TOKYO J. MATH.
VOL. 20, NO. 1, 1997

Realization of Spaces $E_6/(U(1)Spin(10))$, $E_7/(U(1)E_6)$, $E_8/(U(1)E_7)$ and Their Volumes

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(Communicated by T. Nagano)

Freudenthal [3], [4] introduced cross operations $\times$ on the spaces $\mathfrak{J}^C$, $\mathfrak{P}^C$ and $\mathfrak{e}_8{}^C$, respectively. Using these operations, we give definite expressions of the exceptional symmetric spaces *EIII*, *EVII* and the twister space $Z(EIX)$ of the exceptional symmetric space *EIX* as follows:

$$EIII=\{X\in\mathfrak{J}^C \mid X\times X=0, X\neq 0\}/C^*\cong E_6/(U(1)\times Spin(10))/\mathbf{Z}_4\,,$$

$$EVII=\{P\in\mathfrak{P}^C \mid P\times P=0, P\neq 0\}/C^*\cong E_7/(U(1)\times E_6)/\mathbf{Z}_3\,,$$

$$Z(EIX)=\{R\in\mathfrak{e}_8{}^C \mid R\times R=0, R\neq 0\}/C^*\cong E_8/(U(1)\times E_7)/\mathbf{Z}_2\,.$$

We then compute the first Chern classes by using the above expressions and calculate the volumes $\mu(M, g)$ of those spaces M with respect to the metric g indicated in 1.2, 2.2 and 3.2. This follows from the cohomology ring structures of the spaces together with the Borel-Hirzebruch formula. The results enable us calculating the volumes of the symmetric spaces of exceptional type [1]. Our results are as follows:

$$\mu(EIII, g)=\frac{78}{16!}\pi^{16}\,,\qquad \mu(EVII, g)=\frac{13110}{27!}\pi^{27}\,,$$

$$\mu(Z(EIX), g)=\frac{2^{12}\,3^2\,5^2\,7\,31\,37\,41\,43\,47\,53}{57!}\pi^{57}\,.$$

1. *EIII*.

1.1. The Lie group E_6 and its Lie algebra $\mathfrak{e}_6$. Let $\mathfrak{C}$ be the Cayley algebra and $\mathfrak{J}=\{X\in M(3, \mathfrak{C}) \mid X^*=X\}$ be the exceptional Jordan algebra with the Jordan multiplication $X\circ Y=\frac{1}{2}(XY+YX)$ and the Freudenthal multiplication

$$X\times Y=\tfrac{1}{2}(2X\circ Y-\mathrm{tr}(X)Y-\mathrm{tr}(Y)X+(\mathrm{tr}(X)\mathrm{tr}(Y)-(X, Y))E)\,.$$

Received June 23, 1995

We give an inner product of $\mathfrak{J}$ by $(X, Y)=\mathrm{tr}(X\circ Y)$. Let $\mathfrak{J}^C=\mathfrak{J}\oplus i\mathfrak{J}$ be the complexification of $\mathfrak{J}$ and we define the Hermitian inner product $\langle X, Y\rangle$ of $\mathfrak{J}^C$ by $(\tau X, Y)$, where τ is the complex conjugation in $\mathfrak{J}^C$. For $x\in\mathfrak{C}^C=\mathfrak{C}\oplus i\mathfrak{C}$ we define the following matrices which belong to $\mathfrak{J}^C$:

$$E_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad E_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad E_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$F_1(x)=\begin{pmatrix}0&0&0\\0&0&x\\0&\bar{x}&0\end{pmatrix},\quad F_2(x)=\begin{pmatrix}0&0&\bar{x}\\0&0&0\\x&0&0\end{pmatrix},\quad F_3(x)=\begin{pmatrix}0&x&0\\\bar{x}&0&0\\0&0&0\end{pmatrix}.$$

Now the simply connected compact Lie group E_6 is obtained by [10] as

$$E_6=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C)\mid \alpha X\times\alpha X=\tau\alpha\tau(X\times Y),\ \langle\alpha X, \alpha Y\rangle=\langle X, Y\rangle\}.$$

The Lie algebra $\mathfrak{e}_6$ of E_6 is given by

$$\mathfrak{e}_6=\{\phi\in\mathrm{Hom}_C(\mathfrak{J}^C, \mathfrak{J}^C)\mid(\phi X, X\times X)=0,\ \langle\phi X, Y\rangle+\langle X, \phi Y\rangle=0\}.$$

Each element $\phi\in\mathfrak{e}_6$ has the representation as

$$\phi=\delta+i\tilde{T},\qquad \delta\in\mathfrak{f}_4,\ T\in\mathfrak{J}_0.$$

Here $\mathfrak{f}_4=\{\delta\in\mathfrak{e}_6\mid\delta E=0\}$, $\mathfrak{J}_0=\{T\in\mathfrak{J}\mid\mathrm{tr}(T)=0\}$ and $\tilde{T}$ is the C-linear mapping of $\mathfrak{J}^C$ defined by $\tilde{T}X=T\circ X$, $X\in\mathfrak{J}^C$.

1.2. The compact Hermitian symmetric space *EIII*. We define the C-linear transformation σ of $\mathfrak{J}^C$ by

$$\sigma\begin{pmatrix}\xi_1&x_3&\bar{x}_2\\\bar{x}_3&\xi_2&x_1\\x_2&\bar{x}_1&\xi_3\end{pmatrix}=\begin{pmatrix}\xi_1&-x_3&-\bar{x}_2\\-\bar{x}_3&\xi_2&x_1\\-x_2&\bar{x}_1&\xi_3\end{pmatrix}.$$

Then $\sigma\in E_6$ and $\sigma^2=1$. σ induces the involutive automorphism $\tilde{\sigma}$ of E_6 given by

$$\tilde{\sigma}(\alpha)=\sigma\alpha\sigma,\qquad \alpha\in E_6.$$

We verified that the subgroup $(E_6)^\sigma=\{\alpha\in E_6\mid\sigma\alpha\sigma=\alpha\}$ of E_6 is

$$(E_6)^\sigma=U(1)Spin(10)\cong(U(1)\times Spin(10))/\mathbf{Z}_4,$$

where $U(1)$ and $Spin(10)$ are subgroups of E_6 defined by [10] as

$$U(1)=\{\phi(\theta)\mid\theta\in C,\ (\tau\theta)\theta=1\},$$

$$Spin(10)=(E_6)_{E_1}=\{\alpha\in E_6\mid\alpha E_1=E_1\}.$$

Here $\phi(\theta)$ is the C-linear transformation of $\mathfrak{J}^C$ defined by

$$\phi(\theta)\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}=\begin{pmatrix} \theta^4\xi_1 & \theta x_3 & \theta\bar{x}_2 \\ \theta\bar{x}_3 & \theta^{-2}\xi_2 & \theta^{-2}x_1 \\ \theta x_2 & \theta^{-2}\bar{x}_1 & \theta^{-2}\xi_3 \end{pmatrix}.$$

Now, we define the space $EIII$ by

$$EIII=\{X\in\mathfrak{J}^C \mid X\times X=0,\ X\neq 0\}/\sim\ ,$$

where $X\sim Y$ if $Y=aX$ for some $a\in C$, $a\neq 0$.

