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# On Algebraic Unknotting Numbers of Knots

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**Abstract.** We show that the algebraic unknotting number of a classical knot $K$, defined by Murakami [9], is equal to the minimum number of unknotting operations necessary to transform $K$ to a knot with trivial Alexander polynomial. Furthermore, we define a new operation, called an elementary twisting operation, for smooth $(2n-1)$-knots with $n\geq 1$ and odd, and show that this is an unknotting operation for simple $(2n-1)$-knots. Moreover, the unknotting number of a simple $(2n-1)$-knot defined by using the elementary twisting operation is equal to the algebraic unknotting number of the $S$-equivalence class of its Seifert matrix if $n\geq 3$.

## 1. Introduction.

Let $\mathscr{S}$ denote the set of all integral square matrices $V$ such that $\det(V-V^T)=1$, where the zero matrix is also included and $V^T$ denotes the transpose of $V$. Note that $\mathscr{S}$ is the set of Seifert matrices for oriented (tame) knots in $S^3$, where $S^3$ is also oriented (for a precise definition, see §3). In [9], Murakami has defined an *algebraic unknotting operation* on an element $V$ of $\mathscr{S}$ (for details, see §2). In fact, such an operation is well defined on the set $\tilde{\mathscr{S}}$ of all $S$-equivalence classes (see [11]) of elements of $\mathscr{S}$. Then Murakami has shown that every two elements of $\tilde{\mathscr{S}}$ can always be transformed to each other by a finite iteration of algebraic unknotting operations. The *algebraic unknotting number* $u^a(\mathscr{V})$ of an element $\mathscr{V}\in\tilde{\mathscr{S}}$ is defined to be the minimum number of algebraic unknotting operations necessary to transform $\mathscr{V}$ to the $S$-equivalence class of the zero matrix. Furthermore, the *algebraic unknotting number* $u^a(K)$ of a knot $K$ in $S^3$ is defined to be the algebraic unknotting number $u^a([K])$ of the $S$-equivalence class $[K]$ of a Seifert matrix for $K$. These notions have been first defined and studied by Murakami [9]. In [1], there has been obtained a characterization of knots in $S^3$ which have algebraic unknotting number one.

For two knots $K$ and $K'$ in $S^3$, we denote by $d_G(K, K')$ the usual *Gordian distance* between $K$ and $K'$ (see [8]). Furthermore, for a knot $K$, $\Delta_K(t)$ denotes its Alexander polynomial. In this paper, first we will show the following.

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THEOREM 1.1. *For a knot $K$ in $S^3$, we have*

$$u^a(K) = \min\{d_G(K, K') : \Delta_{K'}(t) = 1\}.$$

*In other words, the algebraic unknotting number of $K$ coincides with the minimum number of (geometric) unknotting operations necessary to transform $K$ to a knot with trivial Alexander polynomial.*

As a corollary, we will obtain an estimate from above of the topological 4-ball genus of a classical knot (see §4).

Let us consider an oriented homotopy $(2n-1)$-sphere $K$ smoothly embedded in $S^{2n+1}$ $(n \geq 1)$, where $S^{2n+1}$ is also oriented. We call such a $K$ a $(2n-1)$-*knot*. In §5, as a generalization of the unknotting operation for classical knots, we will define an operation, called an *elementary twisting operation*, which creates a new $(2n-1)$-knot from an old one for $n \geq 1$ and odd. Unfortunately, for $n \geq 3$, this operation will turn out to be insufficient for transforming a given knot to the trivial knot, even if applied successively, since this operation preserves the lower dimensional homotopy groups of the complement. However, if we restrict ourselves to simple $(2n-1)$-knots (see [7]), for which the $i$-th homotopy groups of the complements vanish for all $i \leq n-1$, we will see that this is in fact an unknotting operation and thus we can define the unknotting number $u(K)$ for every simple $(2n-1)$-knot $K$ with $n \geq 1$ and odd (Theorem 5.9). The second main result of this paper is Theorem 5.10 which states that the unknotting number of a simple $(2n-1)$-knot with $n \geq 3$ and odd is equal to the algebraic unknotting number of the $S$-equivalence class of its Seifert matrix.

Throughout the paper, we work in the smooth category except in §4. The homology groups are always with integral coefficients and the symbol "$\cong$" denotes a diffeomorphism between smooth manifolds.

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## 2. Preliminaries.

Let us consider two matrices $V$ and $W \in \mathscr{S}$. We say that $V$ and $W$ are *congruent* and write $V \sim_c W$, if $W = PVP^T$ for some integral square matrix $P$ with $\det P = \pm 1$. We say that $W$ is a *row enlargement* of $V$ (or $V$ is a *row reduction* of $W$), if

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 1 & x & u \\ 0 & v^T & V \end{pmatrix}$$

for some integer $x$ and some integral row vectors $u$ and $v$. We say that $W$ is a *column enlargement* of $V$ (or $V$ is a *column reduction* of $W$), if

$$W=\begin{pmatrix} 0 & 1 & 0 \\ 0 & x & (u')^T \\ 0 & v' & V \end{pmatrix}$$

for some integer $x$ and some integral column vectors $u'$ and $v'$. When $W$ is a row (or column) enlargement of $V$, we write $V \nearrow W$ or $W \searrow V$. The *S-equivalence* on $\mathscr{S}$ is defined to be the equivalence relation generated by congruences, row enlargements and column enlargements (see [11]). We denote by $\tilde{\mathscr{S}}$ the set of $S$-equivalence classes of elements of $\mathscr{S}$.

For two elements $V$ and $W \in \mathscr{S}$ and $\varepsilon = \pm 1$, we say that $W$ is obtained from $V$ by an ($\varepsilon$-)*algebraic unknotting operation* and write $V \xrightarrow{\varepsilon} W$, if

$$W=\begin{pmatrix} \varepsilon & 0 & 0 \\ 1 & x & M \\ 0 & N^T & V \end{pmatrix}$$

for some integer $x$ and some integral row vectors $M$ and $N$. An algebraic unknotting operation on $\tilde{\mathscr{S}}$ assigns $\mathscr{V} \in \tilde{\mathscr{S}}$ to the $S$-equivalence class of

$$\begin{pmatrix} \varepsilon & 0 & 0 \\ 1 & x & M \\ 0 & N^T & V \end{pmatrix} \in \mathscr{S},$$

where $V$ is some representative of $\mathscr{V}$. Note that when a knot $K$ is transformed to a knot $K'$ by an ($\varepsilon$-)unknotting operation, then the $S$-equivalence class $[K']$ of a Seifert matrix for $K'$ is obtained from $[K]$ by an ($\varepsilon$-)algebraic unknotting operation.

By [9, Lemma 1], we have the following.

LEMMA 2.1. *For two elements $V$ and $W \in \mathscr{S}$, there always exists a sequence of matrices $V_0, V_1, \cdots, V_k$ in $\mathscr{S}$ such that*

(1) *$V_0 = V$ and $V_k = W$,*

(2) *for each $i$ with $0 \le i \le k-1$, we have $V_i \sim_c V_{i+1}$, or $V_i \nearrow V_{i+1}$, or $V_i \searrow V_{i+1}$ or $V_i \xrightarrow{\varepsilon} V_{i+1}$ for some $\varepsilon$.*

## 3. Proof of Theorem 1.1.

Let $F$ be an oriented compact connected surface embedded in $S^3$ and $\{\alpha_1, \cdots, \alpha_r\}$ a basis of $H_1(F)$. A *Seifert matrix* for $F$ with respect to the above basis is defined to be the matrix $(\mathrm{lk}(\alpha_i, \alpha_j^+))$, where lk denotes the linking number in $S^3$ and $\alpha_j^+$ is the push-off of $\alpha_j$ in the positive normal direction of $F$ in $S^3$. Let $K$ be an oriented knot in $S^3$. Then a Seifert matrix for an oriented Seifert surface of $K$ is called a Seifert matrix of $K$.

