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On a Generalization of the Conjecture of Jeśmanowicz

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1. Introduction.

In 1956, Sierpiński [S1] showed that the equation $3^x+4^y=5^z$ has only the positive integral solution $(x, y, z)=(2, 2, 2)$. Jeśmanowicz [J] conjectured that if a, b, c are Pythagorean numbers, i.e. positive integers satisfying $a^2+b^2=c^2$, then the equation $a^x+b^y=c^z$ has only the positive integral solution $(x, y, z)=(2, 2, 2)$ (cf. [S2]). It has been verified that this conjecture holds for many other Pythagorean numbers (cf. Lu [Lu], Takakuwa and Asaeda [Ta1], [Ta2], Takakuwa [Ta3], [Ta4], Adachi [A], Le [Le]).

As an analogy to this conjecture, in Terai [Te1], [Te2], [Te3] and [Te4], we proposed the following conjecture and proved it under some conditions when $p=2$, $q=2$ and r is an odd prime.

CONJECTURE. *If a, b, c, p, q, r are fixed positive integers satisfying $a^p+b^q=c^r$ with $p, q, r\geq 2$ and $(a, b)=1$, then the Diophantine equation*

$$a^x+b^y=c^z$$

has only the positive integral solution $(x, y, z)=(p, q, r)$.

In Terai [Te5], using a lower bound for linear forms in two logarithms, due to Laurent, Mignotte and Nesterenko [LMN], we showed the following (cf. Terai [Te4]):

THEOREM A. *Let l be a prime $\equiv 3 \pmod 8 < 11908$. Let a, b, c be fixed positive integers satisfying $a^2+lb^2=c^r$ with $(a, b)=1$ and r odd ≥ 3. Suppose that*

$$a\equiv 3 \pmod 8, \quad 2\| b, \quad \left(\frac{b}{a}\right)=-1, \quad \left(\frac{l}{a}\right)=1 \quad \text{and} \quad a\geq\lambda b,$$

where $\left(\frac{*}{*}\right)$ denotes the Jacobi symbol and

$$\lambda=\sqrt{l}\left\{\exp\left(\frac{2}{\log l+3235}\right)-1\right\}^{-1/2}.$$

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Then the Diophantine equation

$$a^x + lb^y = c^z \tag{1}$$

has only the positive integral solution $(x, y, z) = (2, 2, r)$.

In this paper, by an argument similar to that used in Theorem A, we prove the following:

THEOREM. *Let l be a prime* $\equiv 3 \pmod 8 < 23865310019$. *Let a, b, c be fixed positive integers satisfying $a^2 + lb^2 = c^2$ with $(a, b) = 1$. Suppose that*

$$a \equiv 3 \pmod 8, \quad 4 \| b, \quad \left(\frac{b}{a}\right) = -1 \quad \textit{and} \quad a \geq \lambda b,$$

where

$$\lambda = \sqrt{l}\left\{\exp\left(2\left(\frac{\log l + 2}{\log 5} + 3231\right)^{-1}\right) - 1\right\}^{-1/2}.$$

Then equation (1) *has only the positive integral solution* $(x, y, z) = (2, 2, 2)$.

In §5, we also give a corollary to Theorem.

2. Preliminary lemmas.

LEMMA 1 (Terai [Te5], Lemma 1). *Let l be an odd prime, a odd and b even. The positive integral solutions of the equation $a^2 + lb^2 = c^2$ with $(a, b) = 1$ are given by*

$$a = \pm(u^2 - lv^2), \quad b = 2uv, \quad c = u^2 + lv^2,$$

where u, v are positive integers such that $(u, v) = 1$ and $u \not\equiv v \pmod 2$.

LEMMA 2 (Nagell [N], Theorem 117). *Let l be a prime* $\equiv 3 \pmod 8$. *The Diophantine equation*

$$x^4 - y^4 = lz^2 \tag{2}$$

has no solutions in positive integers x, y and z.

REMARK. In Lemma 2, the condition $l \equiv 3 \pmod 8$ is essential. For in each of the cases $l \equiv 1, 5, 7 \pmod 8$, equation (2) may be solvable for certain values of the prime l. For example, when $l = 41, 5, 7$, equation (2) has the solution $(x, y, z) = (5, 4, 3), (3, 1, 4), (4, 3, 5)$, respectively.

