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Generalized Q-Functions and UC Hierarchy of B-Type

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Abstract. A generalization of the Schur Q-function is introduced. This generalization, called *generalized Q-function*, is indexed by any pair of strict partitions, and can be expressed by the Pfaffian. A connection to the theory of integrable systems is clarified. Firstly, the bilinear identities satisfied by the generalized Q-functions are given and proved to be equivalent to a system of partial differential equations of infinite order. This system is called the *UC hierarchy of B-type* (*BUC hierarchy*). Secondly, the algebraic structure of the BUC hierarchy is investigated from the representation theoretic viewpoint. Some new kind of the boson-fermion correspondence is established, and a representation of an infinite dimensional Lie algebra, denoted by $\mathfrak{go}_{2\infty}$, is obtained. The bilinear identities are translated to the language of neutral fermions, which turn out to characterize a $\boldsymbol{G}$-orbit of the vacuum vector, where $\boldsymbol{G}$ is the group corresponding to $\mathfrak{go}_{2\infty}$.

1. Introduction

1.1. Schur Q-functions and BKP hierarchy. The Schur Q-functions arise from the theory of projective representations of the symmetric and alternating groups[6], which play similar roles to Schur S-functions for irreducible characters of the general linear groups. The Schur Q-functions are also understood as a $t = -1$-specialization of the Hall-Littlewood symmetric function [14].

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2r})$ be any strict partition, i.e., all the λ_i are non-negative integers such that $\lambda_1 > \lambda_2 > \cdots > \lambda_{2r} \geq 0$. Let $q_n(\boldsymbol{x})$ $(n \in \mathbf{Z})$ be elementary Q-functions which are polynomials in variables $\boldsymbol{x} = (x_1, x_3, x_5, \ldots)$ (see (2.1)). We set for all $m, n \in \mathbf{Z}$,

$$q_{m,n}(\boldsymbol{x})=q_m(\boldsymbol{x})q_n(\boldsymbol{x}) + 2\sum_{k\geq 1}(-1)^k q_{m+k}(\boldsymbol{x})q_{n-k}(\boldsymbol{x})$$

which include $q_n(\boldsymbol{x})$ as $q_{n,0}(\boldsymbol{x}) = q_n(\boldsymbol{x})$ and satisfy $q_{m,n}(\boldsymbol{x}) + q_{n,m}(\boldsymbol{x}) = 2(-1)^m\delta_{m+n,0}$. The Schur Q-function $Q_\lambda(\boldsymbol{x})$ indexed by the strict partition λ is then expressed as a Pfaffian [6, 14]

$$Q_\lambda(\boldsymbol{x}) = \mathrm{Pf}\,[M_\lambda] \tag{1.1}$$

for a skew-symmetric matrix $M_\lambda = (m_{i,j})_{1\leq i,j\leq 2r}$ with entries $m_{i,j} = q_{\lambda_i,\lambda_j}(\boldsymbol{x})$ for $i \neq j$ and $m_{i,j} = 0$ for $i = j$. Notice that the original definition of the Schur Q-function as a

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symmetric function can be obtained by setting the variables as $x_n = 2p_n/n$ where p_n is the n-th power sum.

The Schur Q-functions appear as polynomial solutions of the BKP hierarchy of partial differential equations [1, 2, 3, 7], which is one of the variants of the famous KP (Kadomtsev-Petviashivili) hierarchy [16]. From the viewpoint of infinite dimensional Lie algebras [9], the BKP hierarchy corresponds to a Lie algebra of B-type, while the KP hierarchy corresponds to A-type. It was discovered by Y. You [19] that $Q_\lambda(\boldsymbol{x})$ for any strict partition λ gives a solution of the BKP hierarchy.

As was proved by M. Sato, the Schur S-functions give solutions of the KP hierarchy [16]. It is interesting that solutions for both KP and BKP hierarchies include special functions ("character polynomials") related to the representation theory of the finite groups.

1.2. Universal characters and UC hierarchy. The *universal* (*rational*) *character*, defined by K. Koike [12], is a generalization of the Schur S-function, which gives an irreducible rational representation of the general linear groups. (The Schur S-function describes a polynomial representation). The universal character is defined for each *pair* of (ordinary) partitions, while the S-function is defined for a single partition. Moreover the universal character has a (twisted) Jacobi-Trudi formula (determinant expression).

Recently T. Tsuda [18] proposed an extension of the KP hierarchy called the *UC hierarchy*. The UC hierarchy is an integrable system corresponding to the universal characters. The UC hierarchy may be somewhat strange among many classical integrable systems, because all the differential equations included in the hierarchy have infinite order. However many interesting properties of the UC hierarchy were revealed remarkably similar in style with those for the KP hierarchy, e.g., boson-fermion correspondence, Lie algebra symmetry, Plücker relations, etc.

1.3. Purpose. The aim of this paper, motivated by the fact mentioned above, is two-hold. Firstly we define a generalization of the Schur Q-function, as an analogue of the universal character. This generalization, called the *generalized Q-function*, is defined for any pair of strict partitions and possesses a Pfaffian structure which gives a natural generalization of (1.1). Secondly we present a system of differential equations satisfied by the generalized Q-functions. Since this system may be considered as a B-type analogue of the UC hierarchy, we call it the *UC hierarchy of B-type* (*BUC hierarchy*). We also investigate an algebraic structure of the BUC hierarchy, i.e., an infinitesimal symmetry of a certain infinite dimensional Lie algebra, by using the language of neutral fermions.

1.4. Summary of results. The results of this paper are summarized as follows.

By a strict partition, we mean a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of non-negative integers such that $\lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 0$. Let $(\boldsymbol{x}, \boldsymbol{y}) = (x_1, x_3, x_5, \dots, y_1, y_3, y_5, \dots)$ be a set of independent variables. We first define the generalized Q-functions.