Let $V \in \mathscr{S}$ be a Seifert matrix for a Seifert surface $F$ of a given knot $K$ in $S^3$. By Lemma 2.1, there exists a sequence of matrices $V_0, V_1, \cdots, V_k$ in $\mathscr{S}$ such that

(1) $V_0=V$ and $V_k$ is the zero matrix,

(2) for each $i$ with $0 \leq i \leq k-1$, we have $V_i \sim_c V_{i+1}$, or $V_i \nearrow V_{i+1}$, or $V_i \searrow V_{i+1}$, or $V_i \xrightarrow{\varepsilon} V_{i+1}$ for some $\varepsilon$,

(3) the number of elements in the set $\{0 \leq i \leq k-1 : V_i \xrightarrow{\varepsilon} V_{i+1}\}$ is equal to $u^a(K)$.

LEMMA 3.1. *Modifying $V_i$ and $k$ if necessary, we may assume that there exists no $i$ with $V_i \searrow V_{i+1}$, replacing the above condition* (1) *with the condition*

(1)′ *$V_0=V$ and $V_k$ is S-equivalent to the zero matrix.*

PROOF. Suppose that $V_{i_0} \searrow V_{i_0+1}$ for some $i_0$ with $0 \leq i_0 \leq k-1$ and that there exists no $i$ with $i_0 < i \leq k-1$ with $V_i \searrow V_{i+1}$. We may assume that

$$V_{i_0}=\begin{pmatrix} 0 & 0 & 0 \\ 1 & x & u \\ 0 & v^T & V_{i_0+1} \end{pmatrix}$$

for some integer $x$ and some integral row vectors $u$ and $v$ (when $V_{i_0}$ is a column enlargement of $V_{i_0+1}$, the argument is similar).

First, define $V'_{i_0+1}=V_{i_0}$. Note that, in particular, $V_{i_0} \sim_c V'_{i_0+1}$. We will define $V'_j$ for $j$ with $i_0+2 \leq j \leq k$ inductively so that $V'_j$ is a row enlargement of $V_j$ as follows. Note that $V'_{i_0+1}$ is a row enlargement of $V_{i_0+1}$. We suppose that

$$V'_{j-1}=\begin{pmatrix} 0 & 0 & 0 \\ 1 & x_{j-1} & u_{j-1} \\ 0 & (v_{j-1})^T & V_{j-1} \end{pmatrix}.$$

When $V_{j-1} \sim_c V_j$, suppose that $V_j=PV_{j-1}P^T$ for an integral square matrix $P$ with $\det P=\pm 1$. Then we set

$$V'_j=\begin{pmatrix} 0 & 0 & 0 \\ 1 & x_{j-1} & u_{j-1}P^T \\ 0 & P(v_{j-1})^T & V_j \end{pmatrix}.$$

Note that $V'_j$ is a row enlargement of $V_j$ and that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & P \end{pmatrix} V'_{j-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & P \end{pmatrix}^T = V'_j.$$

In particular, we have $V'_{j-1} \sim_c V'_j$.

When $V_{j-1} \nearrow V_j$, suppose that

$$V_j=\begin{pmatrix} 0 & 0 & 0 \\ 1 & y & u_1 \\ 0 & v_1^T & V_{j-1} \end{pmatrix}$$

for some integer $y$ and integral row vectors $u_1$ and $v_1$ (when $V_j$ is a column enlargement of $V_{j-1}$, the argument is similar). Then we set

$$V_j' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & x_{j-1} & 0 & 0 & u_{j-1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & y & u_1 \\ 0 & (v_{j-1})^T & 0 & v_1^T & V_{j-1} \end{pmatrix}.$$

Note that $V_j'$ is a row enlargement of $V_j$. Furthermore, we have

$$V_{j-1}' = \begin{pmatrix} 0 & 0 & 0 \\ 1 & x_{j-1} & u_{j-1} \\ 0 & (v_{j-1})^T & V_{j-1} \end{pmatrix} \sim_c \begin{pmatrix} V_{j-1} & 0 & (v_{j-1})^T \\ 0 & 0 & 0 \\ u_{j-1} & 1 & x_{j-1} \end{pmatrix} = W_j'$$

$$\nearrow W_j'' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & y & u_1 & 0 & 0 \\ 0 & v_1^T & V_{j-1} & 0 & (v_{j-1})^T \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_{j-1} & 1 & x_{j-1} \end{pmatrix} \sim_c \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & x_{j-1} & 0 & 0 & u_{j-1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & y & u_1 \\ 0 & (v_{j-1})^T & 0 & v_1^T & V_{j-1} \end{pmatrix} = V_j'.$$

When $V_{j-1} \xrightarrow{\varepsilon} V_j$, we define $V_j'$ similarly. Then, in this case also we have two matrices $W_j'$ and $W_j'' \in \mathscr{S}$ such that $V_{j-1}' \sim_c W_j' \xrightarrow{\varepsilon} W_j'' \sim_c V_j'$.

Then, in the sequence $V_0, V_1, \cdots, V_k$, replacing $V_j$ with $V_j'$ or with the sequence $W_j', W_j'', V_j'$, we obtain a new sequence such that the number of row reductions and column reductions is strictly smaller than that for the original sequence and that the condition (1)$'$ is satisfied. Then, repeating this procedure, we finally get a sequence with the required properties. This completes the proof of Lemma 3.1. □

LEMMA 3.2. *Let $K$ be a knot in $S^3$ and $V$ a Seifert matrix for a Seifert surface $F$ of $K$. Suppose that $V \sim_c W$ or $V \nearrow W$. Then $W$ is a Seifert matrix for some Seifert surface $F'$ of the same knot $K$.*

PROOF. When $V \sim_c W$, by a base change, we get the result.

When $V \nearrow W$, we may suppose that

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 1 & x & u \\ 0 & v^T & V \end{pmatrix}$$

(when $W$ is a column enlargement, the argument is similar). We have only to show that some matrix congruent to $W$ satisfies the result of the lemma.

Let $\{\alpha_1', \beta_1', \alpha_2', \beta_2', \cdots, \alpha_g', \beta_g'\}$ be a basis of $H_1(F)$ such that $V$ is the Seifert matrix with respect to the basis, where $g$ is the genus of the surface $F$. Let $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g\}$ be a basis of $H_1(F)$ such that $\alpha_i$ and $\beta_j$ are represented by the oriented simple closed curves in $F$ as depicted in Figure 1. Then there exists a square integral matrix $P$ with $\det P = \pm 1$ such that

$$(\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g) = (\alpha'_1, \beta'_1\ \alpha'_2, \beta'_2, \cdots, \alpha'_g, \beta'_g)P .$$

Note that $V_1 = P^T V P$ is the Seifert matrix for $F$ with respect to the basis $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g\}$.

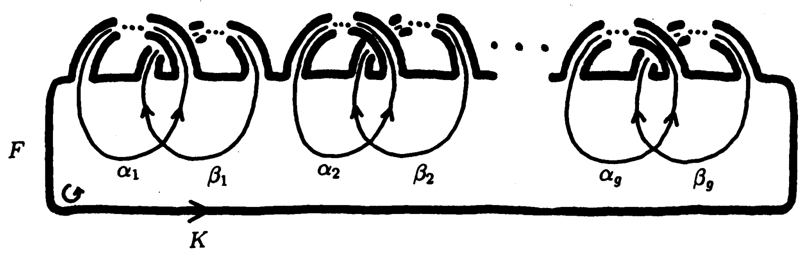


FIGURE 1

Since we have

$$\begin{pmatrix} 1 & 0 & (v-u)P \\ 0 & 1 & 0 \\ 0 & 0 & P \end{pmatrix}^T W \begin{pmatrix} 1 & 0 & (v-u)P \\ 0 & 1 & 0 \\ 0 & 0 & P \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & x & vP \\ 0 & P^T v^T & V_1 \end{pmatrix},$$

we may assume, from the beginning, that $V$ is the Seifert matrix for $F$ with respect to the basis $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g\}$ and that $u = v = (u_1, u_2, \cdots, u_{2g})$.