PROPOSITION 1.1. $EIII\cong E_6/(E_6)^\sigma=E_6/(U(1)Spin(10))$.

PROOF. Let $[X]$ denote an element of $EIII$ which represents $X\in\mathfrak{J}^C$ satisfying $X\times X=0$, $X\neq 0$. The subgroup E_6 acts transitively on $EIII$. In fact, each element $[X]\in EIII$ can be transformed to $[E_1]\in EIII$ by some $\alpha\in E_6$. And the isotropy subgroup of E_6 at $[E_1]$ is $(E_6)^\sigma=U(1)Spin(10)$ (see [10]). Thus Proposition 1 follows.

The tangent space $T_o(EIII)$ of $EIII$ at $o=[E_1]$ is written as

$$\begin{aligned} T_o(EIII)&=\{X\in\mathfrak{J}^C \mid X\times E_1=0,\ \langle X, E_1\rangle=0\} \\ &=\{F_2(x_2)+F_3(x_3) \mid x_2, x_3\in\mathfrak{C}^C\}\,. \end{aligned}$$

For $x\in\mathfrak{C}$ let

$$A_2(x)=\begin{pmatrix} 0 & 0 & -\bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix},\quad A_3(x)=\begin{pmatrix} 0 & x & 0 \\ -\bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and define $\tilde{A}_k(x)X=A_k(x)X-XA_k(x)$ for $X\in\mathfrak{J}^C$ $(k=2, 3)$. Then $\tilde{A}_k(x)\in\mathfrak{f}_4$. Put

$$\mathfrak{k}=(\mathfrak{e})_\sigma=\{\phi\in\mathfrak{e}_6 \mid \sigma_*\phi=\phi\}\,,$$

$$\begin{aligned} \mathfrak{m}&=(\mathfrak{e}_6)_{-\sigma}=\{\phi\in\mathfrak{e}_6 \mid \sigma_*\phi=-\phi\} \\ &=\{(\tilde{A}_2(x_2)+\tilde{A}_3(x_3))+i(\tilde{F}_2(x_2')+\tilde{F}_3(x_3')) \mid x_k, x_k'\in\mathfrak{C}\ (k=2, 3)\}\,. \end{aligned}$$

Then the involutive automorphism $\tilde{\sigma}$ of the group E_6 induces the canonical decomposition $\mathfrak{e}_6=\mathfrak{k}\oplus\mathfrak{m}$. By Proposition 1 the natural projection p: $E_6\to EIII$ induces an isomorphism $p_*:\mathfrak{m}\to T_o(EIII)$. We see that $p_*(\phi)=\phi(E_1)$ and

$$p_*((\tilde{A}_2(x_2)-\tilde{A}_3(x_3))+2i(\tilde{F}_2(x_2')+\tilde{F}_3(x_3')))=F_2(x_2+ix_2')+F_3(x_3+ix_3')$$

for $x_k, x_k'\in\mathfrak{C}$ $(k=2, 3)$. We give the Hermitian metric on $T_o(EIII)$ induced from $\mathfrak{J}^C$. Since the metric is $U(1)Spin(10)$-invariant, this metric can be extended to an E_6-invariant metric on $EIII$. Thus $EIII$ has an Hermitian symmetric structure with this metric and the Riemannian metric g is given by the real restriction: $g(X, Y)=\mathrm{Re}\langle X, Y\rangle$, $X, Y\in T_o(EIII)$.

1.3. The first Chern form c_1 of $EIII$. We shall calculate the first Chern form c_1

of $EIII$. It suffices to compute c_1 at o. Let R be the curvature of $EIII$ and $\{Z_j \mid j=1, \cdots, 16\}$ be an orthonormal C-basis of $T_o(EIII)$. For $X, Y \in T_o(EIII)$, we have

$$c_1(X, Y) = -\frac{1}{2\pi i} \sum_{j=1}^{16} \langle R(X, Y)_0 Z_j, Z_j \rangle = \frac{1}{2\pi i} \sum_{j=1}^{16} \langle p_*([[\hat{X}, \hat{Y}], \hat{Z}_j]), Z_j \rangle ,$$

where $\hat{X}, \hat{Y}, \hat{Z}_j \in \mathfrak{m}$ such that $p_*(\hat{X}) = X$, $p_*(\hat{Y}) = Y$, $p_*(\hat{Z}_j) = Z_j$.

Let $\{e_0, e_1, \cdots, e_7\}$ be the canonical basis of $\mathfrak{C}$. Hereafter we take $\{(1/\sqrt{2})F_k(e_j) \mid k=2, 3, j=0, 1, \cdots, 7\}$ as an orthonormal C-basis of $T_o(EIII)$. Let

$$X = F_2(x_2 + ix_2') + F_3(x_3 + ix_3') , \qquad Y = F_2(y_2 + iy_2') + F_3(y_3 + iy_3') .$$

Put

$$A = A_2(x_2) - A_3(x_3) , \qquad B = A_2(y_2) - A_3(y_3) ,$$
$$F = F_2(x_2') + F_3(x_3') , \qquad G = F_2(y_2') + F_3(y_3') .$$

Then

$$\hat{X} = \tilde{A} + 2i\tilde{F} , \qquad \hat{Y} = \tilde{B} + 2i\tilde{G} .$$

Put $Z = (1/\sqrt{2})F_k(e_j)$ and $\hat{Z} = (1/\sqrt{2})\tilde{A}_k(e_j)$ $(k=2, 3, j=0, 1, \cdots, 7)$. Then $p_*(\hat{Z}) = Z$. Note that

$$\langle p_*([[\hat{X}, \hat{Y}], \hat{Z}]), Z \rangle = \langle [[\tilde{A} + 2i\tilde{F}, \tilde{B} + 2i\tilde{G}], \hat{Z}_j]E_1, Z \rangle$$
$$= i\langle (([\tilde{A}, 2\tilde{G}] - [\tilde{B}, 2\tilde{F}])\hat{Z} - \hat{Z}([\tilde{A}, 2\tilde{G}] - [\tilde{B}, 2\tilde{F}])E_1), Z \rangle .$$