DEFINITION 1.1. Let λ and μ be arbitrary strict partitions. We define the *generalized Q-function* $Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ by

$$Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = Q_\lambda(\boldsymbol{x} - 2\widetilde{\partial}_{\boldsymbol{y}})Q_\mu(\boldsymbol{y} - 2\widetilde{\partial}_{\boldsymbol{x}}) \cdot 1 \tag{1.2}$$

where $\widetilde{\partial}_{\boldsymbol{x}} = \left(\partial_{x_1}, \partial_{x_3}/3, \partial_{x_5}/5, \ldots\right)$.

Since we have $Q_{[\lambda,\emptyset]}(\boldsymbol{x},\boldsymbol{y}) = Q_\lambda(\boldsymbol{x})$ and $Q_{[\emptyset,\mu]}(\boldsymbol{x},\boldsymbol{y}) = Q_\mu(\boldsymbol{y})$ where $\emptyset = (0)$, the generalized Q-function is regarded as a generalization of the Schur Q-function. This definition (1.2) is motivated from a similar relation between the Schur S-function and the universal character (see [18], Lemma 4.7).

The generalized Q-function has a Pfaffian structure as follows (see Theorem 2.8).

THEOREM 1.2. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2r})$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{2s})$ be strict partitions. Let M_λ be a skew-symmetric matrix given in* (1.1) *and $\bar{M}_\mu = (\bar{m}_{i,j})_{1\le i,j\le 2s}$ a skew-symmetric matrix with entries $\bar{m}_{i,j} = q_{\mu_{2s-j+1},\mu_{2s-i+1}}(\boldsymbol{x})$ if $i \neq j$ and $\bar{m}_{i,j} = 0$ if $i = j$. We furthermore define the matrix $N_{\lambda,\mu} = (r_{\mu_{2s-i+1},\lambda_j}(\boldsymbol{x},\boldsymbol{y}))_{1\le i\le 2s, 1\le j\le 2r}$, where $r_{m,n}(\boldsymbol{x},\boldsymbol{y})$ $(m, n \in \mathbf{Z})$ are defined by*

$$r_{m,n}(\boldsymbol{x},\boldsymbol{y}) = q_m(\boldsymbol{y})q_n(\boldsymbol{x}) + 2\sum_{k\ge 1}(-1)^k q_{m-k}(\boldsymbol{y})q_{n-k}(\boldsymbol{x}).$$

Then the generalized Q-function $Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ can be expressed as a Pfaffian of the form

$$Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{Pf}\begin{bmatrix} \bar{M}_\mu & N_{\lambda,\mu} \\ -N^{\mathrm{T}}_{\lambda,\mu} & M_\lambda \end{bmatrix} \tag{1.3}$$

where $N^{\mathrm{T}}_{\lambda,\mu}$ denotes the transpose of $N_{\lambda,\mu}$.

We discuss a relationship between the generalized Q-functions and the following linear differential operators:

$$\begin{aligned} X(z) &= X(z; \boldsymbol{x}, \boldsymbol{y}, \partial_{\boldsymbol{x}}, \partial_{\boldsymbol{y}}) = e^{\xi(\boldsymbol{x}-2\widetilde{\partial}_{\boldsymbol{y}},z)}e^{-2\xi(\widetilde{\partial}_{\boldsymbol{x}},z^{-1})} \\ \bar{X}(z) &= \bar{X}(z; \boldsymbol{x}, \boldsymbol{y}, \partial_{\boldsymbol{x}}, \partial_{\boldsymbol{y}}) = e^{\xi(\boldsymbol{y}-2\widetilde{\partial}_{\boldsymbol{x}},z)}e^{-2\xi(\widetilde{\partial}_{\boldsymbol{y}},z^{-1})} \end{aligned} \tag{1.4}$$

where $\xi(\boldsymbol{x}, z) = \sum_{n\ge 1} x_{2n-1}z^{2n-1}$ and z is a non-zero complex number. In physics, operators of these types are called *vertex operators*.

We express the vertex operators as

$$X(z) = \sum_{n\in\mathbf{Z}} X_n z^n \qquad \bar{X}(z) = \sum_{n\in\mathbf{Z}} \bar{X}_n z^n .$$

The coefficients $X_n = X_n(\boldsymbol{x}, \boldsymbol{y}, \partial_{\boldsymbol{x}}, \partial_{\boldsymbol{y}})$, $\bar{X}_n = \bar{X}_n(\boldsymbol{x}, \boldsymbol{y}, \partial_{\boldsymbol{x}}, \partial_{\boldsymbol{y}})$ $(n \in \mathbf{Z})$ are differential operators on the polynomial algebra $\mathbf{C}[\boldsymbol{x}, \boldsymbol{y}]$, which have the following property (see Proposition 3.1).

PROPOSITION 1.3. *Let $\lambda = (\lambda_1, \dots, \lambda_{2r})$ and $\mu = (\mu_1, \dots, \mu_{2s})$ be strict partitions. Then we have the formula*

$$Q_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}) = X_{\lambda_1} \cdots X_{\lambda_{2r}} \bar{X}_{\mu_1} \cdots \bar{X}_{\mu_{2s}} \cdot 1 . \tag{1.5}$$

We now introduce the BUC hierarchy. Consider the following quadratic relations for an unknown function $\tau = \tau(\boldsymbol{x}, \boldsymbol{y})$:

$$\sum_{n\in\mathbf{Z}}(-1)^n X_n\tau \otimes X_{-n}\tau = \sum_{n\in\mathbf{Z}}(-1)^n \bar{X}_n\tau \otimes \bar{X}_{-n}\tau = \tau \otimes \tau . \tag{1.6}$$