Let $c$ be an oriented simple close curve in $S^3 - F$ such that $\operatorname{lk}(\alpha_i, c) = u_{2i-1}$ and $\operatorname{lk}(\beta_i, c) = u_{2i}$ for all $i$ with $1 \le i \le g$, where lk denotes the linking number in $S^3$. Such an oriented simple closed curve $c$ always exists. Let $A(c)$ be an oriented annulus embedded in $S^3 - F$ such that $c$ is the center circle of $A(c)$ and that $\operatorname{lk}(c, c^+) = x$, where $c^+$ is the push-off of $c$ in the positive normal direction of $A(c)$. Then consider the surface $F'$ constructed from $F$ and $A(c)$ as depicted in Figure 2. Note that $\partial F$ and $\partial F'$ are isotopic to each other in $S^3$ and hence $F'$ can be regarded as a Seifert surface of $K$.

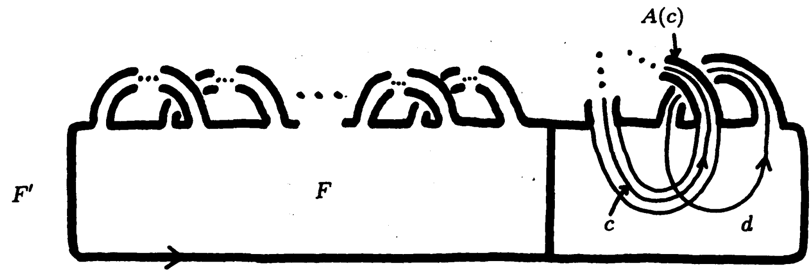


FIGURE 2

Let $\gamma \in H_1(F')$ be the homology class represented by $c$ and $\delta \in H_1(F')$ that represented by the oriented simple closed curve $d$ as in Figure 2. Then it is not difficult to check that the Seifert matrix for $F'$ with respect to the basis $\{\delta, \gamma, \alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g\}$

is equal to $W$. This completes the proof of Lemma 3.2. □

LEMMA 3.3. *Let $K$ be a knot in $S^3$ and $V$ a Seifert matrix for a Seifert surface $F$ of $K$. Suppose that $V \xrightarrow{\varepsilon} W$. Then $W$ is a Seifert matrix for some Seifert surface $F'$ of a knot $K'$ which is obtained from $K$ by applying an $\varepsilon$-unknotting operation once.*

PROOF. Let $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g\}$ be a standard basis of $H_1(F)$ as in the proof of Lemma 3.2. By an argument similar to that in the proof of Lemma 3.2, we may assume that $V$ is the Seifert matrix for $F$ with respect to the basis $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g\}$. Set $V=(v_{ij})$. Recall that $V-V^T$ coincides with the intersection matrix for $F$ with respect to the same basis. Since the intersection numbers satisfy $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ and

$$\alpha_j \cdot \beta_j = \begin{cases} 1 & \text{if } i=j\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

we have

$$v_{ij} - v_{ji} = \begin{cases} 1 & \text{if } j=i+1 \text{ and } i \text{ is odd}\,, \\ -1 & \text{if } j=i-1 \text{ and } i \text{ is even}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Since $V \xrightarrow{\varepsilon} W$, we have

$$W = \begin{pmatrix} \varepsilon & 0 & 0 \\ 1 & x & M \\ 0 & N^T & V \end{pmatrix}$$

for some integer $x$ and integral row vectors $M=(m_1, m_2, \cdots, m_{2g})$ and $N=(n_1, n_2, \cdots, n_{2g})$. Set

$$a_1 = m_2 - n_2\,, \quad a_2 = n_1 - m_1, \cdots, a_{2g-1} = m_{2g} - n_{2g}\,, \quad a_{2g} = n_{2g-1} - m_{2g-1}\,, \quad \text{and}$$

$$c_i = m_i - \sum_{j=1}^{2g} a_j v_{ji}$$

for $i=1, 2, \cdots, 2g$. When $i$ is odd, we have

$$\begin{aligned} m_i - n_i &= -a_{i+1} = -\sum_{j=1}^{2g} a_j (v_{ij} - v_{ji}) + c_i - c_i \\ &= \left( \sum_{j=1}^{2g} a_j v_{ji} + c_i \right) - \left( \sum_{j=1}^{2g} a_j v_{ij} + c_i \right) \\ &= m_i - \left( \sum_{j=1}^{2g} a_j v_{ij} + c_i \right) \end{aligned}$$

and hence

$$n_i = \sum_{j=1}^{2g} a_j v_{ij} + c_i \,. \tag{1}$$

When $i$ is even, we see that the equation (1) also holds by using a similar argument.

Let $c$ be an oriented simple closed curve in $S^3 - F$ such that $\mathrm{lk}(\alpha_i, c) = c_{2i-1}$ and $\mathrm{lk}(\beta_i, c) = c_{2i}$ for all $i$ with $1 \le i \le g$. Such an oriented simple closed curve $c$ always exists. Let $A(c)$ be an oriented annulus embedded in $S^3 - F$ such that $c$ is the center circle of $A(c)$. Then consider the surface $F'$ constructed from $F$ and $A(c)$ as depicted in Figure 3. Note that the knot $\partial F'$ is isotopic to a knot obtained from $K$ by applying an $\varepsilon$-unknotting operation exactly once. The number $l$ of full twists on $A(c)$ as in Figure 3 is determined so that $\mathrm{lk}(\gamma', (\gamma')^+) = x$, where

$$\gamma' = \gamma + a_1\alpha_1 + a_2\beta_2 + \cdots + a_{2g-1}\alpha_g + a_{2g}\beta_g \in H_1(F')$$

and $\gamma \in H_1(F')$ is the homology class represented by $c$.

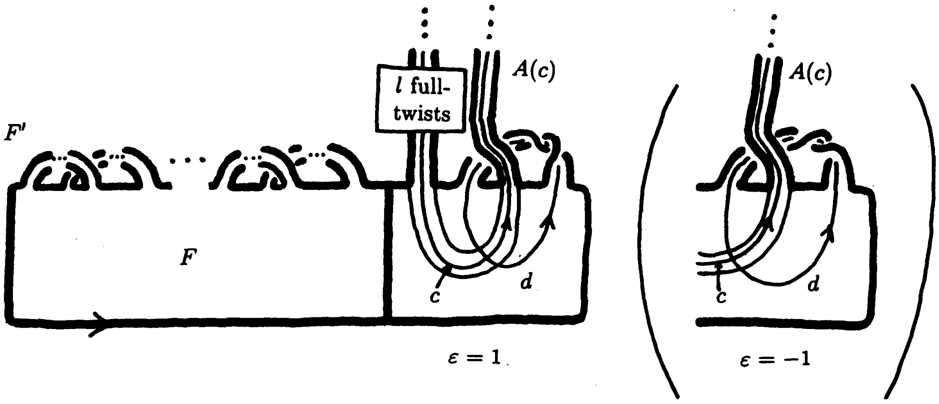


FIGURE 3

Let $\delta \in H_1(F')$ be the homology class represented by the oriented simple closed curve $d$ as in Figure 3. Then we have

$$\mathrm{lk}(\delta, \delta^+) = \varepsilon \,,$$

$$\mathrm{lk}(\delta, (\gamma')^+) = \mathrm{lk}(\delta, \alpha_i^+) = \mathrm{lk}(\delta, \beta_i^+) = 0 \,,$$