First consider the case $k=2$. We have

$$\begin{aligned}
&\langle i[\tilde{A}, 2\tilde{G}]\hat{Z}E_1, Z \rangle \\
&\quad = -i\langle [\tilde{A}_2(x_2) - \tilde{A}_3(x_3), 2\tilde{F}_2(y_2') + 2\tilde{F}_3(y_3')]Z, Z \rangle \\
&\quad = -(i/2)\langle -F_2((e_j y_3')\bar{x}_3 + (e_j x_3)\bar{y}_3') + F_3(***), F_2(e_j) \rangle \\
&\quad = -(i/2)(-2(e_j y_3', e_j x_3) - 2(e_j x_3, e_j y_3')) = 2i(x_3, y_3') ,
\end{aligned}$$

$$\begin{aligned}
&\langle -i\hat{Z}[\tilde{A}, 2\tilde{G}]E_1, Z \rangle \\
&\quad = i\langle A_2(e_j)[\tilde{A}_2(x_2) - \tilde{A}_3(x_3), 2\tilde{F}_2(y_2') + 2\tilde{F}_3(y_3')]E_1, F_2(e_j) \rangle \\
&\quad = (i/2)\langle -4(x_2, y_2')F_2(e_j) - 4(x_3, y_3')F_2(e_j) + F_3(***), F_2(e_j) \rangle \\
&\quad = (i/2)(-8(x_2, y_2') - 8(x_3, y_3'))(e_j, e_j) = -4i((x_2, y_2') + (x_3, y_3')) .
\end{aligned}$$

In the same way we have

$$\langle -i[\tilde{B}, 2\tilde{F}]\hat{Z}E_1, Z \rangle = -2i(y_3, x_3') ,$$
$$\langle i\hat{Z}[\tilde{B}, 2\tilde{F}]E_1, Z \rangle = 4i((y_2, x_2') + (y_3, x_3')) .$$

Thus

$$\langle p_*([[\hat{X}, \hat{Y}], \hat{Z}]), Z\rangle$$
$$= -4i(x_2, y_2') - 2i(x_3, y_3') + 4i(y_2, x_2') + 2i(y_3, x_3') .$$

Similarly we can calculate in the case of $k=3$ as follows:

$$\langle p_*([[\hat{X}, \hat{Y}], \hat{Z}]), Z\rangle$$
$$= -4i(x_3, y_3') - 2i(x_2, y_2') + 4i(y_3, x_3') + 2i(y_2, x_2') .$$

Sum up them we have

$$\begin{aligned} c_1(X, Y) &= -(1/2\pi i) \sum_{k=2}^{3} \sum_{j=0}^{7} \langle p_*([[\hat{X}, \hat{Y}], (1/\sqrt{2})\tilde{A}_k(e_j)]), (1/\sqrt{2})F_k(e_j)\rangle \\ &= (1/2\pi i) \sum_{j=0}^{7} 6i(-(x_2, y_2') - (x_3, y_3') + (x_2', y_2) + (x_3', y_3)) \\ &= -(24/\pi)(-(x_2, y_2') - (x_3, y_3') + (x_2', y_2) + (x_3', y_3)) \\ &= -(12/\pi)\mathrm{Im}\langle X, Y\rangle . \end{aligned}$$

Let $\{dx_{kj}, dx_{kj}' \mid k=2, 3, j=0, 1, \cdots, 7\}$ be the dual basis of $\{(1/\sqrt{2})F_k(e_j), (i/\sqrt{2})F_k(e_j) \mid k=2, 3, j=0, 1, \cdots, 7\}$ which is a real basis of $T_o(EIII)$. Then the first Chern form c_1 at o is written as follows:

$$c_1 = -\frac{12}{\pi} \sum_{k=2}^{3} \sum_{j=0}^{7} dx_{kj} \wedge dx_{kj}' .$$

1.4. The volume of *EIII*. Toda-Watanabe [8] determined the integral cohomology ring of *EIII* as

$$H^*(EIII) = \mathbf{Z}[t, w]/(\rho_1, \rho_2) ,$$

where $t \in H^2$, $w \in H^8$ and $\rho_1 = t^9 - 3w^2t$, $\rho_2 = w^3 + 15w^2t^4 - 9wt^8$. Combining the relations $\rho_1 = 0$ and $\rho_2 = 0$ we have

$$78wt^{12} = 45t^{16} , \qquad 78w^2t^8 = 26t^{16} .$$

It follows that $m = (1/78)t^{16}$ gives a generator of $H^{32}(EIII)$. (Explicitly $m = 11wt^{12} - 19w^2t^8$).

According to Borel-Hirzebruch [5], §16, 2-form $(1/12)c_1$ represents a generator of $H^2(EIII)$. Using 1.3 the form

$$\frac{1}{78\ 12^{16}} c_1^{16} = \frac{16!}{78\pi^{16}} dx_{20} \wedge dx_{20}' \wedge \cdots \wedge dx_{37} \wedge dx_{37}'$$

represents the cohomology class m.

Remark. By Borel-Hirzebruch formula [5], Theorem 24.10 the Chern number $c_1^{16}[EIII]$ is calculated as

$$c_1^{16}[EIII]=16!\prod_{b\in\psi}\frac{(d,b)}{(a/2,b)}=78\ 12^{16},$$

where ψ is the set of positive complementary roots of E_6 to those of $U(1)Spin(10)$, a the sum of positive roots of E_6 and $d=\sum_{b\in\psi}b$.

From the above argument, we get

$$1=\int_{EIII}\frac{16!}{78\pi^{16}}dx_{21}\wedge dx'_{21}\wedge\cdots\wedge dx_{37}\wedge dx'_{37}=\frac{16!}{78\pi^{16}}\mu(EIII,g).$$

Thus we have

THEOREM 1.2. $\mu(EIII,g)=\dfrac{78}{16!}\pi^{16}$.