We call this equation *bilinear identities*. The bilinear identities are equivalent to a system of differential equations for τ. In fact, by virtue of a calculus on "Hirota differentials", we can rewrite (1.6) as follows (see Proposition 3.4)

$$\begin{aligned}
&\sum_{n,m\geq 0} q_n(2\boldsymbol{a})q_{n+m}(-2\widetilde{D}_{\boldsymbol{x}})\, q_m(-2\widetilde{D}_{\boldsymbol{y}})e^{\langle \boldsymbol{a}, D_x\rangle}e^{\langle \boldsymbol{b}, D_y\rangle}\tau\cdot\tau = e^{\langle \boldsymbol{a}, D_x\rangle}e^{\langle \boldsymbol{b}, D_y\rangle}\tau\cdot\tau \\
&\sum_{n,m\geq 0} q_n(2\boldsymbol{b})q_m(-2\widetilde{D}_{\boldsymbol{x}})\, q_{n+m}(-2\widetilde{D}_{\boldsymbol{y}})\, e^{\langle \boldsymbol{a}, D_x\rangle}e^{\langle \boldsymbol{b}, D_y\rangle}\tau\cdot\tau = e^{\langle \boldsymbol{a}, D_x\rangle}e^{\langle \boldsymbol{b}, D_y\rangle}\tau\cdot\tau
\end{aligned} \tag{1.7}$$

where $\boldsymbol{a} = (a_1, a_3, a_5, \dots)$, $\boldsymbol{b} = (b_1, b_3, b_5, \dots)$ are newly introduced variables, and we have used notation $\langle \boldsymbol{a}, D_{\boldsymbol{x}}\rangle = \sum_{n\geq 1} a_{2n-1}D_{x_{2n-1}}$ for Hirota differentials $\widetilde{D}_{\boldsymbol{x}} = (D_{x_1},\ D_{x_3}/3,\ D_{x_5}/5, \dots)$. This equation produces an infinite number of Hirota bilinear equations, when we regard (1.7) as multiple Taylor series with variables $(\boldsymbol{a}, \boldsymbol{b})$.

DEFINITION 1.4. A system of the Hirota bilinear equations (1.7) is called the *UC hierarchy of B-type* (or the *BUC hierarchy* for short).

Note that the BUC hierarchy reduces to the (bilinear) BKP hierarchy when τ is independent of $\boldsymbol{y}$.

We have a class of polynomial solutions of the BUC hierarchy (see Theorem 3.7).

THEOREM 1.5. *The generalized Q-functions are solutions of the BUC hierarchy.*

We can also obtain soliton-type solutions, which can be expressed by Pfaffians (see Theorem 3.9 and Proposition 3.11).

We next investigate an algebraic structure of the BUC hierarchy. We use the language of *neutral fermions*. Let $\mathcal{A}$ be an associative $\mathbf{C}$-algebra with generators $\phi_m, \bar{\phi}_m$ $(m \in \mathbf{Z})$ (neutral fermions) satisfying fundamental relations

$$[\phi_m, \phi_n]_+ = [\bar{\phi}_m, \bar{\phi}_n]_+ = (-1)^m\delta_{m+n,0} \quad [\phi_m, \bar{\phi}_n] = 0$$

(the former is "anti-commutative" and the latter is "commutative"). The *Fock representation* of $\mathcal{A}$ is an irreducible representation on the vector space $\mathcal{F} = \mathcal{A}|0\rangle = \{a|0\rangle \mid a \in \mathcal{A}\}$, where $|0\rangle$ satisfies the relations $\phi_n|0\rangle = \bar{\phi}_n|0\rangle = 0$ for $n < 0$. The representation space $\mathcal{F}$ is called the *fermionic Fock space*.

The Fock representation has an explicit realization in the polynomial algebra with infinite variables $\boldsymbol{x}, \boldsymbol{y}, q, \bar{q}$. Let $\mathcal{I}$ be an ideal of $\mathbf{C}[\boldsymbol{x}, \boldsymbol{y}, q, \bar{q}\ ]$ generated by $q^2 - 1/2$ and $\bar{q}^{\,2} - 1/2$.

We define the *bosonic Fock space* by $\mathcal{B} = \mathbf{C}[\boldsymbol{x}, \boldsymbol{y}, q, \bar{q}\ ]/\mathcal{T}$. Then we have the following theorem (see Theorem 4.8).

THEOREM 1.6. *There exists a vector space isomorphism* $\sigma : \mathcal{F} \cong \mathcal{B}$.

The map σ is called the *boson-fermion correspondence*, which gives a new version of a similar correspondence due to DJKM [2, 3, 7] for single component neutral fermions. An explicit description of the map σ can be given in an analogous way to the case of the UC hierarchy [18] (see Section 4.2). Through the boson-fermion correspondence, we can realize the neutral fermion on $\mathcal{B}$ by means of the vertex operators (see Theorem 4.11).

We now discuss a relationship between the BUC hierarchy and an infinite dimensional Lie algebra. Consider any element of the form

$$X = \sum_{i,j \in \mathbf{Z}} (a_{ij} : \phi_i \phi_{-j} : + \bar{a}_{ij} : \bar{\phi}_i \bar{\phi}_{-j} :) + c \quad (c \in \mathbf{C})$$

where : : denotes a normal product of neutral fermions, and $a_{ij}, \bar{a}_{ij}$ are assumed to be subject to the condition $a_{ij} = \bar{a}_{ij} = 0$ if $|i - j| \gg 0$. The set of such elements forms an infinite dimensional Lie algebra, denoted by $\mathfrak{go}_{2\infty}$, which is realized as an infinite rank matrix Lie algebra (see Section 5.1 and 5.2). We define the (formal) Lie group $\boldsymbol{G}$ corresponding to $\mathfrak{go}_{2\infty}$:

$$\boldsymbol{G} = \{e^{X_1} \cdots e^{X_k} \mid X_i \in \mathfrak{go}_{2\infty} : \text{locally nilpotent}\}$$

and consider $\boldsymbol{G}|0\rangle = \{g|0\rangle \mid g \in \boldsymbol{G}\}$. By the boson-fermion correspondence, $\boldsymbol{G}|0\rangle$ is mapped to the subspace of $\mathbf{C}[\boldsymbol{x}, \boldsymbol{y}]$, which turns out to give a description of polynomial solutions of the BUC hierarchy (see Theorem 5.4).