$$\mathrm{lk}(\gamma', \delta^+) = 1 \,,$$

$$\mathrm{lk}(\gamma', (\gamma')^+) = x \,,$$

$$\begin{aligned}\mathrm{lk}(\gamma', \alpha_i^+) &= \mathrm{lk}(c, \alpha_i) + \mathrm{lk}(a_1\alpha_1 + a_2\beta_2 + \cdots + a_{2g-1}\alpha_g + a_{2g}\beta_g, \alpha_i^+) \\ &= c_{2i-1} + \sum_{j=1}^{2g} a_j v_{j,2i-1} = m_{2i-1} \,,\end{aligned}$$

$$\mathrm{lk}(\gamma', \beta_i^+) = \mathrm{lk}(c, \beta_i) + \mathrm{lk}(a_1\alpha_1 + a_2\beta_2 + \cdots + a_{2g-1}\alpha_g + a_{2g}\beta_g, \beta_i^+)$$

$$=c_{2i}+\sum_{j=1}^{2g} a_j v_{j,2i}=m_{2i}\,,$$

$$\mathrm{lk}(\alpha_i, \delta^+)=\mathrm{lk}(\beta_i, \delta^+)=0\,,$$

$$\mathrm{lk}(\alpha_i, (\gamma')^+)=\mathrm{lk}(\alpha_i, c)+\mathrm{lk}(\alpha_i, a_1\alpha_1^+ +a_2\beta_2^+ +\cdots+a_{2g-1}\alpha_g^+ +a_{2g}\beta_g^+)$$

$$=c_{2i-1}+\sum_{j=1}^{2g} a_j v_{2i-1,j}=n_{2i-1}\,,$$

$$\mathrm{lk}(\beta_i, (\gamma')^+)=\mathrm{lk}(\beta_i, c)+\mathrm{lk}(\beta_i, a_1\alpha_1^+ +a_2\beta_2^+ +\cdots+a_{2g-1}\alpha_g^+ +a_{2g}\beta_g^+)$$

$$=c_{2i}+\sum_{j=1}^{2g} a_j v_{2i,j}=n_{2i}\,.$$

Then the Seifert matrix for $F'$ with respect to the basis $\{\delta, \gamma', \alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g\}$ is equal to $W$. This completes the proof of Lemma 3.3. □

PROOF OF THEOREM 1.1. Suppose $n=u^a(K)$. Then by Lemmas 3.1, 3.2 and 3.3, we can obtain a knot $K'$ with trivial Alexander polynomial by applying $n$ unknotting operations to $K$. Thus we have

$$u^a(K)\geq\min\{d_G(K, K') : \Delta_{K'}(t)=1\}\,.$$

Conversely, if $K$ is transformed to a knot with trivial Alexander polynomial by applying $k$ unknotting operations, then we have $u^a(K)\leq k$, since each unknotting operation corresponds to an algebraic unknotting operation for the corresponding Seifert matrix (for details, see Lemma 5.6 of §5). Thus we have

$$\mathrm{u}^a(K)\leq\min\{d_G(K, K') : \Delta_{K'}(t)=1\}\,.$$

Hence we have the required equality. This completes the proof of Theorem 1.1. □

## 4. An application.

Let $K$ be a tame knot in $S^3=\partial D^4$. Then it is known that $K$ always bounds a properly and topologically locally flatly embedded compact connected orientable surface $F$ in $D^4$. Define the *topological 4-ball genus* $g^*(K)$ of $K$ to be the minimum genus of such embedded surfaces bounded by $K$. Note that $g^*(K)=0$ if and only if $K$ is a topologically slice knot.

PROPOSITION 4.1. *For a tame knot $K$ in $S^3$, we always have*

$$g^*(K)\leq u^a(K)\,.$$

PROOF. Set $n=u^a(K)$. By Theorem 1.1, we can transform $K$ to a knot $K'$ with trivial Alexander polynomial by applying $n$ unknotting operations. Then by a standard argument, we can construct a properly and smoothly embedded compact connected orientable surface $F'$ of genus $n$ in $S^3 \times [0, 1]$ such that $F' \cap (S^3 \times \{0\}) = K \times \{0\}$ and $F' \cap (S^3 \times \{1\}) = K' \times \{1\}$. On the other hand, since $K'$ has trivial Alexander polynomial, by a result of Freedman [2], $K'$ bounds a properly and topologically locally flatly embedded 2-disk $\Delta$ in $D^4$. Then the compact surface $F = F' \cup \Delta$ embedded in $(S^3 \times [0, 1]) \bigcup_{S^3 \times \{1\}} D^4 \cong D^4$ is a properly and topologically locally flatly embedded compact connected orientable surface of genus $n$ bounded by $K$, where we identify $\partial D^4$ with $S^3 \times \{1\}$. Thus we have $g^*(K) \leq n = u^a(K)$. This completes the proof. □

REMARK 4.2. By a similar argument, we can show that the *smooth* 4-ball genus of a smooth knot in $S^3$ is less than or equal to its (geometric) unknotting number, which is well-known.

REMARK 4.3. We have some examples of knots $K$ with $g^*(K) < u^a(K)$. For example, let $K_1$ be an arbitrary knot with nontrivial Alexander polynomial. Then $K = K_1 \sharp (-K_1)$ is a smooth slice knot and hence $g^*(K) = 0$, where $-K_1$ denotes the mirror image of $K_1$. However, $u^a(K)$ does not vanish, since $K$ has a nontrivial Alexander polynomial.

REMARK 4.4. There have been obtained some estimates from below for the *smooth* 4-ball genus for knots in $S^3$. As has been pointed out in [10, p. 594], some of these estimates are valid for *topological* 4-ball genus as well, as long as at most the $G$-signature theorem is used in the proof. For example, a result of Gilmer [3, §4] gives such an estimate.

REMARK 4.5. Rudolph [10] has shown that for almost all torus knots, the topological 4-ball genus is strictly smaller than the usual genus. On the other hand, it is known that the unknotting number of a torus knot is equal to its genus (see [5, p. 775]). We do not know if the algebraic unknotting number of a torus knot is equal to the usual unknotting number or not.

## 5. Unknotting high dimensional knots.

Let $K$ be a smoothly embedded homotopy $(2n-1)$-sphere in $S^{2n+1}$. We call such a $K$ a $(2n-1)$-*knot*. In this section, we define an operation for $(2n-1)$-knots when $n$ is odd and study its relationship to the algebraic unknotting numbers for the corresponding Seifert matrices.

Let $K$ be a $(2n-1)$-knot with $n$ odd. Let $\Delta$ be a smoothly embedded $(n+1)$-disk in $S^{2n+1}$ with the following properties:

(1) $\partial\Delta \cap K = \emptyset$,

(2) $\Delta$ intersects $K$ transversely along an $(n-1)$-dimensional sphere,

(3) there exists a smoothly embedded $n$-disk $\Delta_0$ in $\Delta$ such that $\partial\Delta_0 = \Delta \cap K$, and

(4) when $n = 1$, the linking number of $\partial\Delta$ and $K$ is equal to zero.

We call the pair $(\Delta, \Delta_0)$ a *transverse disk pair* for $K$. Note that the $(n-1)$-sphere $\Delta \cap K$ is standard in the homotopy $(2n-1)$-sphere $K$. Let $N(\Delta) \cong \Delta \times D^n$ be a normal disk bundle of $\Delta$, which we consider to be embedded in $S^{2n+1}$. We may assume that $N(\Delta)\cap K\cong(\Delta\cap K)\times D^n=\partial\Delta_0\times D^n$. We fix a diffeomorphism $\xi:\Delta\to D^{n+1}$ with $D^{n+1}=\{(x_1,\cdots,x_{n+1})\in\mathbf{R}^{n+1}:x_1^2+\cdots+x_{n+1}^2\le 1\}$ such that $\Delta_0$ corresponds to $D_0=\{(x_1,\cdots,x_n,0)\in D^{n+1}:x_1^2+\cdots+x_n^2\le 1/4\}$. Set $x_0=(0,\cdots,0,1)\in S^n=\partial D^{n+1}$.