2. *EVII.*

2.1. The Lie group E_7 and its Lie algebra $\mathfrak{e}_7$. Let $\mathfrak{P}^C$ denote the C-vector space $\mathfrak{J}^C\oplus\mathfrak{J}^C\oplus C\oplus C$. For $\phi\in\mathfrak{e}_6^C$, $A,B\in\mathfrak{J}^C$, $\nu\in C$, we define the C-linear transformation $\Phi(\phi,A,B,\nu)$ of $\mathfrak{P}^C$ by

$$\Phi(\phi,A,B,\nu)\begin{pmatrix}X\\Y\\\xi\\\eta\end{pmatrix}=\begin{pmatrix}\phi X-\frac{1}{3}\nu X+2B\times Y+\eta A\\2A\times X-{}^t\phi Y+\frac{1}{3}\nu Y+\xi B\\(A,Y)+\nu\xi\\(B,X)-\nu\eta\end{pmatrix}.$$

For $P=(X,Y,\xi,\eta)$, $Q=(Z,W,\zeta,\omega)\in\mathfrak{P}^C$, we define the C-linear transformation $P\times Q$ of $\mathfrak{P}^C$ by

$$P\times Q=\Phi(\phi,A,B,\nu),\qquad\begin{cases}\phi=-\frac{1}{2}(X\vee W+Z\vee Y)\\A=-\frac{1}{4}(2Y\times W-\xi Z-\zeta X)\\B=\frac{1}{4}(2X\times Z-\eta W-\omega Y)\\\nu=\frac{1}{8}((X,W)+(Z,W)-3(\xi\omega+\zeta\eta)),\end{cases}$$

where $X\vee W\in\mathfrak{e}_6^C$ is defined as

$$(X\vee W)Z=\frac{1}{2}(W,Z)X+\frac{1}{6}(X,W)Z-2W\times(X\times Z),\qquad Z\in\mathfrak{e}_6^C.$$

Finally we define the Hermitian inner product $\langle P,Q\rangle$ of $\mathfrak{P}^C$ by

$$\langle P,Q\rangle=\langle X,Z\rangle+\langle Y,W\rangle+(\tau\xi)\zeta+(\tau\eta)\omega$$

for $P=(X,Y,\xi,\eta)$, $Q=(Z,W,\zeta,\omega)\in\mathfrak{P}^C$.

Now the simply connected compact Lie group E_7 is obtained by [6] as

$$E_7=\{\alpha\in \mathrm{Iso}_C(\mathfrak{P}^C)\,|\,\alpha(P\times Q)\alpha^{-1}=\alpha P\times \alpha Q,\ \langle\alpha P, \alpha Q\rangle=\langle P, Q\rangle\}$$

and its Lie algebra $\mathfrak{e}_7$ is given by

$$\mathfrak{e}_7=\{\Phi(\phi, A,\ -\tau A, \nu)\in \mathrm{Hom}_C(\mathfrak{P}^C, \mathfrak{P}^C)\,|\,\phi\in\mathfrak{e}_6,\ A\in\mathfrak{J}^C,\ \nu\in i\mathbf{R}\}$$

with the Lie bracket

$$[\Phi(\phi_1, A_1, -\tau A_1, \nu_1), \Phi(\phi_2, A_2, -\tau A_2, \nu_2)]=\Phi(\phi, A, -\tau A, \nu)\,,$$

$$\begin{cases}\phi=[\phi_1, \phi_2]-2A_1\vee\tau A_2+2A_2\vee\tau A_1\\ A=(\phi_1+\frac{2}{3}\nu_1)A_2-(\phi_2+\frac{2}{3}\nu_2)A_1\\ \nu\ =\langle A_1, A_2\rangle-\langle A_2, A_1\rangle\,.\end{cases}$$

2.2. The compact Hermitian symmetric space *EVII.* We define the C-linear transformation ι of $\mathfrak{P}^C$ by

$$\iota(X, Y, \xi, \eta)=(-iX, iY, -i\xi, i\eta)\,.$$

Then $\iota\in E_7$ and $\iota^2=-1$, $\iota^4=1$. ι induces the involutive automorphism $\tilde{\iota}$ of E_7 given by

$$\tilde{\iota}(\alpha)=\iota\alpha\iota^{-1}\,,\qquad \alpha\in E_7\,.$$

We verified that the subgroup $(E_7)^\iota=\{\alpha\in E_7\,|\,\iota\alpha\iota^{-1}=\alpha\}$ of E_7 is

$$(E_7)^\iota=U(1)E_6\cong(U(1)\times E_6)/\mathbf{Z}_3\,,$$

where $U(1)$ and E_6 are subgroups of E_7 defined by [6] as

$$U(1)=\{\phi(\theta)\,|\,\theta\in C,\ (\tau\theta)\theta=1\}\,,$$
$$E_6=(E_7)_{\dot{1}}=\{\alpha\dot{1}=\dot{1}\}\,.$$

Here $\phi(\theta)$ is the C-linear transformation of $\mathfrak{P}^C$ defined by

$$\phi(\theta)(X, Y, \xi, \eta)=(\theta^{-1}X, \theta Y, \theta^3\xi, \theta^{-3}\eta)$$

and $\dot{1}=(0, 0, 1, 0)$.

Now, we define the space *EVII* by

$$EVII=\{P\in\mathfrak{P}^C\,|\,P\times P=0,\ P\neq 0\}/\sim\,,$$

where $P\sim Q$ if $P=aQ$ for some $a\in C$, $a\neq 0$.

PROPOSITION 2.1. $EVII\cong E_7/(E_7)^\iota=E_7/U(1)E_6$.

PROOF. Let $[P]$ denote an element of *EVII* which represents $P\in\mathfrak{P}^C$ satisfying $P\times P=0$ and $P\neq 0$. The group E_7 acts transitively on *EVII*. In fact each element $[P]\in EVII$ can be transformed to $[\dot{1}]\in EVII$ by some $\alpha\in E_7$. And the isotropy subgroup of E_7 at $[\dot{1}]$ is $U(1)E_6$ (see [6]). Thus Proposition 2.1 follows.