Let α be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (x - \alpha_i),$$

where the a_i's are relatively prime integers with $a_0>0$ and the α_i's are conjugates of α. Then

$$h(\alpha)=\frac{1}{d}(\log a_0+\sum_{i=1}^{d}\log\max(1,|\alpha_i|))$$

is called the *absolute logarithmic height* of α. In particular, if $\alpha\in\mathbf{Q}$, say $\alpha=p/q$ as a fraction in lowest terms, then we have $h(\alpha)=\log\max(|p|,|q|)$.

We use the following result [LMN] (Corollary 2 and Table 2, p. 320) to prove Key lemma, which plays an important role in the proof of Theorem.

LEMMA 3 ([LMN]). *Let α_1, α_2 be positive real algebraic numbers and $D=[\mathbf{Q}(\alpha_1,\alpha_2):\mathbf{Q}]$. Let A_1, A_2 denote real numbers >1 such that $\log A_j\geq\max\{h(\alpha_j), |\log\alpha_j|/D, 1/D\}$ for $j=1,2$. If $\Lambda=b_2\log\alpha_2-b_1\log\alpha_1\neq 0$ for some positive integers b_1, b_2, then we have*

$$\log|\Lambda|\geq -32.31\,D^4(\max\{\log B+0.18,\,10/D,\,1/2\})^2(\log A_1)(\log A_2),$$

where $B=b_1/(D\log A_2)+b_2/(D\log A_1)$.

KEY LEMMA. *Let a, b, c, d be fixed positive integers satisfying $a^2+db^2=c^2$ with $(a,b)=1$, $b>1$ and $d\leq 23865310019$. Let $y=1$ or 2. If $x\geq 2$ and $a\geq\lambda b$, then the Diophantine equation*

$$a^x+db^y=c^z \tag{3}$$

has only the positive integral solution $(x,y,z)=(2,2,2)$, where

$$\lambda=\sqrt{d}\left\{\exp\left(2\left(\frac{\log d+2}{\log 5}+3231\right)^{-1}\right)-1\right\}^{-1/2}.$$

REMARK. The table of values of λ for some d's is as follows:

λ	d
$40.19479\cdots$	1
$69.62677\cdots$	3
$133.34177\cdots$	11
$175.25463\cdots$	19
$263.67018\cdots$	43

3. Proof of Key lemma.

Suppose that our assumptions are all satisfied.

Let $x=2$. If $y=1$, then $c^z=a^x+db^y=a^2+db<a^2+db^2=c^2$. Thus $z=1$ and so $a^2+db=c$. But we have

$$a^2+db^2=c^2=(a^2+db)^2=a^4+2a^2db+d^2b^2>a^2+db^2\,,$$

which is impossible. If $y=2$, then $c^z=a^x+db^y=a^2+db^2=c^2$. Hence $z=2$.

Let $x\geq 3$. From $a^2+db^2=c^2$ and $a^x+db^y=c^z$, we now consider the following linear forms in two logarithms:

$$\Lambda_1=2\log c-2\log a\ (>0)\,,\quad \Lambda_2=z\log c-x\log a\ (>0)\,.$$

Using the inequality $\log(1+t)<t$ for $t>0$, we have

$$0<\Lambda_2=\log\left(\frac{c^z}{a^x}\right)=\log\left(1+\frac{db^y}{a^x}\right)<\frac{db^y}{a^x}\leq\frac{db^2}{a^x}\,.$$

Hence we obtain

$$\log\Lambda_2<\log d+2\log b-x\log a\,. \tag{4}$$

On the other hand, we use Lemma 3 to obtain a lower bound for Λ_2. It follows from Lemma 3 that

$$\log\Lambda_2\geq -32.31H^2(\log a)(\log c)\,, \tag{5}$$

where $H=\max\{\log B+0.18,\ 10\}$ and $B=x/\log c+z/\log a$.

Now we distinguish two cases: (i) $B\leq e^{9.82}\,(=18398.050741\cdots)$ and (ii) $B>e^{9.82}$.