THEOREM 1.7. *A polynomial function* $\tau \in \mathbf{C}[\boldsymbol{x}, \boldsymbol{y}]$ *gives a solution of the BUC hierarchy if and only if there exists a* $g|0\rangle \in \boldsymbol{G}|0\rangle$ *such that* $\tau = \sigma(g|0\rangle)$.

By the explicit formula of σ, a polynomial solution corresponding to $g|0\rangle$ can be given in terms of a vacuum expectation value

$$\tau = \langle e^{H(\boldsymbol{x},\boldsymbol{y})}\, g\rangle \cdot 1$$

with the Hamiltonian operator $H(\boldsymbol{x}, \boldsymbol{y})$ (defined in Section 4.2). Solution expressed in this form is referred to as a τ*-function*, in soliton theory.

Our discussion will be finished by showing the relationship for solutions between the BUC hierarchy and the BKP hierarchy (see Theorem 5.8).

THEOREM 1.8. $\tau(\boldsymbol{x}, \boldsymbol{y}) \in \mathbf{C}[\boldsymbol{x}, \boldsymbol{y}]$ *is a solution of the BUC hierarchy if and only if there exist solutions* $\tau_1(\boldsymbol{x}), \tau_2(\boldsymbol{x}) \in \mathbf{C}[\boldsymbol{x}]$ *of the BKP hierarchy such that*

$$\tau(\boldsymbol{x}, \boldsymbol{y}) = \tau_1(\boldsymbol{x} - 2\widetilde{\partial}_{\boldsymbol{y}})\, \tau_2(\boldsymbol{y} - 2\widetilde{\partial}_{\boldsymbol{x}}) \cdot 1\,. \tag{1.8}$$

1.5. Contents of the paper. In Section 2, we define the generalized Q-functions and prove their Pfaffian structures. In Section 3, we introduce the BUC hierarchy and construct exact solutions of polynomial-type and soliton-type. In Section 4, we establish the boson-fermion correspondence. In Section 5, we define the Lie algebra $\mathfrak{go}_{2\infty}$ and give a Lie algebraic description of the BUC hierarchy. We also discuss a relation to the BKP hierarchy. Section 6 is devoted to concluding remarks.

2. The generalized Q-functions

In this section, we first recall a definition of the Schur Q-function and then introduce the generalized Q-function. The goal of this section is to prove a Pfaffian formula for the generalized Q-function (Theorem 2.8).

2.1. Schur Q-functions.

DEFINITION 2.1. Let $\boldsymbol{x} = (x_1, x_3, x_5, \dots)$ be a set of independent variables. *Elementary Q-functions* $q_n(\boldsymbol{x})$ $(n \in \mathbf{Z})$ are polynomials in $\boldsymbol{x}$ defined by the generating series:

$$\sum_{n\in\mathbf{Z}} q_n(\boldsymbol{x})z^n = e^{\xi(\boldsymbol{x},z)} \quad \text{where} \quad \xi(\boldsymbol{x},z) = \sum_{n\geq 1} x_{2n-1}z^{2n-1}\,. \tag{2.1}$$

Explicitly, $q_0(\boldsymbol{x}) = 1$, $q_n(\boldsymbol{x}) = 0$ $(n < 0)$, and

$$q_n(\boldsymbol{x}) = \sum_{k_1+3k_3+5k_5+\cdots=n} \frac{x_1^{k_1}x_3^{k_3}x_5^{k_5}\cdots}{k_1!k_3!k_5!\cdots} \quad \text{for} \quad n \geq 1\,.$$

The n-th elementary Q-function $q_n(\boldsymbol{x})$ is a homogeneous polynomial of degree n when we put $\deg x_n = n$ $(n \geq 1)$. Since

$$1 = e^{\xi(\boldsymbol{x},z)}e^{-\xi(\boldsymbol{x},z)} = \sum_{m,n\in\mathbf{Z}} q_m(\boldsymbol{x})q_n(-\boldsymbol{x})z^{m+n}$$

we obtain the following equality:

$$\sum_{m+n=k} (-1)^n q_m(\boldsymbol{x})q_n(\boldsymbol{x}) = \delta_{k,0} \tag{2.2}$$

where we have used $q_n(-\boldsymbol{x}) = (-1)^n q_n(\boldsymbol{x})$ by homogeneity. For each $m, n \in \mathbf{Z}$, we define

$$q_{m,n}(\boldsymbol{x}) = q_m(\boldsymbol{x})q_n(\boldsymbol{x}) + 2\sum_{k\geq 1}(-1)^k q_{m+k}(\boldsymbol{x})q_{n-k}(\boldsymbol{x}) \tag{2.3}$$

which include $q_n(\boldsymbol{x})$ as $q_{n,0}(\boldsymbol{x}) = q_n(\boldsymbol{x})$. It is proved by using (2.2) that for all $m, n \in \mathbf{Z}$

$$q_{m,n}(\boldsymbol{x}) + q_{n,m}(\boldsymbol{x}) = 2(-1)^m \delta_{m+n,0}\,. \tag{2.4}$$

A sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of non-negative integers is called a *strict* (or *distinct*) partition if $\lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 0$. The *length* of λ denoted by $l(\lambda)$ is a number of non-zero λ_i in λ, and the sum $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_{l(\lambda)}$ is called the *weight* of λ. We may write any strict partition as $(\lambda_1, \lambda_2, \dots, \lambda_{2r})$ by adding $\lambda_{2r} = 0$ (if necessary). For example, $(1, 0)$, $(2, 0)$, $(3, 2, 1, 0)$ and so on.

DEFINITION 2.2 ([6]). For any strict partition λ, we define the Schur Q-function $Q_\lambda(\boldsymbol{x})$ by the following inductive formulae.

1. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k+1})$ be a strict partition with odd length, i.e., $l(\lambda) = 2k+1$, then

$$Q_{(\lambda_1,\lambda_2,\ldots,\lambda_{2k+1})}(\boldsymbol{x}) = \sum_{i=1}^{2k+1} (-1)^{i+1} q_{\lambda_i}(\boldsymbol{x}) Q_{(\lambda_1,\ldots,\widehat{\lambda_i},\ldots,\lambda_{2k+1})}(\boldsymbol{x}) . \tag{2.5a}$$

2. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$ be a strict partition with even length, i.e., $l(\lambda) = 2k$, then

$$Q_{(\lambda_1,\lambda_2,\ldots,\lambda_{2k})}(\boldsymbol{x}) = \sum_{i=2}^{2k} (-1)^{i+1} q_{\lambda_1,\lambda_i}(\boldsymbol{x}) Q_{(\lambda_2,\ldots,\widehat{\lambda_i},\ldots,\lambda_{2k})}(\boldsymbol{x}). \tag{2.5b}$$

In the above, the hat $\widehat{}$ denotes the absence of the corresponding letter and we put $Q_\emptyset(\boldsymbol{x})=1$, $Q_{(\lambda_1,\lambda_2)}(\boldsymbol{x})=q_{\lambda_1,\lambda_2}(\boldsymbol{x})$ for $\lambda_1 > \lambda_2 \geq 0$.

EXAMPLE 2.3. The following is a list of $Q_\lambda(\boldsymbol{x})$ with degree $|\lambda| \leq 5$:

	λ	$Q_\lambda(\boldsymbol{x})$		λ	$Q_\lambda(\boldsymbol{x})$
degree 0	$\emptyset$	1	degree 4	(4)	$\frac{1}{24}x_1^4 + x_1x_3$
degree 1	(1)	x_1		$(3, 1)$	$\frac{1}{12}x_1^4 - x_1x_3$
degree 2	(2)	$\frac{1}{2}x_1^2$	degree 5	(5)	$\frac{1}{120}x_1^5 + \frac{1}{2}x_1^2x_3 + x_5$
degree 3	(3)	$\frac{1}{6}x_1^3 + x_3$		$(4, 1)$	$\frac{1}{40}x_1^5 - 2x_5$
	$(2, 1)$	$\frac{1}{6}x_1^3 - 2x_3$		$(3, 2)$	$\frac{1}{60}x_1^5 - \frac{1}{2}x_1^2x_3 + 2x_5$

Another useful definition of $Q_\lambda(\boldsymbol{x})$ is given in terms of Pfaffian. For each strict partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2r})$, define the matrix

$$M_\lambda=(m_{i,j})_{1\leq i,j\leq 2r}$$

by putting each (i, j)-th entry as $m_{i,j} = q_{\lambda_i,\lambda_j}(\boldsymbol{x})$ for $i \neq j$ and $m_{i,j} = 0$ for $i = j$. Notice that M_λ is a skew-symmetric matrix by the relation (2.4). The inductive definition of $Q_\lambda(\boldsymbol{x})$ given in (2.5) can be rewritten in terms of Pfaffian [6, 14]:

$$Q_\lambda(\boldsymbol{x}) = \mathrm{Pf}\,[M_\lambda] . \tag{2.6}$$

Here recall that Pfaffian for a skew-symmetric matrix $A = (a_{ij})_{1\leq i,j\leq 2r}$ is defined by

$$\mathrm{Pf}[A] = {\sum}' \operatorname{sgn} \begin{pmatrix} 1 & 2 & \cdots & 2r \\ i_1 & i_2 & \cdots & i_{2r} \end{pmatrix} a_{i_1i_2}a_{i_3i_4}\cdots a_{i_{2r-1}i_{2r}}$$

where ${\sum}'$ means the summation running through $i_1 < i_3 < \cdots < i_{2r-1}$ and $i_1 < i_2, i_3 < i_4, \ldots, i_{2r-1} < i_{2r}$. The equivalence between (2.6) and (2.5) follows from the expansion rule for Pfaffian (cf. [5]):

$$\mathrm{Pf}\,[A] = \sum_{i=1}^{2r} (-1)^{i+j-1} a_{ij}\, \mathrm{Pf}[A^*_{ij}] \quad (1 \le j \le 2r) \tag{2.7}$$

where A^*_{ij} denotes the submatrix of A obtained by deleting the i-th, j-th rows and i-th, j-th columns.

2.2. Definition of generalized Q-functions

DEFINITION 2.4. Let $(\boldsymbol{x}, \boldsymbol{y}) = (x_1, x_3, x_5, \dots, y_1, y_3, y_5, \dots)$ be a set of independent variables. Let λ and μ be arbitrary strict partitions. We define the *generalized Q-function* $Q_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y})$ by the following formula:

$$Q_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}) = Q_\lambda(\boldsymbol{x} - 2\widetilde{\partial}_{\boldsymbol{y}}) Q_\mu(\boldsymbol{y} - 2\widetilde{\partial}_{\boldsymbol{x}}) \cdot 1 \tag{2.8}$$

where $\widetilde{\partial}_{\boldsymbol{x}}$ stands for $(\partial_{x_1}, \partial_{x_3}/3, \partial_{x_5}/5, \dots)$ $(\partial_{x_n} = \frac{\partial}{\partial_{x_n}})$.

Notice that (2.8) resembles a similar relation between the Schur S-function and the universal character (see [18], Lemma 4.7).

EXAMPLE 2.5. If either λ or μ is chosen to be $\emptyset = (0)$, then $Q_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y})$ reduces to the Schur Q-function:

$$Q_{[\lambda,\emptyset]}(\boldsymbol{x}, \boldsymbol{y}) = Q_\lambda(\boldsymbol{x}) \quad Q_{[\emptyset,\mu]}(\boldsymbol{x}, \boldsymbol{y}) = Q_\mu(\boldsymbol{y})\,.$$

If we count the degree of each variables as $\deg x_n = n$ and $\deg y_n = -n$, then $Q_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y})$ is a homogeneous polynomial of degree $|\lambda| - |\mu|$.

