)\tag{5.10}$$

$$+\frac{\alpha}{16}(5-\gamma-2\alpha)|\lambda|^3(1,v^2)+\frac{\alpha}{4}\frac{5-\gamma-2\alpha}{8-(3+\gamma)\alpha}\frac{1}{|\lambda|}(\exp[-2\lambda\alpha x]v^{2\alpha},v_t^2)$$

$$\leq|(\exp[\lambda x]G,v_x)|+\frac{1}{|\lambda|}|(\exp[\lambda(1-\alpha)x]G,v^\alpha v_t)|$$

$$+\frac{1}{|\lambda|}|(\exp[\lambda x]G,v^{-1}v_x^2)|+\frac{1}{4}|\lambda||(\exp[\lambda x]G,v)|$$

$$+\frac{1}{4}|\lambda||I_1|+\frac{1}{2}|I_2|+\frac{1}{|\lambda|}|I_3|.$$

Since $|u_x|\le\exp[-\lambda x](|v_x|+|\lambda|v)$, we get

$$\exp[\lambda x]|G|\le 2(K+aM^\alpha)(|v_x|+|\lambda|v).$$

And

$$|\lambda|(|v_x|+|\lambda|v)v,\ (|v_x|+|\lambda|v)|v_x|\le 2(v_x^2+\lambda^2v^2).$$

Hence

$$|\lambda||(\exp[\lambda x]G, v)|,\ |(\exp[\lambda x]G, v_x)|\le 4(K+aM^\alpha)[(1, v_x^2)+\lambda^2(1, v^2)].$$

Further

$$\begin{aligned}&\frac{1}{|\lambda|}|(\exp[\lambda(1-\alpha)x]G, v^\alpha v_t)|\\&\le\tfrac{1}{2}\lambda^{-2}(\exp[-2\lambda\alpha x]v^{2\alpha}, v_t^2)+4(K+aM^\alpha)^2[(1, v_x^2)+\lambda^2(1, v^2)],\\&\frac{1}{|\lambda|}|(\exp[\lambda x]G, v^{-1}v_x^2)|\\&\le\tfrac{1}{2}\lambda^{-2}(v^{-2}, v_x^4)+4(K+aM^\alpha)^2[(1, v_x^2)+\lambda^2(1, v^2)].\end{aligned}$$

From the above inequalities we infer

$$\begin{aligned}&|\lambda||(\exp[\lambda x]G, v)|+|(\exp[\lambda x]G, v_x)|\\&\quad+\frac{1}{|\lambda|}|(\exp[\lambda x]G, v^{-1}v_x^2)|+\frac{1}{|\lambda|}|(\exp[\lambda(1-\alpha)x]G, v^\alpha v_t)|\\&\le 8(K+aM^\alpha)(1+K+aM^\alpha)[(1, v_x^2)+\lambda^2(1, v^2)]\\&\quad+\tfrac{1}{2}\lambda^{-2}(v^{-2}, v_x^4)+\tfrac{1}{2}\lambda^{-2}(\exp[-2\lambda\alpha x]v^{2\alpha}, v_t^2).\end{aligned}$$

Combining this with (5.10), we obtain

$$\begin{aligned}(5.11)\qquad&\frac{1}{|\lambda|}\left[\frac{\alpha}{4}\left(\frac{1}{3}-\frac{1}{\gamma}\right)-\frac{1}{2|\lambda|}\right](v^{-2}, v_x^4)\\&+\left[\frac{\alpha}{16}(5-\gamma-2\alpha)|\lambda|-8(K+aM^\alpha)(1+K+aM^\alpha)\right][(1, v_x^2)+\lambda^2(1, v^2)]\\&+\frac{1}{2|\lambda|}\left(\frac{\alpha}{2}\frac{5-\gamma-2\alpha}{8-(3+\gamma)\alpha}-\frac{1}{|\lambda|}\right)(\exp[-2\lambda\alpha x]v^{2\alpha}, v_t^2)\\&\le|\lambda||I_1|+|I_2|+|\lambda|^{-1}|I_3|.\end{aligned}$$

Now we estimate the line integrals I_1, I_2 and I_3. We note that $\cos\langle\nu, t\rangle=0$ on L_a.