Case (i): $B\leq e^{9.82}$. Then we show that making Λ_1 small yields a contradiction. (In case (ii), we do not use Λ_1.) Since $H=10$, combining (4) and (5) leads to

$$\log d+2\log b-x\log a>-3231(\log a)(\log c)\,,$$

so

$$\frac{\log d}{\log a}+2\cdot\frac{\log b}{\log a}+3231\log c>x\,.$$

Note that since $\lambda>1$, we have $a>b>1$. Hence we have

$$\log d+2+3231\log c>x\,.$$

We want to obtain a lower bound for x. We now show $x-z>0$. Other wise, $(a^x+db^y)^2=c^{2z}\geq c^{2x}=(a^2+db^2)^x$. But since $x\geq 3$ and $y=1,2$, we have $(a^x+db^y)^2<(a^2+db^2)^x$. Thus we have $x-z>0$ and so $2x-2z\geq 2$.

Eliminating a from the defining equations for Λ_1, Λ_2 yields

$$x\Lambda_1-2\Lambda_2=(2x-2z)\log c\,,$$

so

$$x=\frac{2x-2z}{\Lambda_1}\cdot\log c+\frac{2\Lambda_2}{\Lambda_1}>\frac{2}{\Lambda_1}\cdot\log c\,,$$

since $2x-2z\geq 2$ and Λ_1, $\Lambda_2>0$.

Therefore we obtain

$$\log d + 2 + 3231 \log c > \frac{2}{\Lambda_1} \cdot \log c ,$$

thus

$$\Lambda_1 = \log\left(1 + \frac{db^2}{a^2}\right) > \frac{2 \log c}{\log d + 2 + 3231 \log c} = \frac{2}{(\log d + 2)/\log c + 3231}$$

$$\geq \frac{2}{(\log d + 2)/\log 5 + 3231} ,$$

since $c \geq 5$. Hence we have

$$\frac{db^2}{a^2} > \exp\left(2\left(\frac{\log d + 2}{\log 5} + 3231\right)^{-1}\right) - 1 ,$$

which implies

$$a < \sqrt{d}\left\{\exp\left(2\left(\frac{\log d + 2}{\log 5} + 3231\right)^{-1}\right) - 1\right\}^{-1/2} b =: \lambda b .$$

Therefore if $a \geq \lambda b$, then equation (3) has no positive integral solutions x, y, z with $x \geq 3$.

Case (ii): $B > e^{9.82}$. Then $H = \log B + 0.18$. Combining (4) and (5) leads to

$$\log d + 2 \log b - x \log a > -32.31 H^2 (\log a)(\log c) ,$$

so

$$\frac{\log d + 2 \log b}{(\log a)(\log c)} - \frac{x}{\log c} > -32.31 H^2 .$$

Since $\Lambda_2 = z \log c - x \log a$, we have $B = 2x/\log c + \Lambda_2/((\log a)(\log c))$. From this, we have

$$\frac{2(\log d + 2 \log b)}{(\log a)(\log c)} - B + \frac{\Lambda_2}{(\log a)(\log c)} > -64.62 H^2 .$$

In view of $a \geq \lambda b$, $b > 1$ and $a^2 + db^2 = c^2$, we have $a \geq 2\lambda$, $c \geq 2\sqrt{\lambda^2 + 1}$. Note that $\Lambda_2 < d/(2\lambda^3)$. In fact, $\Lambda_2 < db^2/a^x \leq db^2/a^3 = d(b/a)^2(1/a) \leq d/(2\lambda^3)$, since $x \geq 3$. Thus

$$B < \frac{2 \log d}{(\log a)(\log c)} + \frac{4 \log b}{(\log a)(\log c)} + \frac{\Lambda_2}{(\log a)(\log c)} + 64.62 H^2$$

$$< \frac{2 \log d}{(\log 2\lambda)(\log 2\sqrt{\lambda^2 + 1})} + \frac{4}{\log 2\sqrt{\lambda^2 + 1}} + \frac{d}{2\lambda^3 (\log 2\lambda)(\log 2\sqrt{\lambda^2 + 1})} + 64.62 H^2$$

$$< 0.10392 \log d + 0.91175 + 0.0000005 d + 64.62 H^2$$

$$< 11936.05 + 64.62(\log B + 0.18)^2 ,$$

since $d \leq 23865310019$ and $\lambda > 40.19479$. Hence we obtain $B < 18398.05$, which contradicts $B > e^{9.82}$. This completes the proof of Key lemma. □