EXAMPLE 2.6. Let $[\lambda, \mu] = [(2, 1), (1, 0)]$, then

$$Q_{[(2,1),(1,0)]}(\boldsymbol{x}, \boldsymbol{y}) = \left(\frac{1}{6}x_1^3 - 2x_3\right)\Bigg|_{\boldsymbol{x} \to \boldsymbol{x} - 2\widetilde{\partial}_{\boldsymbol{y}}} \left(y_1 - 2\partial_{x_1}\right) \cdot 1 = \left(\frac{1}{6}x_1^3 - 2x_3\right) y_1 - x_1^2$$

which has homogeneous degree $|\lambda| - |\mu| = 2$. Furthermore by using $Q_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}) = Q_{[\mu,\lambda]}(\boldsymbol{y}, \boldsymbol{x})$ (see Remark 2.7 below), we have

$$Q_{[(1,0),(2,1)]}(\boldsymbol{x}, \boldsymbol{y}) = Q_{[(2,1),(1,0)]}(\boldsymbol{y}, \boldsymbol{x}) = \left(\frac{1}{6}y_1^3 - 2y_3\right) x_1 - y_1^2$$

which has homogeneous degree $|\lambda| - |\mu| = -2$.

REMARK 2.7. Eq.(2.8) can be equivalently rewritten as

$$\begin{aligned} Q_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}) &= \exp\left(-\sum_n \frac{2}{n}\partial_{x_n}\partial_{y_n}\right) Q_\lambda(\boldsymbol{x}) Q_\mu(\boldsymbol{y}) \exp\left(\sum_n \frac{2}{n}\partial_{x_n}\partial_{y_n}\right) \cdot 1 \\ &= \exp\left(-\sum_n \frac{2}{n}\partial_{x_n}\partial_{y_n}\right) [Q_\lambda(\boldsymbol{x}) Q_\mu(\boldsymbol{y})]\,. \end{aligned}$$

so that we have in particular

$$Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = Q_{[\mu,\lambda]}(\boldsymbol{y},\boldsymbol{x}) .$$

2.3. Pfaffian representation. The generalized Q-function can be expressible in terms of Pfaffian described as follows. Let $r_{m,n}(\boldsymbol{x},\boldsymbol{y})$ $(m, n \in \mathbf{Z})$ be polynomials in $\boldsymbol{x}$ and $\boldsymbol{y}$ defined by

$$r_{m,n}(\boldsymbol{x},\boldsymbol{y}) = q_m(\boldsymbol{y})q_n(\boldsymbol{x}) + 2\sum_{k\geq 1}(-1)^k q_{m-k}(\boldsymbol{y})q_{n-k}(\boldsymbol{x}) . \tag{2.9}$$

For strict partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2r})$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{2s})$, define the matrix

$$N_{\lambda,\mu} = (r_{\mu_{2s-i+1},\lambda_j}(\boldsymbol{x},\boldsymbol{y}))_{1\leq i\leq 2s, 1\leq j\leq 2r} .$$

We also define the skew-symmetric matrix

$$\bar{M}_\mu = \left(\bar{m}_{i,j}\right)_{1\leq i,j\leq 2s}$$

by putting each (i, j)-th entry as $\bar{m}_{i,j} = q_{\mu_{2s-j+1},\mu_{2s-i+1}}(\boldsymbol{y})$ for $i \neq j$ and $\bar{m}_{i,j} = 0$ for $i = j$.

We will prove the following theorem.

THEOREM 2.8. *The generalized Q-function $Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ for any pair of strict partitions $[\lambda,\mu]$ can be expressed as a single Pfaffian of the form*:

$$Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{Pf}\begin{bmatrix} \bar{M}_\mu & N_{\lambda,\mu} \\ -N^{\mathrm{T}}_{\lambda,\mu} & M_\lambda \end{bmatrix} \tag{2.10}$$

where $N^{\mathrm{T}}_{\lambda,\mu}$ denotes the transposed matrix of $N_{\lambda,\mu}$.

EXAMPLE 2.9. If $[\lambda,\mu] = [(2, 1), (1, 0)]$, then

$$M_\lambda = \begin{bmatrix} 0 & q_{2,1} \\ q_{1,2} & 0 \end{bmatrix} \quad \bar{M}_\mu = \begin{bmatrix} 0 & \bar{q}_{1,0} \\ \bar{q}_{0,1} & 0 \end{bmatrix} \quad N_{\lambda,\mu} = \begin{bmatrix} r_{0,2} & r_{0,1} \\ r_{1,2} & r_{1,1} \end{bmatrix}$$

and hence

$$Q_{[(2,1),(1,0)]}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{Pf}\begin{bmatrix} 0 & \bar{q}_{1,0} & r_{0,2} & r_{0,1} \\ \bar{q}_{0,1} & 0 & r_{1,2} & r_{1,1} \\ -r_{0,2} & -r_{1,2} & 0 & q_{2,1} \\ -r_{0,1} & -r_{1,1} & q_{1,2} & 0 \end{bmatrix} = \bar{q}_{1,0}q_{2,1} - r_{0,2}r_{1,1} + r_{0,1}r_{1,2}$$

where we have denoted $q_{m,n} = q_{m,n}(\boldsymbol{x})$, $\bar{q}_{m,n} = q_{m,n}(\boldsymbol{y})$ and $r_{m,n} = r_{m,n}(\boldsymbol{x},\boldsymbol{y})$ for simplicity. Substituting

$$\bar{q}_{1,0} = y_1 \quad q_{2,1} = x_1^3/6 - 2x_3$$

$$r_{0,1} = x_1 \quad r_{0,2} = x_1^2/2 \quad r_{1,1} = x_1y_1 - 2 \quad r_{1,2} = x_1^2y_1/2 - 2x_1$$

yields the same result given in Example 2.6.

EXAMPLE 2.10. If $[\lambda, \mu] = [(m, 0), (n, 0)]$, then

$$Q_{[(m,0),(n,0)]}(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{Pf}\begin{bmatrix} 0 & \bar{q}_{n,0} & r_{0,m} & r_{0,0} \\ \bar{q}_{0,n} & 0 & r_{n,m} & r_{n,0} \\ -r_{0,m} & -r_{n,m} & 0 & q_{m,0} \\ -r_{0,0} & -r_{n,0} & q_{0,m} & 0 \end{bmatrix} = r_{n,m} .$$

Here notice that $r_{0,m} = q_m(\boldsymbol{x})$ and $r_{n,0} = q_n(\boldsymbol{y})$.