First we see that

$$
\begin{aligned}
|I_1| &\le |\lambda| \int_{\partial D_a} v^2 d\sigma + \int_{\partial D_a} v|v_x| d\sigma + \frac{1}{2} M^\alpha \int_{\Gamma_a} v^2 d\sigma \\
&\le \kappa\left[\int_{\partial D_a} \exp[2\lambda x](\lambda^2 u^2 + u_x^2) d\sigma + M^\alpha \int_{\Gamma_a} u^2 d\sigma\right] \\
&\le \kappa\left[\int_{\Gamma_a} (\lambda^2 u^2 + u_x^2) d\sigma + M^\alpha \int_{\Gamma_a} u^2 d\sigma + a\lambda^2 \exp[2\lambda a^2] M^2\right],
\end{aligned}
$$

$$
\begin{aligned}
|I_2| &\le \lambda^2 \int_{\partial D_a} v^2 d\sigma + \int_{\partial D_a} v_x^2 d\sigma \\
&\le \kappa \int_{\partial D_a} \exp[2\lambda x](\lambda^2 u^2 + u_x^2) d\sigma \\
&\le \kappa\left[\int_{\Gamma_a} (\lambda^2 u^2 + u_x^2) d\sigma + a\lambda^2 \exp[2\lambda a^2] M^2\right].
\end{aligned}
$$

Let $\tilde{a} = (a_0 - a^2)^{-1}$. Then from (5.7) it follows that

$$
v^{-1} v_x^2 \le \kappa \exp[\lambda x][M(K + M^\alpha + \tilde{a}) + \lambda^2 u],
$$

$$
v^{-1} |v_x|^3 \le \kappa \exp[2\lambda x][M(K + M^\alpha + \tilde{a})(|\lambda| u + |u_x|) + \lambda^4 u^2 + u_x^2].
$$

Hence

$$
\begin{aligned}
|I_3| \le \kappa &\int_{\partial D_a} \exp[2\lambda x][M(K + M^\alpha + \tilde{a})(|\lambda| u + |u_x|) + \lambda^4 u^2 + u_x^2] d\sigma \\
&+ \kappa M^\alpha \int_{\partial D_a} \exp[2\lambda x](\lambda^2 u^2 + u_x^2 + u_t^2) d\sigma + \kappa M^\alpha \int_{\Gamma_a} (\lambda^2 u^2 + u_x^2) d\sigma \\
\le \kappa &\left[M(K + M^\alpha + \tilde{a}) \int_{\Gamma_a} (|\lambda| u + |u_x|) d\sigma \right. \\
&\left. + \int_{\Gamma_a} (\lambda^4 u^2 + u_x^2) d\sigma + M^\alpha \int_{\Gamma_a} (\lambda^2 u^2 + u_x^2 + u_t^2) d\sigma \right] \\
&+ \kappa [a|\lambda| \exp[2\lambda a^2] \cdot M^2 (K + M^\alpha + \tilde{a}) \\
&\quad + a\lambda^4 \exp[2\lambda a^2] \cdot M^2 + a\lambda^2 \exp[2\lambda a^2] \cdot M^{2+\alpha}].
\end{aligned}
$$

Let ε be the number in the statement of Theorem 2. We set $J = |\lambda||I_1| + |I_2| + |\lambda|^{-1}|I_3|$. Then from the above argument,

$$
\begin{aligned}
J \le \kappa &[\varepsilon^2 |\lambda|^3 + \varepsilon^2 |\lambda| M^\alpha + \varepsilon M(K + M^\alpha + \tilde{a})] \\
&+ \kappa a \exp[2\lambda a^2] \cdot M^2 [|\lambda|^3 + (K + \tilde{a}) + |\lambda| M^\alpha]
\end{aligned}
$$

$$\leq \kappa\varepsilon[|\lambda|^3+|\lambda|M^\alpha+M(K+M^\alpha+\tilde{a})]$$
$$+\kappa a\exp[2\lambda a^2]\cdot M^2[|\lambda|^3+(K+\tilde{a})+|\lambda|M^\alpha].$$