4. Proof of Theorem.

We first prove the following lemma:

LEMMA 4. *Let a, b, c, l be as in Theorem. If equation* (1) *has positive integral solutions* (x, y, z), *then*

(i) *x is even, $y=1$, z is odd,*

or

(ii) *x is even, $y=2$, z is even.*

PROOF. From Lemma 1, we have

$$a = \pm(u^2 - lv^2), \quad b = 2uv, \quad c = u^2 + lv^2,$$

where u, v are positive integers such that $(u, v)=1$ and $u \not\equiv v \pmod 2$. Since $4 \| b$, we have $2 \| u$ or $2 \| v$. If $2 \| u$, then $a \equiv \pm(4-3\cdot 1) \equiv \pm 1 \pmod 8$, which contradicts $a \equiv 3 \pmod 8$. Thus $2 \| v$. Then we have $c \equiv 5 \pmod 8$.

Case (i): $y=1$. Then it follows from (1) that $3^x + 4 \equiv 5^z \pmod 8$. Thus x is even and z is odd.

Case (ii): $y>1$. Then from (1), we have $3^x \equiv 5^z \pmod 8$. Hence x and z are even. Note that $\left(\frac{l}{a}\right)=1$, since $a^2 + lb^2 = c^2$. From (1) and $\left(\frac{b}{a}\right) = -1$, we have $(-1)^y = 1$ and so y is even. Thus it follows from Lemma 1 that

$$a^X = \pm(u^2 - lv^2), \quad b^Y = 2uv, \quad c^Z = u^2 + lv^2,$$

where $x=2X$, $y=2Y$, $z=2Z$ and u, v are positive integers such that $(u, v)=1$ and $u \not\equiv v \pmod 2$.

If $Y>1$, then $uv \equiv 0 \pmod 4$. If $u \equiv 0 \pmod 4$, then we have $c^Z \equiv 3 \pmod 8$, which is impossible, since $c \equiv 5 \pmod 8$. If $v \equiv 0 \pmod 4$, then we have

$$a^X \equiv \pm 1 \pmod 8, \quad c^Z \equiv 1 \pmod 8.$$

In view of $a \equiv 3 \pmod 8$ and $c \equiv 5 \pmod 8$, we see that X and Z are even. Then equation (1) leads to

$$(c^{z/4})^4 - (a^{x/4})^4 = l(b^{y/2})^2,$$

which has no non-trivial solutions from Lemma 2. Hence $Y=1$ and so $y=2$. □

Therefore Theorem now follows from Key lemma. □

REMARK. When $l=1$, the results corresponding to Lemmas 1, 2 are known (cf. Ribenboim [R], pp. 36, 38). Thus in view of the proof, Theorem also holds for the case $l=1$ under the conditions. Hence Theorem generalizes Theorem 2 in Terai [Te4].

5. Corollary.

In this section, we show the following Corollary to Theorem (cf. Lemma 1).

COROLLARY. *Let l be a prime $\equiv 3 \pmod 8 < 23865310019$. Let $a=lv^2-u^2>0$, $b=2uv$, $c=u^2+lv^2$ with $2\|v$, $(u,v)=1$, $u\equiv -1 \pmod l$ and $a\geq \lambda b$. Then equation* (1) *has only the positive integral solution* $(x,y,z)=(2,2,2)$.

PROOF. It follows from $2\|v$ that $a\equiv 3 \pmod 8$ and $4\|b$.

We show $\left(\frac{u}{a}\right)=-1$. If $u\equiv 1 \pmod 4$, then we have $\left(\frac{u}{a}\right)=\left(\frac{a}{u}\right)=\left(\frac{l}{u}\right)=\left(\frac{u}{l}\right)=\left(\frac{-1}{l}\right)=-1$, since $l\equiv -1 \pmod 4$. If $u\equiv -1 \pmod 4$, then we have $\left(\frac{u}{a}\right)=-\left(\frac{a}{u}\right)=-\left(\frac{l}{u}\right)=\left(\frac{u}{l}\right)=\left(\frac{-1}{l}\right)=-1$.