The rest of this section is devoted to the proof of Theorem 2.8.

2.4. Generating function. In order to prove Theorem 2.8, we here give a supplementary discussion on the generating function for generalized Q-functions.

First we consider a generating function $E(\boldsymbol{x}; z_1, z_2) = \sum_{m,.n\in\mathbf{Z}} q_{m,n}(\boldsymbol{x}) z_1^m z_2^n$ for the polynomials $q_{m,n}(\boldsymbol{x})$ defined in (2.3). It is easy to check that this function can be written equivalently as

$$E(\boldsymbol{x}; z_1, z_2) = \frac{1 - z_2/z_1}{1 + z_2/z_1} e^{\xi(\boldsymbol{x}, z_1)} e^{\xi(\boldsymbol{x}, z_2)} \tag{2.11}$$

where the rational function on the right hand side should be understood as the Laurent series with $|z_1| > |z_2|$, i.e.,

$$\frac{1 - z_2/z_1}{1 + z_2/z_1} = 1 + 2 \sum_{k \geq 1} (-z_2/z_1)^k .$$

Now by introducing more complex variables $\underline{z} = (z_1, z_2, \ldots, z_{2r})$, we define the function

$$G_1(\boldsymbol{x}; \underline{z}) = \prod_{1 \leq i < j \leq 2r} E(\boldsymbol{x}; z_i, z_j) \tag{2.12}$$

or equivalently

$$G_1(\boldsymbol{x}; \underline{z}) = \prod_{1 \leq i < j \leq 2r} \frac{1 - z_j/z_i}{1 + z_j/z_i} \prod_{i=1}^{2r} e^{\xi(\boldsymbol{x}, z_i)} \tag{2.13}$$

where as remarked above, the right hand side should be considered as the Laurent series with $|z_1| > |z_2| > \cdots > |z_{2r}|$. Then we have

LEMMA 2.11 ([14]). *The coefficient of* $\underline{z}^\lambda = z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_{2r}^{\lambda_{2r}}$ *in* (2.13) *is equal to* $Q_\lambda(\boldsymbol{x})$.

PROOF. In general, we have the factorization formula

$$\prod_{1 \leq i < j \leq 2r} \frac{z_i - z_j}{z_i + z_j} = \mathrm{Pf}\left[\frac{z_i - z_j}{z_i + z_j}\right]_{1 \leq i, j \leq 2r} \tag{2.14}$$

(cf. [14]). Using this, we can rewrite (2.13) as

$$G_1(\boldsymbol{x};\underline{z}) = \mathrm{Pf}\left[\frac{1-z_j/z_i}{1+z_j/z_i}e^{\xi(\boldsymbol{x},z_i)}e^{\xi(\boldsymbol{x},z_j)}\right]_{1\le i,j\le 2r} = \mathrm{Pf}\left[E(\boldsymbol{x};z_i,z_j)\right]_{1\le i,j\le 2r}$$

so that, by definition of Pfaffians, $G_1(\boldsymbol{x};\underline{z})$ is a sum of terms of the form

$$\pm E(\boldsymbol{x};z_{i_1},z_{i_2})E(\boldsymbol{x};z_{i_3},z_{i_4})\cdots E(\boldsymbol{x};z_{i_{2r-1}},z_{i_{2r}})$$

where $i_1 < i_3 < \cdots < i_{2r-1}$ and $i_1 < i_2,\ldots,i_{2r-1} < i_{2r}$. The coefficient of $\underline{z}^\lambda = z_1^{\lambda_1}z_2^{\lambda_2}\cdots z_{2r}^{\lambda_{2r}}$ in the term just written yields

$$\pm q_{\lambda_{i_1},\lambda_{i_2}}(\boldsymbol{x})q_{\lambda_{i_3},\lambda_{i_4}}(\boldsymbol{x})\cdots q_{\lambda_{i_{2r-1}},\lambda_{i_{2r}}}(\boldsymbol{x})\,.$$

Hence the coefficient of $\underline{z}^\lambda$ in $G_1(\boldsymbol{x};\underline{z})$ equals

$$\sum{}' \pm q_{\lambda_{i_1},\lambda_{i_2}}(\boldsymbol{x})q_{\lambda_{i_3},\lambda_{i_4}}(\boldsymbol{x})\cdots q_{\lambda_{i_{2r-1}},\lambda_{i_{2r}}}(\boldsymbol{x}) = \mathrm{Pf}[M_\lambda] = Q_\lambda(\boldsymbol{x})$$

as required. □

Thanks to this lemma, we can introduce a generating function for the generalized Q-functions. Introduce the sets of complex variables $\underline{z} = (z_1,z_2,\ldots,z_{2r})$, $\underline{w} = (w_1,w_2,\ldots,w_{2s})$, and define

$$G_2(\boldsymbol{x},\boldsymbol{y};\underline{z},\underline{w}) = G_1(\boldsymbol{x}-2\widetilde{\partial}_{\boldsymbol{y}};\underline{z})G_1(\boldsymbol{y}-2\widetilde{\partial}_{\boldsymbol{x}};\underline{w})\cdot 1\,. \tag{2.15}$$

By Lemma A.1 in the appendix, this function has an expression

$$G_2(\boldsymbol{x},\boldsymbol{y};\underline{z},\underline{w}) = \prod_{1\le i<j\le 2r}\frac{1-z_j/z_i}{1+z_j/z_i}\prod_{1\le i<j\le 2s}\frac{1-w_j/w_i}{1+w_j/w_i}\prod_{\substack{1\le i\le 2r\\1\le j\le 2s}}\frac{1-z_iw_j}{1+z_iw_j}$$
$$\times\prod_{i=1}^{2r}e^{\xi(\boldsymbol{x},z_i)}\prod_{i=1}^{2s}e^{\xi(\boldsymbol{y},w_i)}\,. \tag{2.16}$$

where the right hand side is considered as the Laurent series with $|w_{2s}^{-1}| > \cdots > |w_1^{-1}| > |z_1| > \cdots > |z_{2r}|$.