The tangent space $T_o(EVII)$ of $EVII$ at $o=[\dot{1}]$ is written as

$$T_o(EVII)=\{P\in\mathfrak{P}^C\,|\,P\times\dot{1}=0,\ \langle P,\dot{1}\rangle=0\}$$
$$=\{(0,Y,0,0)\in\mathfrak{P}^C\,|\,Y\in\mathfrak{J}^C\}\cong\mathfrak{J}^C.$$

Put

$$\mathfrak{k}=(\mathfrak{e}_7)_\iota=\{\Phi\in\mathfrak{e}_7\,|\,\iota_*\Phi=\Phi\}$$
$$=\{\Phi(\phi,0,0,\nu)\in\mathfrak{e}_7\,|\,\phi\in\mathfrak{e}_6,\ \nu\in i\mathbf{R}\},$$
$$\mathfrak{m}=(\mathfrak{e}_7)_{-\iota}=\{\Phi\in\mathfrak{e}_7\,|\,\iota_*\Phi=-\Phi\}$$
$$=\{\Phi(0,A,-\tau A,0)\in\mathfrak{e}_7\,|\,A\in\mathfrak{J}^C\}.$$

Then the involutive automorphism $\tilde{\iota}$ of the group E_7 induces the canonical decomposition $\mathfrak{e}_7=\mathfrak{k}\oplus\mathfrak{m}$. By Proposition 3 the natural projection $p: E_7\to EIII$ induces an isomorphism $p_*:\mathfrak{m}\to T_o(EVII)=\mathfrak{J}^C$. We shall identify the C-vector space $T_o(EVII)$ with $\mathfrak{J}$. Then we verify that

$$p_*(\Phi(0,-\tau X,X,0))=\Phi(0,-\tau X,X,0)(\dot{1})=X.$$

We give the Hermitian metric on $T_o(EVII)$ from $\mathfrak{J}^C$. Since the metric is $U(1)E_6$-invariant, this metric can be extended to E_7-invariant metric on $EVII$. Thus $EVII$ has an Hermitian structure with this metric and the Riemannian metric g is given by the real restriction.

2.3. The first Chern form c_1 of $EVII$. We shall calculate the first Chern class of $EVII$ similar way as 1.3. We take $\{E_k,(1/\sqrt{2})F_k(e_j)\,|\,j=0,\cdots,7,\ k=2,3\}$ as an orthonormal C-basis of $T_o(EVII)$. For $X,Y\in\mathfrak{J}^C$ put

$$\phi=-2\tau X\vee Y+2\tau Y\vee X,\qquad \nu=\langle Y,X\rangle-\langle X,Y\rangle.$$

Then, for $Z\in T_o(EVII)$, we have

$$\begin{aligned}
&[[\Phi(0,-\tau X,X,0),\Phi(0,-\tau Y,Y,0)],\Phi(0,-\tau Z,Z,0)]\dot{1}\\
&=[\Phi(\phi,0,0,\nu),\Phi(0,-\tau Z,Z,0)]\dot{1}\\
&=\Phi(0,-(\phi+\tfrac{2}{3}\nu)\tau Z,\tau((\phi+\tfrac{2}{3}\nu)\tau Z),0)\dot{1}\\
&=\tau((\phi+\tfrac{2}{3}\nu)\tau Z)\\
&=\tau(-(Y,\tau Z)\tau X-\tfrac{1}{3}(\tau X,Y)\tau Z+4Y\times(\tau X\times\tau Z)+(X,\tau Z)\tau Y\\
&\quad+\tfrac{1}{3}(\tau Y,X)\tau Z-4X\times(\tau Y\times\tau Z)+\tfrac{2}{3}(((\tau Y,X)-(\tau X,Y))\tau Z)\\
&=-(\tau Y,Z)X-(X,\tau Y)Z+4\tau Y\times(X\times Z)+(\tau X,Z)Y\\
&\quad+(Y,\tau X)Z-4\tau X\times(Y\times Z).
\end{aligned}$$

If $\tau Z=Z$ and $(Z,Z)=1$, then

$$\begin{aligned}
\langle R\langle X,Y\rangle_o Z,Z\rangle=&-(Y,Z)(\tau X,Z)-(\tau X,Y)+4(\tau X\times Z,Y\times Z)\\
&+(X,Z)(\tau Y,Z)+(\tau Y,X)-4(\tau Y\times Z,X\times Z)
\end{aligned}$$

$$= -2i\,\mathrm{Im}((\tau X, Z)(Y, Z) + (\tau X, Y) + 4(\tau Y \times Z, X \times Z)) .$$

We verify that

$$\sum_{k=1}^{3} (\tau Y \times E_k, X \times E_k) + \frac{1}{2} \sum_{k=1}^{3} \sum_{j=0}^{7} (\tau Y \times F_k(e_j), X \times F_k(e_j)) = \frac{5}{2} .$$

Then we have

$$c_1(X, Y) = -\frac{1}{2\pi i}\left(\sum_{k=1}^{3} \langle R(X, Y)_o E_k, E_k\rangle + \sum_{k=1}^{3} \sum_{j=0}^{7} \langle R(X, Y)_o F_k(e_j), F_k(e_j)\rangle\right)$$

$$= \frac{1}{\pi}\mathrm{Im}((\tau X, Y) + 27(\tau X, Y) + 10(\tau Y, X)) = \frac{18}{\pi}(\tau X, Y) .$$

Let $\{d\xi_k, d\xi'_k, dx_{kj}, dx'_{kj} \mid k = 1, 2, 3, j = 0, 1, \cdots, 7\}$ be the dual basis of $\{E_k, iE_k, (1/\sqrt{2})F_k(e_j), (1/\sqrt{2})iF_k(e_j) \mid k = 1, 2, 3, j = 0, 1, \cdots, 7\}$, then the first Chern form c_1 is written as

$$c_1 = \frac{18}{\pi}\left(\sum_{k=1}^{3} d\xi_k \wedge d\xi'_k + \sum_{k=1}^{3} \sum_{j=0}^{7} dx_{kj} \wedge dx'_{kj}\right) .$$

2.4. The volume of ***EVII.*** Watanabe [9] determined the integral cohomology ring of *EVII* as

$$H^*(EVII) = \mathbf{Z}[u, v, w]/(\rho_1, \rho_2, \rho_3) ,$$

where $u \in H^2$, $v \in H^{10}$, $w \in H^{18}$ and $\rho_1 = v^2 - 2wu$, $\rho_2 = -2wv + 18wu^5 - 6vu^9 + u^{14}$, $\rho_3 = w^2 + 20wvu^4 - 18wu^9 + 2vu^{13}$. Combining the relations $\rho_1 = 0$, $\rho_2 = 0$ and $\rho_3 = 0$ we have

$$13110wvu^{13} = 507u^{27} , \quad 13110wu^{18} = 1190u^{27} , \quad 13110vu^{22} = 5586u^{27} .$$

It follows that $m = (1/13110)u^{27}$ is a generator of $H^{54}(EVII)$. (Explicitly $m = 89wvu^{13} - 52wu^{18} + 8vu^{22}$). According to Borel-Hirzebruch [5], §16, the 2-form $(1/18)c_1$ represents a generator of $H^2(EVII)$. Using 2.3 the form

$$\frac{1}{13110\ 18^{27}} c_1^{27} = \frac{27!}{13110\pi^{27}} d\xi_1 \wedge d\xi'_1 \wedge \cdots \wedge d\xi_3 \wedge d\xi'_3 \wedge dx_{10} \wedge dx'_{10}$$
$$\wedge \cdots \wedge dx_{37} \wedge dx'_{37}$$

represents the cohomology class m.