We also show $\left(\frac{v}{a}\right)=-1$. Put $v=2t$ (t is odd). If $t\equiv 1 \pmod 4$ (>1), then we have $\left(\frac{v}{a}\right)=\left(\frac{2}{a}\right)\left(\frac{t}{a}\right)=-\left(\frac{a}{t}\right)=-\left(\frac{-1}{t}\right)=-1$. If $t\equiv -1 \pmod 4$, then we have $\left(\frac{v}{a}\right)=\left(\frac{2}{a}\right)\left(\frac{t}{a}\right)=\left(\frac{a}{t}\right)=\left(\frac{-1}{t}\right)=-1$.

Hence we have $\left(\frac{b}{a}\right)=\left(\frac{2}{a}\right)\left(\frac{u}{a}\right)\left(\frac{v}{a}\right)=(-1)\cdot(-1)\cdot(-1)=-1$.

Therefore by Theorem, if $u\equiv -1 \pmod l$, $2\|v$ and $a\geq \lambda b$, then equation (1) has only the positive integral solution $(x,y,z)=(2,2,2)$ (cf. Table in §2). □

REMARK. By Corollary, we see that when l is a prime $\equiv 3 \pmod 8 < 23865310019$, there are infinitely many a, b, c satisfying the conditions of Theorem.

References

[A] N. ADACHI, An application of Frey's idea to exponential Diophantine equations, Proc. Japan Acad. Ser. A **70** (1994), 261–263.

[J] L. JEŚMANOWICZ, Some remarks on Pythagorean numbers (in Polish), Wiadom. Mat. **1** (1956), 196–202.

[LMN] M. LAURENT, M. MIGNOTTE et Y. NESTERENKO, Formes linéairies en deux logarithmes et déterminants d'interpolation, J. Number Theory **55** (1995), 285–321.

[Le] M.-H. LE, On Jeśmanowicz' conjecture concerning Pythagorean numbers, Proc. Japan Acad. Ser. A **72** (1996), 97–98.

[Lu] LU WEN-TWAN, On Pythagorean numbers $4n^2-1$, $4n$, $4n^2+1$ (in Chinese), Acta Sci. Nat. Univ. Szechuan **2** (1959), 39–42.

[N] T. Nagell, *Introduction to Number Theory*, Chelsea (1981).
[R] P. Ribenboim, *13 Lectures on Fermat's Last Theorem*, Springer (1979).
[S1] W. Sierpiński, On the equation $3^x+4^y=5^z$ (in Polish), Wiadom. Mat. **1** (1956), 194–195.
[S2] W. Sierpiński, *Elementary Theory of Numbers*, PWN-Polish Scientific Publishers (1988).
[Ta1] K. Takakuwa and Y. Asaeda, On a conjecture on Pythagorean numbers, Proc. Japan Acad. Ser. A **69** (1993), 252–255.
[Ta2] K. Takakuwa and Y. Asaeda, On a conjecture on Pythagorean numbers II, Proc. Japan Acad. Ser. A **69** (1993), 287–290.
[Ta3] K. Takakuwa, On a conjecture on Pythagorean numbers III, Proc. Japan Acad. Ser. A **69** (1993), 345–349.
[Ta4] K. Takakuwa, A remark on Jeśmanowicz' conjecture, Proc. Japan Acad. Ser. A **72** (1996), 109–110.
[Te1] N. Terai, The Diophantine equation $a^x+b^y=c^z$, Proc. Japan Acad. Ser. A **70** (1994), 22–26.
[Te2] N. Terai, The Diophantine equation $a^x+b^y=c^z$ II, Proc. Japan Acad. Ser. A **71** (1995), 109–110.
[Te3] N. Terai, The Diophantine equation $a^x+b^y=c^z$ III, Proc. Japan Acad. Ser. A **72** (1996), 20–22.
[Te4] N. Terai, Applications of a lower bound for linear forms in two logarithms to exponential Diophantine equations, submitted.
[Te5] N. Terai, A note on certain exponential Diophantine equations, (to appear in Proc. of Number Theory Conference in Eger, Hungary).

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