We construct a smooth embedding $\varphi:\Delta_0\times D^n\to\Delta\times D^n$ as follows. Let $\psi:D^n\to S^n$ be a smooth map such that $\psi(U)=x_0$ for a neighborhood $U$ of $\partial D^n$ in $D^n$ and that $\psi$ represents a generator of $\pi_n(S^n, x_0)$ (note that there are two choices for $\psi$ up to homotopy relative to $\partial D^n$). Then define the smooth embedding $\varphi': D_0\times D^n\to D^{n+1}\times D^n$ by $\varphi'(z, w)=(\varphi_w(z), w)$, where $\varphi_w: D_0\to D^{n+1}$ ($w\in D^n$) is a smooth family of embeddings such that for each $w\in D^n$, its image coincides with the $n$-disk of radius 1/2 centered at the origin lying on the hyperplane in $\mathbf{R}^{n+1}$ perpendicular to $\psi(w)\in S^n=\partial D^{n+1}$. Then we define $\varphi:\Delta_0\times D^n\to\Delta\times D^n\cong N(\Delta)$ by $\varphi=(\xi^{-1}\times\mathrm{id}_{D^n})\circ\varphi'\circ((\xi\,|\,\Delta_0)\times\mathrm{id}_{D^n})$, where $\mathrm{id}_{D^n}$ denotes the identity map of $D^n$. Note that $\varphi(\partial\Delta_0\times\partial D^n)=\partial\Delta_0\times\partial D^n\subset\Delta\times D^n\cong N(\Delta)$.

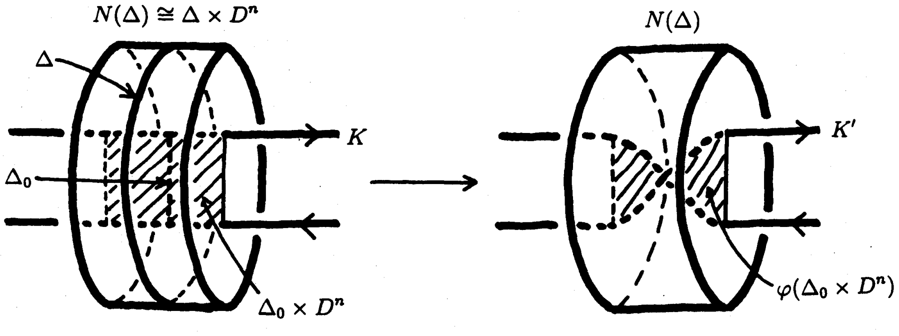


FIGURE 4

Then set $K'=(K-(N(\Delta)\cap K))\cup\varphi(\partial\Delta_0\times D^n)$ (see Figure 4). Note that $K'$ is a $(2n-1)$-dimensional closed manifold smoothly embedded in $S^{2n+1}$. We say that $K'$ is obtained from $K$ by an *elementary twisting operation* with respect to the transverse disk pair $(\Delta, \Delta_0)$. Note that when $n=1$, this is nothing but a usual unknotting operation. Furthermore, note that, for a given transverse disk pair $(\Delta, \Delta_0)$, we have essentially two choices for the elementary twisting operation, depending on the generator of $\pi_n(S^n, x_0)$ chosen for the construction of $\varphi$.

LEMMA 5.1. *If $n$ is odd, then $K'$ is again a homotopy $(2n-1)$-sphere.*

PROOF. When $n=1$, the result is obvious. Let us assume $n\ge 3$. Then it is not difficult to show that the closure of $K-(N(\Delta)\cap K)$ is diffeomorphic to $D^n\times S^{n-1}$. In other words, $K$ is identified with the smooth closed $(2n-1)$-dimensional manifold $(D^n\times S^{n-1})\bigcup_h(S^{n-1}\times D^n)$ obtained by attaching $D^n\times S^{n-1}$ and $S^{n-1}\times D^n\cong\partial\Delta_0\times D^n$ by using a diffeomorphism $h:\partial(S^{n-1}\times D^n)\to\partial(D^n\times S^{n-1})$. Then we see that $K'$ is diffeomorphic to the smooth $(2n-1)$-dimensional closed manifold $(D^n\times S^{n-1})\bigcup_{h\circ h'}(S^{n-1}\times D^n)$ obtained by attaching $D^n\times S^{n-1}$ and $S^{n-1}\times D^n$ by the diffeomorphism $h\circ h'$, where $h':\partial(S^{n-1}\times D^n)\to\partial(S^{n-1}\times D^n)$ is a diffeomorphism of the form

$h'(a, b) = (h'_b(a), b)$ $((a, b) \in S^{n-1} \times \partial D^n)$ for some smooth family of diffeomorphisms $h'_b: S^{n-1} \to S^{n-1}$ $(b \in \partial D^n)$.

Let $\alpha$ and $\beta \in H_{n-1}(S^{n-1} \times S^{n-1})$ be the homology classes represented by $S^{n-1} \times \{*\}$ and $\{*\} \times S^{n-1}$ respectively. Replacing $\alpha$ with $-\alpha$ if necessary, we may assume that the intersection matrix for $S^{n-1} \times S^{n-1}$ on the $(n-1)$-st homology group with respect to the basis $\{\alpha, \beta\}$ is equal to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

since $n-1$ is even. As every orientation preserving diffeomorphism of $S^{n-1} \times S^{n-1}$ preserves the intersection form, it is easy to show that its induced isomorphism on the $(n-1)$-st homology group must have one of the following as its representation matrix with respect to the basis $\{\alpha, \beta\}$:

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $K \cong (D^n \times S^{n-1}) \bigcup_h (S^{n-1} \times D^n)$ is a homotopy $(2n-1)$-sphere, we see that the matrix corresponding to $h_*: H_{n-1}(S^{n-1} \times S^{n-1}) \to H_{n-1}(S^{n-1} \times S^{n-1})$ must coincide with

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

On the other hand, since $h'$ is of the form $h'(a, b) = (h'_b(a), b)$ $((a, b) \in S^{n-1} \times \partial D^n)$, the matrix corresponding to $h'_*: H_{n-1}(S^{n-1} \times S^{n-1}) \to H_{n-1}(S^{n-1} \times S^{n-1})$ must also coincide with either of the above matrices. Then it follows from an easy argument using the Mayer-Vietoris exact sequence that

$$K' \cong (D^n \times S^{n-1}) \bigcup_{h \circ h'} (S^{n-1} \times D^n)$$

is a homology $(2n-1)$-sphere. Since it is also simply connected, we see that $K'$ is a homotopy $(2n-1)$-sphere. This completes the proof. □

REMARK 5.2. When $n$ is even, the consequence of the above lemma does not hold in general.

REMARK 5.3. For $n \geq 3$ and odd, there exist at most two diffeomorphism types of homotopy $(2n-1)$-spheres which can be embedded in $S^{2n+1}$ (see [4, §8]). More precisely, we have the following. For a $(2n-1)$-knot $K$ in $S^{2n+1}$, let $\Delta_K(t)$ denote its Alexander polynomial (see [6]). Then we always have $\Delta_k(-1) \equiv 1$ or $5 \pmod 8$ and if $\Delta_K(-1) \equiv 1 \pmod 8$, then $K$ is diffeomorphic to the standard $(2n-1)$-sphere, while if $\Delta_K(-1) \equiv 5 \pmod 8$, then $K$ is diffeomorphic to the unique homotopy $(2n-1)$-sphere possibly not diffeomorphic to the standard $S^{2n-1}$ which bounds a compact parallelizable $2n$-dimensional manifold (see [6, §3] and [4, §8]).