REMARK. Using Borel-Hirzebruch formula [5], Theorem 24.10, we verify that $c_1^{27}[EVII] = 13110\ 18^{27}$.

From the above argument, we get

$$1 = \int_{EVII} \frac{27!}{13110\pi^{27}} d\xi_1 \wedge d\xi'_1 \wedge \cdots \wedge d\xi_3 \wedge d\xi'_3 \wedge dx_{10} \wedge dx'_{10}$$

$$\wedge \cdots \wedge dx_{37} \wedge dx'_{37}$$

$$= \frac{27!}{13110\pi^{27}} \mathrm{vol}(EVII) .$$

Thus we have

THEOREM 2.2. $\mu(EVII, g) = \dfrac{13110}{27!} \pi^{27}$.

3. $Z(EIX)$.

3.1. The Lie group E_8 and its Lie algebra $\mathfrak{e}_8$. Let $\mathfrak{e}_8^C = \mathfrak{e}_7^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C$ be the complex Lie algebra of type E_8 with the Lie bracket

$$[(\Phi_1, P_1, Q_1, r_1, s_1, t_1), (\Phi_2, P_2, Q_2, r_2, s_2, t_2)] = (\Phi, P, Q, r, s, t) ,$$

$$\begin{cases} \Phi = [\Phi_1, \Phi_2] + P_1 \times Q_2 - P_2 \times Q_1 \\ P = \Phi_1 P_2 - \Phi_2 P_1 + r_1 P_2 - r_2 P_1 + s_1 Q_2 - s_2 Q_1 \\ Q = \Phi_1 Q_2 - \Phi_2 Q_1 - r_1 Q_2 + r_2 Q_1 + t_1 P_2 - t_2 P_1 \\ r \;= -\frac{1}{8}\{P_1, Q_2\} + \frac{1}{8}\{P_2, Q_1\} + s_1 t_2 - s_2 t_1 \\ s \;= \frac{1}{4}\{P_1, P_2\} + 2r_1 s_2 - 2r_2 s_1 \\ t \;= -\frac{1}{4}\{Q_1, Q_2\} - 2r_1 t_2 + 2r_2 t_1 . \end{cases}$$

Here $\{P_1, P_2\}$ is defined as

$$\{P_1, P_2\} = (X_1, Y_2) - (X_2, Y_1) + \xi_1 \eta_2 - \xi_2 \eta_1$$

for $P_i = (X_i, Y_i, \xi_i, \eta_i) \in \mathfrak{P}^C$, $i = 1, 2$. The Killing form B_8 of $\mathfrak{e}_8^C$ is given by

$$B_8(R_1, R_2) = \tfrac{5}{3} B_7(\Phi_1, \Phi_2) + 15\{Q_1, P_2\} - 15\{P_1, Q_2\} + 120 r_1 r_2 + 60 t_1 s_2 + 60 s_1 t_2$$

for $R_i = (\Phi_i, P_i, Q_i, r_i, s_i, t_i) \in \mathfrak{e}_8^C$, $i = 1, 2$, where B_7 is the Killing form of the Lie algebra $\mathfrak{e}_7^C$. We define C-linear transformations $\lambda : \mathfrak{P}^C \to \mathfrak{P}^C$, $\tilde{\lambda} : \mathfrak{e}_8^C \to \mathfrak{e}_8^C$ by $\lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi)$, $\tilde{\lambda}(\Phi, P, Q, r, s, t) = (\lambda \Phi \lambda^{-1}, \lambda Q, -\lambda P, -r, -t, -s)$, respectively, and define an Hermitian inner product $\langle , \rangle$ of $\mathfrak{e}_8^C$ by $\langle R_1, R_2 \rangle = -(1/15) B_8(\tau \tilde{\lambda} R_1, R_2)$. Then the group

$$E_8 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{e}_8^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2], \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\}$$

is the simply connected compact Lie group of type E_8. The Lie algebra of E_8 has the form

$$\mathfrak{e}_8 = \{(\Phi, P, -\tau\lambda P, r, s, -\tau s) \mid \Phi \in \mathfrak{e}_7, Q \in \mathfrak{P}^C, r \in i\mathbf{R}, s \in C\} .$$

3.2. The compact quaternionic symmetric space EIX and its twister space $Z(EIX)$. Let $\upsilon : \mathfrak{e}_8 \to \mathfrak{e}_8$ be the C-linear transformation given by $\upsilon(\Phi, P, Q, r, s, t) = (\Phi, -P, -Q, r, s, t)$. Then $\upsilon \in E_8$ and $\upsilon^2 = 1$. We verified that the subgroup $(E_8)^\upsilon = \{\alpha \in E_8 \mid \upsilon\alpha = \alpha\upsilon\}$ is

$$(E_8)^{\upsilon}=SU(2)E_7\cong(SU(2)\times E_7)/\mathbf{Z}_2\,,$$

where $U(1)$ and E_7 are subgroups of E_8 defined as

$$SU(2)=\{\phi(A)\,|\,A\in SU(2)\}\,,$$
$$E_7=(E_8)_{1^-}=\{\alpha\in E_8\,|\,\alpha 1^-=1^-\}\,.$$

Here $\phi(A)$ is the C-linear transformation of $\mathfrak{e}_8{}^C$ defined by

$$\phi(A)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a1 & -\tau b1 & 0 & 0 & 0 \\ 0 & b1 & \tau a1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\tau a)a-(\tau b)b & -(\tau a)b & a(\tau b) \\ 0 & 0 & 0 & 2a(\tau b) & a^2 & -(\tau b)^2 \\ 0 & 0 & 0 & 2(\tau a)b & -b^2 & (\tau a)^2 \end{pmatrix}$$

for $A=\begin{pmatrix} a & -\tau b \\ b & \tau a \end{pmatrix}\in SU(2)$ and $1^-=(0,0,0,0,1,0)$. Now the symmetric quaternion Kähler manifold EIX is defined by

$$EIX=E_8/(SU(2)\times E_7)/\mathbf{Z}_2\,.$$

By [4], for $R_1, R_2, R\in\mathfrak{e}_8{}^C$, the C-linear transformation $R_1\times R_2$ is defined by

$$(R_1\times R_2)R_1=[R_1,[R_2,R]]+[R_2,[R_1,R]]+\tfrac{1}{15}(B_8(R_1,R)R_2+B_8(R_2,R)R_1)\,.$$

We define the space $Z(EIX)$ by

$$Z(EIX)=\{R\in\mathfrak{e}_8{}^C\,|\,R\times R=0,\ R\neq 0\}/\sim\,,$$

where $R_1\sim R_2$ if $R_2=aR_1$ for some $a\in C$, $a\neq 0$. Let $[R]$ denote an element of $Z(EIX)$ which represents $R\in\mathfrak{e}_8{}^C$ satisfying $R\times R=0$, $R\neq 0$.