Let $|\lambda|\geq\max(1, K+\tilde{a}+M+M^\alpha)$. Then we conclude that

$$J\leq\kappa|\lambda|^3(\varepsilon+a\exp[2\lambda a^2]M^2). \tag{5.12}$$

We rewrite (5.11). Then

$$\begin{aligned}&\frac{1}{2|\lambda|}\left(\frac{\alpha(\gamma-3)}{6\gamma}-\frac{1}{|\lambda|}\right)(v^{-2}, v_x^4)\\&+\frac{1}{16}\alpha\beta(|\lambda|-\frac{128}{\alpha\beta}(1+K+aM^\alpha)^2)[\lambda^2(1, v^2)+(1, v_x^2)]\\&+\frac{1}{2|\lambda|}\left(\frac{\alpha\beta}{2(8-(3+\gamma)\alpha)}-\frac{1}{|\lambda|}\right)(u^{2\alpha}, v_t^2)\leq J.\end{aligned}\tag{5.13}$$

Further we impose on λ the assumption

$$|\lambda|\geq\max\left(\frac{6\gamma}{\alpha(\gamma-3)}, \frac{256}{\alpha\beta}(1+K+aM^\alpha)^2\right).$$

Then

$$|\lambda|-\frac{128}{\alpha\beta}(1+K+aM^\alpha)^2>\frac{128}{\alpha\beta}(1+K+aM^\alpha)^2.$$

Since $|\lambda|>32/(\alpha\beta)$, we see that

$$\frac{\alpha\beta}{2(8-(3+\gamma)\alpha)}-\frac{1}{|\lambda|}>\frac{\alpha\beta}{16}-\frac{\alpha\beta}{32}=\frac{\alpha\beta}{32}.$$

Hence (5.13) becomes

$$8(1+K+aM^\alpha)^2[\lambda^2(1, v^2)+(1, v_x^2)]+\frac{1}{64}\frac{\alpha\beta}{|\lambda|}(u^{2\alpha}, v_t^2)\leq J,$$

namely

$$\lambda^2(1, v^2)+(1, v_x^2)+\alpha\beta(u^{2\alpha}, v_t^2)\leq\kappa|\lambda|J.$$

Combining this with (5.12), we finally obtain

$$(\exp[2\lambda x], u^2)+(\exp[2\lambda x], u_x^2)+\alpha\beta(\exp[2\lambda x]u^{2\alpha}, u_t^2)$$
$$\leq\kappa\lambda^4(\varepsilon+M^2\exp[2\lambda a^2]).$$

On the right-hand side of this inequality, we reduce the integral domain D_a to $D_{a/\sqrt{2}}$. Then

$$\exp[\lambda a^2]\cdot\int_{D_{a/\sqrt{2}}}(u^2+u_x^2+\alpha\beta u^{2\alpha}u_t^2)dxdt\le\kappa\lambda^4(\varepsilon+M^2\exp[2\lambda a^2])\,.$$

Using the trivial inequalities

$$x^4\exp[x]\le\kappa\exp[4x/3]\,,\quad x^4\exp[-x]\le\kappa\exp[-2x/3]\qquad(x>0)\,,$$

we have

$$\lambda^4\exp[|\lambda|a^2]\le\kappa a^{-8}\exp[4|\lambda|a^2/3]\,,\qquad|\lambda|^4\exp[-|\lambda|a^2]\le\kappa a^{-8}\exp[-2|\lambda|a^2/3]\,.$$

Thereby

$$\int_{D_{a/\sqrt{2}}}(u^2+u_x^2+\alpha\beta u^{2\alpha}u_t^2)dxdt\le\kappa a^{-8}(\varepsilon\exp[4|\lambda|a^2/3]+M^2\exp[-2|\lambda|a^2/3])\,.$$

Here we set $|\lambda|=\frac{1}{2}a^{-2}\log(M/\varepsilon)$. Then

$$\exp[4|\lambda|a^2/3]=(M/\varepsilon)^{2/3}\,,\qquad\exp[-2|\lambda|a^2/3]=(\varepsilon/M)^{1/3}\,.$$

Therefore we obtain finally

$$\int_{D_{a/\sqrt{2}}}(u^2+u_x^2+\alpha\beta u^{2\alpha}u_t^2)dxdt\le\kappa a^{-8}(\varepsilon^{1/3}M^{2/3}+\varepsilon^{1/3}M^{5/3})\,.$$

We have completed the proof of Theorem 2.

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Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KANAZAWA UNIVERSITY,
KAKUMA-MACHI, KANAZAWA, 920–11 JAPAN.