LEMMA 5.4. *Suppose that $K'$ is obtained from $K$ by an elementary twisting operation. Then the homotopy groups $\pi_i(S^{2n+1}-K)$ and $\pi_i(S^{2n+1}-K')$ are isomorphic to each other for all $i \leq n-1$.*

PROOF. Suppose that $(\Delta, \Delta_0)$ is the transverse disk pair with respect to which the elementary twisting operation is performed. Consider the inclusion map $\imath: S^{2n+1}-(K\cup\Delta_0)\to S^{2n+1}-K$. Since the codimension of $\Delta_0$ in $S^{2n+1}$ is equal to $n+1$, we see that $\imath$ induces an isomorphism on the $i$-th homotopy group for all $i\leq n-1$. Since $S^{2n+1}-(K\cup N(\Delta))$ is a deformation retract of $S^{2n+1}-(K\cup\Delta_0)$, we see that the inclusion map $S^{2n+1}-(K\cup N(\Delta))\to S^{2n+1}-K$ induces an isomorphism on the $i$-th homotopy group for all $i\leq n-1$. By the same argument, we see that the inclusion map $S^{2n+1}-(K'\cup N(\Delta))\to S^{2n+1}-K'$ also induces an isomorphism on the $i$-th homotopy group for all $i\leq n-1$. Since $S^{2n+1}-(K\cup N(\Delta))=S^{2n+1}-(K'\cup N(\Delta))$, we have the conclusion. □

In view of the above lemma, the elementary twisting operation is *not* an unknotting operation for $n\geq 3$ and odd, since it does not affect the fundamental group of the complement. This observation leads us to consider the restricted class of $(2n-1)$-knots as follows.

DEFINITION 5.5. A $(2n-1)$-knot $K$ is called *simple* if $\pi_i(S^{2n+1}-K)$ vanishes for all $i\leq n-1$ (see [7]).

By Lemma 5.4, if $K$ is a simple $(2n-1)$-knot with $n$ odd, then every $(2n-1)$-knot $K'$ obtained from $K$ by an elementary twisting operation is also simple.

Let $K$ be a simple $(2n-1)$-knot and $V$ a Seifert matrix, which is defined to be a representation matrix of the Seifert linking form defined on the $n$-th homology group of a Seifert surface of $K$. It is known that when $n$ is odd, we have $V\in\mathscr{S}$ and that the $S$-equivalence class of such a Seifert matrix is well-defined for a given simple $(2n-1)$-knot $K$ (see [7]). In particular, we can apply algebraic unknotting operations to such a Seifert matrix.

LEMMA 5.6. *Suppose that a simple $(2n-1)$-knot $K'$ is obtained from a simple $(2n-1)$-knot $K$ by an elementary twisting operation. Then a Seifert matrix for an appropriate Seifert surface of $K'$ is obtained from a Seifert matrix for $K$ by applying an algebraic unknotting operation once.*

PROOF. Since $K$ is simple, there exists an $(n-1)$-connected Seifert surface $F$ of $K$ (see [7]). Note that $F$ has a handlebody decomposition consisting of one 0-handle and some $n$-handles. Let $(\Delta, \Delta_0)$ be the transverse disk pair with respect to which the elementary twisting operation on $K$ is performed. Since $F$ has an $n$-dimensional spine, we may assume that $\Delta_0\cap F=\partial\Delta_0$. Then there exists an embedding $\eta: \Delta_0\times D^n\to S^{2n+1}$

such that $\eta(x, 0) = x$ for all $x \in \Delta_0$, that $\eta(\Delta_0 \times D^n) \cap F = \eta(\Delta_0 \times D^n) \cap \partial F = \eta(\partial\Delta_0 \times D^n)$ and that $\Delta \cap \eta(\Delta_0 \times D^n) = \Delta_0$. Furthermore, there exists an embedding $\tilde{\eta} : \Delta_0 \times D^n \times [0, 1] \to S^{2n+1}$ such that $\tilde{\eta} | (\Delta_0 \times D^n \times \{0\}) = \eta$ and that $\tilde{\eta}(\partial\Delta_0 \times D^n \times (0, 1])$ lies on the negative side of the oriented Seifert surface $F$. Set $\tilde{F} = F \cup \tilde{\eta}(\Delta_0 \times \partial(D^n \times [0, 1]))$. Then it is easy to see that $K$ is isotopic to the $(2n-1)$-knot $\tilde{K} = \partial\tilde{F}$, and $\tilde{F}$ is regarded as a Seifert surface of $\tilde{K}$. Moreover, $K \cup \Delta_0$ is isotopic to $\tilde{K} \cup \tilde{\eta}(\Delta_0 \times \{0\} \times \{1\})$. We set $\Delta_0' = \tilde{\eta}(\Delta_0 \times \{0\} \times \{1\})$ and let $\Delta'$ denote the embedded $(n+1)$-disk corresponding to $\Delta$ under the isotopy. We may assume, by a further isotopy, that $\Delta' \cap \tilde{F} = \Delta_0'$ (see Figure 5).

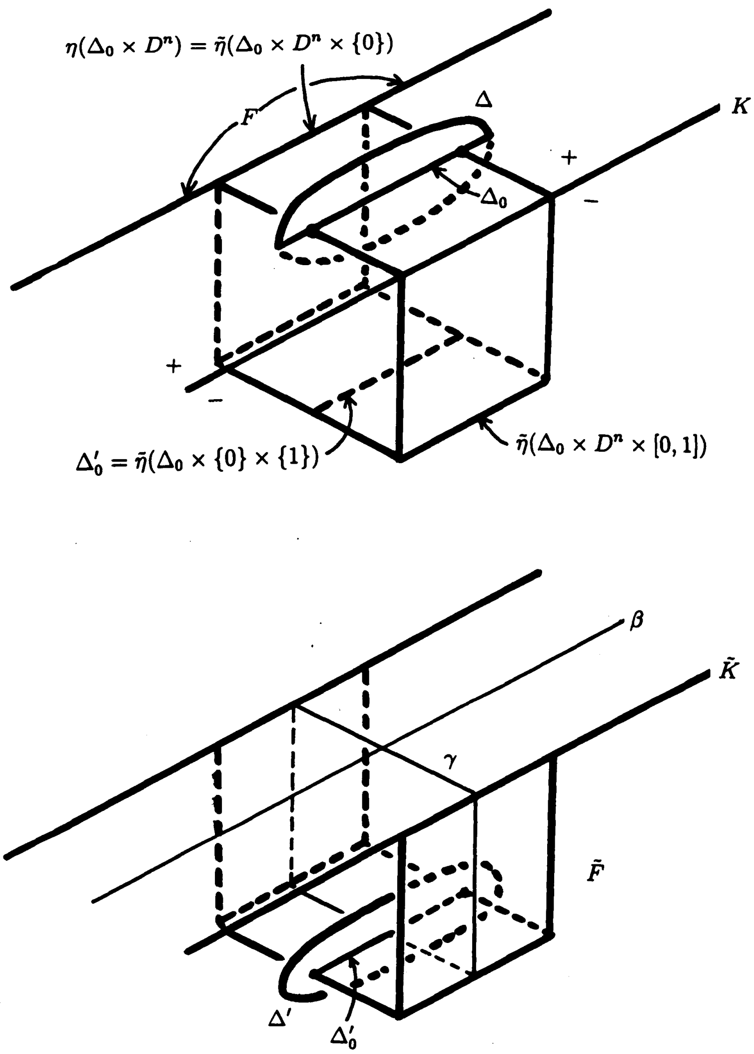


FIGURE 5

Let $N(\Delta') \cong \Delta' \times D^n$ be a normal disk bundle of $\Delta'$ embedded in $S^{2n+1}$. We may assume that $N(\Delta') \cap \tilde{F}$ corresponds to $\Delta_0' \times D^n$. Let $\varphi : \Delta_0' \times D^n \to \Delta' \times D^n \cong N(\Delta')$ be the embedding as defined in the definition of the elementary twisting operation, where we replace $\Delta$ and $\Delta_0$ with $\Delta'$ and $\Delta_0'$ respectively. Then set

$$\tilde{F}' = (\tilde{F} - (N(\Delta') \cap \tilde{F})) \cup \varphi(\Delta'_0 \times D^n)\,.$$

It is not difficult to see that the $(2n-1)$-knot $K'$, which is the result of the elementary twisting operation applied to $K$ with respect to the transverse disk pair $(\Delta, \Delta_0)$, is isotopic to $\partial\tilde{F}'$, which is the result of the elementary twisting operation applied to $\tilde{K}$ with respect to the transverse disk pair $(\Delta', \Delta'_0)$. Thus we have only to show that an appropriate Seifert matrix for $\tilde{F}'$ is obtained from an appropriate Seifert matrix for $F$ by applying an algebraic unknotting operation once.