PROPOSITION 3.1. *$Z(EIX)\cong E_8/(U(1)E_7)\cong E_8/(U(1)\times E_7)/\mathbf{Z}_2$, where $U(1)$ is the subgroup of $SU(2)$ such that $U(1)=\left\{\begin{pmatrix} a & 0 \\ 0 & \bar{a}\end{pmatrix}\right\}$.*

PROOF. By [11], the comact Lie group E_8 acts transitively on the space

$$\mathfrak{m}_1=\{R\in\mathfrak{e}_8{}^C\,|\,R\times R=0,\ \langle R,R\rangle=4\}\,,$$

and the isotropy subgroup of E_8 at 1^- is isomorphic to E_7. Then the space $\mathfrak{m}_1$ is diffeomorphic to E_8/E_7. It follows that E_8 acts transitively on $Z(EIX)$ and the isotropy subgroup of E_8 at $[1^-]$ is isomorphic to $U(1)E_7$. Thus $Z(EIX)\cong E_8/(U(1)E_7)$ and Proposition 3.1 follows.

The tangent space $T_o(Z(EIX))$ of $Z(EIX)$ at $[1^-]$ is written as

$$T_{[1^-]}(Z(EIX))=\{R\in \mathfrak{e}_8{}^C \mid R\times 1^- =0, \langle R, 1^-\rangle=0\}$$

which can be identified with

$$\mathfrak{m}=\{(0, P, -\tau\lambda P, 0, s, -\tau s)\in \mathfrak{e}_8 \mid P\in \mathfrak{P}^C, s\in C\}\cong \mathfrak{P}^C\oplus C\,.$$

Let $J: \mathfrak{m}\to\mathfrak{m}$ be a map defined by

$$J(0, P, -\tau\lambda P, 0, s, -\tau s)=(0, iP, -i\tau\lambda P, 0, is, i\tau s)\,, \qquad P\in\mathfrak{P}^C, s\in C\,.$$

Then J defines an invariant complex structure on $Z(EIX)$. For $P\in\mathfrak{P}^C$ and $s\in C$, put $(P, s)=(0, P, -\tau\lambda P, 0, s, -\tau s)\in T_{[1^-]}(Z(EIX))$. For $(P_1, s_1), (P_2, s_2)\in T_{[1^-]}(Z(EIX))$, let

$$\langle (P_1, s_1), (P_2, s_2)\rangle=\mathrm{Re}(\langle P_1, P_2\rangle+8(\tau s_1)s_2)\,.$$

Then $\langle\,,\rangle$ is a $U(1)\cdot E_7$-invariant Hermitian inner product on $\mathfrak{m}$ and it defines an E_8-invariant Herimitian metric g on $Z(EIX)$. Similarly $\langle\,,\rangle$ defines an E_8-invariant metric g_0 on EIX.

3.3. The first Chern form c_1 of $Z(EIX)$. Let $\mathfrak{k}$ be the Lie algebra of $U(1)E_7$. Then $Z(EIX)$ is a reductive homogeneous space with an $Ad(U(1)E_7)$-invariant decomposition $\mathfrak{e}_8=\mathfrak{m}\oplus\mathfrak{k}$. The Riemannian connection for g is given by

$$\Lambda(R_1)R_2=\tfrac{1}{2}[R_1, R_2]_{\mathfrak{m}}+U(R_1, R_2)\,,$$

where $U(R_1, R_2)$ is the symmetric bilinear mapping $\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ defined by

$$2\langle U(R_1, R_2), R\rangle=\langle R_1, [R, R_2]_{\mathfrak{m}}\rangle+\langle [R, R_1]_{\mathfrak{m}}, R_2\rangle \qquad \text{for all } R_1, R_2, R\in\mathfrak{m}\,.$$

For $R_1=(P_1, s_1)$, $R_2=(P_2, s_2)\in\mathfrak{m}$, we have that

$$\begin{aligned}\Lambda(R_1)R_2&=\tfrac{1}{2}(-s_1(\tau\lambda P_2)+s_2(\tau\lambda P_1), \tfrac{1}{4}\{P_1, P_2\})+\tfrac{1}{2}(s_1(\tau\lambda P_2)+s_2(\tau\lambda P_1), 0)\\ &=(s_2(\tau\lambda P_1), \tfrac{1}{8}\{P_1, P_2\})\,.\end{aligned}$$

Then $\Lambda(R_1)J(R_2)=J(\Lambda(R_1)R_2)$ and g is a Kähler metric. Next we shall show that g is an Einstein metric. By [2], the Ricci curvature r can be expressed by

$$\begin{aligned}r(R, R)=&-\tfrac{1}{2}\sum_i |[R, S_i]_{\mathfrak{m}}|^2-\tfrac{1}{2}B_8(R, R)+\tfrac{1}{4}\sum_{i,j}\langle [S_i, S_j]_{\mathfrak{m}}, R\rangle^2\\ &-\sum_i\langle [U(S_i, S_i), R]_{\mathfrak{m}}, R\rangle^2\,, \qquad R\in\mathfrak{m}\,.\end{aligned}$$

Here $\{S_i\}$ is an orthonormal basis of $\mathfrak{m}$ and $|S|=\sqrt{\langle S, S\rangle}$. For $X\in\mathfrak{J}^C$, we denote elements of $\mathfrak{m}$ as follows:

$$\dot{X}=((X, 0, 0, 0), 0)\,, \quad \underset{\cdot}{X}=((0, X, 0, 0), 0)\,, \quad \dot{1}=((0, 0, 1, 0), 0)\,,$$

$$\underset{\cdot}{X}=((0, 0, 0, 1), 0)\,, \qquad \tilde{1}=\tfrac{1}{2\sqrt{2}}((0, 0, 0, 0), 1)\,.$$

We can choose an orthonormal basis of $\mathfrak{m}$ as follows.