Let $\{\alpha_i\}$ be a basis of $H_n(F)$. Since $F$ is contained in $\tilde{F}$ and $\tilde{F}'$, we may assume that they are elements of $H_n(\tilde{F})$ and $H_n(\tilde{F}')$. Let $\beta \in H_n(\tilde{F})$ be the homology class represented by the $n$-sphere consisting of $\Delta_0$ and a properly embedded $n$-disk in $F$. Furthermore, let $\gamma \in H_n(\tilde{F})$ be the homology class represented by $\eta(\{r\} \times \partial(D^n \times [0, 1]))$, where $r \in \Delta_0$ is the center of the $n$-disk $\Delta_0$ (see Figure 5). We orient $\beta$ and $\gamma$ so that $\mathrm{lk}(\beta, \gamma^+) = 1$. Then it is easily seen that the Seifert matrix for $\tilde{F}$ with respect to the basis $\{\gamma, \beta, \alpha_i\}$ is equal to

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & x & M \\ 0 & N^T & V \end{pmatrix}$$

for some integer $x$ and some integral row vectors $M$ and $N$, where $V$ is the Seifert matrix for $F$ with respect to the basis $\{\alpha_i\}$. Furthermore, we have a basis for $H_n(\tilde{F}')$ corresponding to $\{\gamma, \beta, \alpha_i\}$ and, with respect to this basis, the Seifert matrix for $\tilde{F}'$ is equal to

$$\begin{pmatrix} \varepsilon & 0 & 0 \\ 1 & x & M \\ 0 & N^T & V \end{pmatrix},$$

which is obtained from $V$ by applying an algebraic unknotting operation once, where $\varepsilon = \pm 1$. This completes the proof. $\square$

The following is a generalization of Lemma 3.2. Recall that every simple $(2n-1)$-knot admits a Seifert surface which is $(n-1)$-connected (see [7]).

LEMMA 5.7. *Let $K$ be a simple $(2n-1)$-knot in $S^{2n+1}$ with $n \geq 1$ and odd and $V$ a Seifert matrix for some $(n-1)$-connected Seifert surface $F$ of $K$. Suppose that $V \sim_c W$ or $V \nearrow W$. Then $W$ is a Seifert matrix for some $(n-1)$-connected Seifert surface $F'$ of the same $(2n-1)$-knot $K$.*

PROOF. For $n = 1$, this is nothing but Lemma 3.2. For $n \geq 3$ and odd, we proceed as in the proof of Lemma 3.2. When $V \sim_c W$, the result is obvious. Suppose that $V \nearrow W$ and let $x$, $u$ and $v$ be as in the proof of Lemma 3.2. Since $F$ is $(n-1)$-connected, we may assume that $F$ has a handlebody decomposition consisting of a 0-handle and $2g$ $n$-handles (see [7]). Let $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g\}$ be a basis of $H_n(F)$ such that each element corresponds to the core of an $n$-handle of $F$. By the same argument as in the proof of Lemma 3.2, we may assume that $V$ is the Seifert matrix for $F$ with respect to the above basis and that $u=v=(u_1, u_2, \cdots, u_{2g})$. Then there exists an oriented $n$-sphere $c$ smoothly embedded in $S^{2n+1}-F$ such that $\mathrm{lk}(\alpha_i, c)=u_{2i-1}$ and $\mathrm{lk}(\beta_i, c)=u_{2i}$ for all $i$ with $1\leq i\leq g$, where lk denotes the linking number in $S^{2n+1}$. Such an oriented embedded $n$-sphere can be consructed as follows. Let $D_i$ $(i=1, 2, \cdots, 2g)$ be an embedded $(n+1)$-disk in $S^{2n+1}$ which is disjoint from the 0-handle of $F$ such that $D_i$ does not intersect the $j$-th $n$-handle of $F$ for all $j\neq i$ and that $D_i$ intersects the core of the $i$-th $n$-handle of $F$ transversely in one point. We may assume that $\partial D_i \cap F=\emptyset$. Then set $c_i'=\partial D_i \subset S^{2n+1}-F$ $(i=1, 2, \cdots, 2g)$. We orient $c_i'$ so that the linking number in $S^{2n+1}$ of $c_i'$ and the homology class corresponding to the core of the $i$-th $n$-handle of $F$ is equal to 1. Then consider $u_i$ parallel copies of $c_i'$ for each $i$, where for $i$ with $u_i<0$, we consider $|u_i|$ parallel copies of $c_i'$ with the orientation reversed. Then take the oriented connected sum of these oriented $n$-spheres in $S^{2n+1}-F$. The resulting oriented $n$-sphere embedded in $S^{2n+1}-F$ has the required properties.

We fix an orientation for $S^n\times D^n$ and let $\zeta_1: S^n\times D^n \to S^{2n+1}-F$ be a smooth embedding such that $\zeta_1(S^n\times\{0\})=c$ and that $\mathrm{lk}(c, c^+)=x$, where 0 is the center of $D^n$ and $c^+$ is the push-off of $c$ in the positive normal direction of $\zeta_1(S^n\times D^n)$. We can construct such an embedding by using an operation similar to the elementary twisting operation in order to adjust $\mathrm{lk}(c, c^+)$. Let $D_0^n$ be an $n$-disk embedded in $S^n$ and let $\zeta_2: D_0^n\times D^n\times[0, 1]\to S^{2n+1}-F$ be a smooth embedding such that $\zeta_2 | D_0^n\times D^n\times\{0\}=\zeta_1 | D_0^n\times D^n$ and that $\zeta_2(D_0^n\times D^n\times(0, 1])$ lies on the negative side of $\zeta_1(S^n\times D^n)$. Then set $E=\zeta_1(S^n\times D^n)\cup\zeta_2(D_0^n\times\partial(D^n\times[0, 1]))$. Note that $\partial E$ is diffeomorphic to the $(2n-1)$-sphere. Furthermore, by considering the smooth family of $(2n-1)$-knots $L_t=\partial(\zeta_1(S^n\times D^n)\cup\zeta_2(D_0^n\times\partial(D^n\times[0, t])))$ $(t\in(0, 1])$ and $L_0=\partial(\zeta_1((S^n-\mathrm{Int}\, D_0^n)\times D^n))$, we see that $\partial E=L_1$ is isotopic to $L_0$ in $S^{2n+1}$. In particular, $\partial E$ is the trivial $(2n-1)$-knot.