$$\mathfrak{S}=\Big\{\dot{E}_i,\ \underset{\cdot}{E}_i,\ J(\dot{E}_i),\ J(\underset{\cdot}{E}_i),\ (1/\sqrt{2})\,\dot{F}_i(e_k),\ (1/\sqrt{2})\,\underset{\cdot}{F}_i(e_k),\ (1/\sqrt{2})J(\dot{F}_i(e_k)),$$
$$(1/\sqrt{2})\,J(\underset{\cdot}{F}_i(e_k)),\ \tilde{1},\ J(\tilde{1})\ \Big|\ i=1,2,3,\ k=0,1,\cdots,7\Big\}.$$

Since

$$|[\tilde{1},S]_{\mathrm{m}}|^2=\tfrac{1}{8}\qquad \text{for } S\in\mathfrak{S} \text{ with } S\neq\tilde{1},\ J(\tilde{1}),$$

we have that $\sum_{S\in\mathfrak{S}}|[\tilde{1},S]_{\mathrm{m}}|^2=14$. Let $\mathfrak{A}=\{E_i,\ (1/\sqrt{2})F_i(e_k)\,|\,i=1,2,3,\ k=0,1,\cdots,7\}$. By straightforward computation we have that

$$\begin{aligned}\sum_{S,\,T\in\mathfrak{S}}\langle[S,T]_{\mathrm{m}},\tilde{1}\rangle^2&=2\sum_{X\in\mathfrak{A}}\langle[\dot{X},\underset{\cdot}{X}]_{\mathrm{m}},\tilde{1}\rangle^2+2\sum_{X\in\mathfrak{A}}\langle[J(\dot{X}),J(\underset{\cdot}{X})]_{\mathrm{m}},\tilde{1}\rangle^2\\&\quad+2\langle[\dot{1},\underset{\cdot}{1}]_{\mathrm{m}},\tilde{1}\rangle^2+2\langle[J(\dot{1}),J(\underset{\cdot}{1})]_{\mathrm{m}},\tilde{1}\rangle^2\\&=2(27\cdot\tfrac{1}{2}+27\cdot\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2})=56\,.\end{aligned}$$

Note that $U(S,S)=0$ for $S\in\mathfrak{S}$. Since $B_8(\tilde{1},\tilde{1})=-15$, we have that $r(\tilde{1},\tilde{1})=29/2$. Similarly we have that

$$\begin{aligned}\sum_{S\in\mathfrak{S}}|[\dot{1},S]_{\mathrm{m}}|^2&=|[\dot{1},\tilde{1}]_{\mathrm{m}}|^2+|[\dot{1},J(\tilde{1})]_{\mathrm{m}}|^2+|[\dot{1},\underset{\cdot}{1}]_{\mathrm{m}}|^2+|[\dot{1},J(\underset{\cdot}{1})]_{\mathrm{m}}|^2\\&=\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{2}+\tfrac{1}{2}=\tfrac{5}{4}\,,\end{aligned}$$

$$\begin{aligned}\sum_{S,\,T\in\mathfrak{S}}\langle[S,T]_{\mathrm{m}},\dot{1}\rangle^2&=2\langle[\tilde{1},\underset{\cdot}{1}]_{\mathrm{m}},\dot{1}\rangle^2+2\langle[J(\tilde{1}),J(\underset{\cdot}{1})]_{\mathrm{m}},\dot{1}\rangle^2\\&=2\cdot\tfrac{1}{8}+2\cdot\tfrac{1}{8}=\tfrac{1}{2}\,.\end{aligned}$$

Since $B_8(\dot{1},\dot{1})=-30$, we have that $r(\dot{1},\dot{1})=29/2$. Similarly we have that $r(\underset{\cdot}{1},\underset{\cdot}{1})=29/2$. Let $X\in\mathfrak{A}$. Then

$$\begin{aligned}\sum_{S\in\mathfrak{S}}|[\dot{X},S]_{\mathrm{m}}|^2&=|[\dot{X},\tilde{1}]_{\mathrm{m}}|^2+[\dot{X},J(\tilde{1})]_{\mathrm{m}}|^2+|[\dot{X},\underset{\cdot}{X}]_{\mathrm{m}}|^2\\&\quad+|[J(\dot{X}),J(\underset{\cdot}{X})]_{\mathrm{m}}|^2=\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{2}+\tfrac{1}{2}=\tfrac{5}{4}\,,\end{aligned}$$

$$\begin{aligned}\sum_{S,\,T\in\mathfrak{S}}\langle[S,T]_{\mathrm{m}},\dot{X}\rangle^2&=2\langle[\tilde{1},\underset{\cdot}{X}]_{\mathrm{m}},\dot{X}\rangle^2+2\langle[J(\tilde{1}),J(\underset{\cdot}{X})]_{\mathrm{m}},\dot{X}\rangle^2\\&=2\cdot\tfrac{1}{8}+2\cdot\tfrac{1}{8}=\tfrac{1}{2}\,.\end{aligned}$$

Since $B_{E_8}(\dot{X},\dot{X})=-15$, we have that $r(\dot{X},\dot{X})=29/2$. Similarly we have that $r(\underset{\cdot}{X},\underset{\cdot}{X})=29/2$.

Note that $r(S,S)=r(J(S),J(S))$ for $S\in\mathrm{m}$. Thus $r(S,S)=29/2$ for $S\in\mathfrak{S}$ and g is an Einstein metric with the Einstein constant $29/2$. Consequently $Z(EIX)$ is a Kähler-Einstein manifold. Then the first Chern form c_1 is represented by $(29/4\pi)\omega$, where ω is the Kähler form of the Hermitian metric g.

3.4. The volume of $Z(EIX)$. By Borel-Hirzebruch [5], Theorem 24.10 the Chern number $\mu=c_1^{57}[Z(EIX)]$ is calculated as

$$\mu=57!\prod_{b\in\psi}\frac{(d,b)}{(a/2,b)}.$$

Using 3.3 we have that

$$1=\int_{Z(EIX)}\frac{1}{\mu}c_1^{57}=\frac{1}{\mu}\int_{Z(EIX)}\left(\frac{29}{4\pi}\right)^{57}57!\,\omega^{57}=\frac{1}{\mu}\left(\frac{29}{4\pi}\right)^{57}\mu(Z(EIX),g).$$

Then

THEOREM 3.2. $$\mu(Z(EIX),g)=\frac{2^{12}\,3^2\,5^2\,7\,31\,37\,41\,43\,47\,53}{57!}\pi^{57}.$$

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