Finally let $F'$ be the oriented boundary connected sum of $F$ and $E$ in $S^{2n+1}$. More precisely, let $\zeta_3: D^{2n-1}\times[-1, 1]\to S^{2n+1}$ be a smooth embedding such that $\zeta_3(D^{2n-1}\times[-1, 1])\cap F=\zeta_3(D^{2n-1}\times[-1, 1])\cap\partial F=\zeta_3(D^{2n-1}\times\{-1\})$, $\zeta_3(D^{2n-1}\times[-1, 1])\cap E=\zeta_3(D^{2n-1}\times[-1, 1])\cap\partial E=\zeta_3(D^{2n-1}\times\{1\})$, that $\zeta_3(D^{2n-1}\times[-1, 1])\cap\zeta_2(D_0^n\times D^n\times[0, 1])=\emptyset$, and that the orientations of $\zeta_3(D^{2n-1}\times[-1, 1])$ induced by those of $F$ and $E$ coincide with each other. Then set $F'=F\cup\zeta_3(D^{2n-1}\times[-1, 1])\cup E$. Note that $F'$ is $(n-1)$-connected and that $\partial F'$ is isotopic to $\partial F=K$. In other words, $F'$ can be regarded as a Seifert surface of $K$.

Let $\gamma\in H_n(F')$ be the homology class represented by $c$ and $\delta\in H_n(F')$ that represented by the $n$-sphere $\zeta_2(\{p_0\}\times\partial(D^n\times[0, 1]))$, where $p_0$ is a point in the interior of $D_0^n$. Then we have that the Seifert matrix for $F'$ with respect to the basis $\{\delta, \gamma, \alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g\}$ is equal to $W$, replacing $\delta$ with $-\delta$ if necessary. This completes the proof. □

The following is a generalization of Lemma 3.3.

LEMMA 5.8. *Let $K$ be a simple $(2n-1)$-knot in $S^{2n+1}$ with $n \geq 1$ and odd and $V$ a Seifert matrix for some $(n-1)$-connected Seifert surface $F$ of $K$. Suppose that $V \xrightarrow{\varepsilon} W$. Then $W$ is a Seifert matrix for some $(n-1)$-connected Seifert surface $F'$ of a $(2n-1)$-knot $K'$ which is obtained from $K$ by applying an elementary twisting operation once.*

PROOF. By Poincaré duality, there exists a basis $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g\}$ of $H_n(F)$ such that the intersection numbers satisfy $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ and

$$\alpha_i \cdot \beta_j = \begin{cases} 1 & \text{if } i=j\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

for some $g$. By an argument similar to that in the proof of Lemma 3.2, we may assume that $V$ is the Seifert matrix for $F$ with respect to this basis. Furthermore, when $n \geq 3$ and odd, since $F$ is $(n-1)$-connected, we may assume that $F$ has a handlebody decomposition consisting of a 0-handle and $2g$ $n$-handles and that each element of the above bais corresponds to the core of an $n$-handle, by applying handle slides if necessary. We determine the integers $c_i$ $(i=1, 2, \cdots, 2g)$ as in the proof of Lemma 3.3. Then there exists an oriented $n$-sphere $c$ smoothly embedded in $S^{2n+1}$ such that $\mathrm{lk}(\alpha_i, c) = c_{2i-1}$ and $\mathrm{lk}(\beta_i, c) = c_{2i}$ for all $i$ with $1 \leq i \leq g$.

Then construct $E$ as in the proof of Lemma 5.7 using $c$, where we adjust the linking number $\mathrm{lk}(c, c^+)$ appropriately as in the proof of Lemma 3.3. Let $(\Delta, \Delta_0)$ be a transverse disk pair for $\partial E$ such that $\Delta \subset S^{2n+1} - F$, $\Delta \cap E = \Delta_0 = \zeta_2(D_0^n \times \{0\} \times \{1\})$ and $\Delta \cap \zeta_3(D^{2n-1} \times [-1, 1]) = \varnothing$. Then let $E'$ be the Seifert surface of the $(2n-1)$-knot obtained from $\partial E$ by an elementary twisting operation with respect to the transverse disk pair $(\Delta, \Delta_0)$, where $E'$ is naturally obtained from $E$ as $\tilde{F}'$ is obtained from $\tilde{F}$ in the proof of Lemma 5.6. Then let $F'$ be the oriented boundary connected sum of $F$ and $E'$ in $S^{2n+1}$. Note that $F'$ is $(n-1)$-connected and that $\partial F'$ is isotopic to a $(2n-1)$-knot which is obtained from $K$ by applying an elementary twisting operation once.

Then as in the proof of Lemma 3.3, we obtain the required result. This completes the proof. □

Now we are ready to show the following.

THEOREM 5.9. *The elementary twisting operation for simple $(2n-1)$-knots is an unknotting operation for $n \geq 1$ and odd. In other words, every such simple $(2n-1)$-knot can be transformed to the trivial knot by a finite iteration of elementary twisting oprations.*

PROOF. For $n=1$, the result is obvious, since the elementary twisting operations are nothing but the usual unknotting operations for classical knots.

Let $K$ be a simple $(2n-1)$-knot with $n \geq 3$ and odd. Let $V$ be a Seifert matrix for an $(n-1)$-connected Seifert surface of $K$. Then there exists a sequence of matrices in $\mathscr{S}$ as in Lemma 3.1. By Lemmas 5.7 and 5.8, by a finite iteration of elementary twisting operations, we obtain a simple $(2n-1)$-knot $K_0$ whose Seifert matrix is $S$-equivalent to the zero matrix. By Levine [7], such a knot $K_0$ is trivial. This completes the proof. □

For a simple $(2n-1)$-knot $K$ with $n$ odd, let $u(K)$ denote the minimum number of elementary twisting operations necessary to transform $K$ to the trivial knot. The invariant $u(K)$ is called the *unknotting number* of $K$. This terminology is justified by Theorem 5.9. Then we obtain the following.

THEOREM 5.10. *For a simple $(2n-1)$-knot $K$ with $n \geq 3$ and odd, we have $u(K) = u^a([K])$, where $[K] \in \tilde{\mathscr{S}}$ is the S-equivalence class of a Seifert matrix for $K$. In other words, the unknotting number of $K$ coincides with the algebraic unknotting number of its Seifert matrix.*

PROOF. By the proof of Theorem 5.9, we have

$$u(K) \leq u^a([K]) .$$

On the other hand, by Lemma 5.6, we have

$$u(K) \geq u^a([K]) .$$

Thus we obtain the required equality. This completes the proof. □

REMARK 5.11. Using a result of Fogel [1, Theorem], we can characterize those simple $(2n-1)$-knots with $n \geq 3$ and odd which have unknotting number one, in terms of their Alexander modules and Blanchfield pairings.

We end this paper by posing some problems.

PROBLEM 5.12. (1) Are there some operations which generalize the elementary twisting operation and which can transform an arbitrary $(2n-1)$-knot to the trivial knot when applied successively? How about the case where $n$ is even?

(2) Is the algebraic unknotting number of a knot additive under connected sum? In other words, for $\mathscr{V}$ and $\mathscr{W} \in \tilde{\mathscr{S}}$, if we denote by $\mathscr{V} \oplus \mathscr{W}$ the $S$-equivalence class represented by the matrix $V \oplus W \in \mathscr{S}$ with $V \in \mathscr{V}$ and $W \in \mathscr{W}$, then do we have $u^a(\mathscr{V} \oplus \mathscr{W}) = u^a(\mathscr{V}) + u^a(\mathscr{W})$?

**Added in proof.** After the acceptance of the paper, the author was informed that the results in §3 concerning classical knots had already been obtained independently by Micah Fogel in his thesis (M. E. Fogel, The algebraic unknotting number, PhD thesis, Univ. of California, Berkeley, 1993). Other than the results contained in [1], he also studied algebraic unknotting numbers of genus one two-bridge knots and gave an explicit example of classical knots which show that *the algebraic unknotting number is not additive under connected sum*; i.e., the answer to Problem 5.12 (2) is negative in general. The author would like to thank Micah Fogel for having kindly sent his thesis to the author.

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