Comparing Definition 2.4, Lemma 2.11 and (2.15), we obtain

LEMMA 2.12. *The coefficient of* $\underline{z}^\lambda\underline{w}^\mu = z_1^{\lambda_1}\cdots z_{2r}^{\lambda_{2r}}w_1^{\mu_1}\cdots w_{2s}^{\mu_{2s}}$ *in* (2.16) *is equal to* $Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$.

2.5. Verification of Theorem 2.8. First of all, we express the right hand side of (2.16) as a certain Pfaffian. In what follows, we denote the variables as

$$\underline{z} \stackrel{\text{def}}{=} (z_1,z_2,\ldots,z_{2r}) \quad \underline{w} \stackrel{\text{def}}{=} (z_{-1}^{-1},z_{-2}^{-1},\ldots,z_{-2s}^{-1})\,.$$

Then

$$G_2(\boldsymbol{x}, \boldsymbol{y}; \underline{z}, \underline{w}) = \prod_{\substack{-2s \le i < j \le 2r \\ i,j \ne 0}} \frac{1 - z_j/z_i}{1 + z_j/z_i} \prod_{i=1}^{2s} e^{\xi(\boldsymbol{y}, z_i^{-1})} \prod_{j=1}^{2r} e^{\xi(\boldsymbol{x}, z_j)} .$$

By the same argument as in the proof of Lemma 2.11, the right hand side can be written as a Pfaffian of the form

$$G_2(\boldsymbol{x}, \boldsymbol{y}; \underline{z}, \underline{w}) = \mathrm{Pf}\left[\frac{1 - z_j/z_i}{1 + z_j/z_i} e^{\xi_i} e^{\xi_j}\right]_{\substack{-2s \le i < j \le 2r \\ i,j \ne 0}}$$

where the exponential factor ξ_i denotes $\xi(\boldsymbol{y}, z_i^{-1})$ for negative i and $\xi(\boldsymbol{x}, z_i)$ for positive i. We put

$$C_{i,j} = \frac{1 - z_j/z_i}{1 + z_j/z_i} e^{\xi_i} e^{\xi_j} \quad (-2s \le i, j \le 2r, \ \ i, j \ne 0) .$$

Then by definition of Pfaffians, $G_2(\boldsymbol{x}, \boldsymbol{y}; \underline{z}, \underline{w})$ can be expressed as a sum

$$G_2(\boldsymbol{x}, \boldsymbol{y}; \underline{z}, \underline{w}) = \sum{}' \pm C_{i_{-2s}, i_{-2s+1}} C_{i_{-2s+2}, i_{-2s+3}} \cdots C_{i_{2r-1}, i_{2r}} . \tag{2.17}$$

We now prepare the following lemma.

LEMMA 2.13. *Let $C_{i,j}^{(m,n)}$ be the coefficient of $z_i^m z_j^n$ in $C_{i,j}$, i.e., $C_{i,j} = \sum_{m,n \in \mathbf{Z}} C_{i,j}^{(m,n)} z_i^m z_j^n$. Then*

$$C_{i,j}^{(m,n)} = \begin{cases} q_{m,n}(\boldsymbol{x}) & 0 < i < j \\ -q_{n,m}(\boldsymbol{x}) & 0 < j < i \\ q_{-n,-m}(\boldsymbol{y}) & i < j < 0 \\ -q_{-m,-n}(\boldsymbol{y}) & j < i < 0 \\ r_{-m,n}(\boldsymbol{x}, \boldsymbol{y}) & i < 0 < j \\ -r_{-n,m}(\boldsymbol{x}, \boldsymbol{y}) & j < 0 < i . \end{cases} \tag{2.18}$$

Note that each $C_{i,j}$ has been considered as the Laurent series with $|z_{-2s}| > \cdots > |z_{-1}| > |z_1| > \cdots > |z_{2r}|$.

PROOF. This is a straightforward calculation. □

Let $[\lambda, \mu]$ be a pair of strict partitions, which we denote by

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2r}) \quad \mu = (-\lambda_{-1}, -\lambda_{-2}, \ldots, -\lambda_{-2s})$$

i.e., $\lambda_1 > \lambda_2 > \cdots > \lambda_{2r} \ge 0$ and $\lambda_{-1} < \lambda_{-2} < \cdots < \lambda_{-2s} \le 0$. We already know by Lemma 2.12 that the coefficient of $z_1^{\lambda_1} \cdots z_{2r}^{\lambda_{2r}} z_{-1}^{\lambda_{-1}} \cdots z_{-2s}^{\lambda_{-2s}}$ in the right hand side of (2.17), i.e.,

$$\sum{}' \pm C^{(\lambda_{i_{-2s}},\lambda_{i_{-2s+1}})}_{i_{-2s},i_{-2s+1}} C^{(\lambda_{i_{-2s+2}},\lambda_{i_{-2s+3}})}_{i_{-2s+2},i_{-2s+3}} \cdots C^{(\lambda_{i_{2r-1}},\lambda_{i_{2r}})}_{i_{2r-1},i_{2r}}$$

is equal to $Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$. This sum can be rewritten into a Pfaffian

$$\mathrm{Pf}\left[C^{(\lambda_i,\lambda_j)}_{i,j}\right]_{\substack{-2s\le i,j\le 2r\\ i,j\neq 0}}$$

where we note that this Pfaffian does make sense by Lemma 2.13. By re-denoting μ as $(\mu_1,\mu_2,\ldots,\mu_{2s})$ and substituting the expressions (2.18), we see that the last Pfaffian coincides with the desirous formula (2.10). The proof is completed.

3. The UC hierarchy of B-type (BUC hierarchy)

In this section, we introduce a series of differential equations called the BUC hierarchy. We construct two classes of exact solutions: polynomial solutions and soliton-type solutions.

3.1. Vertex operators. We start with linear differential operators of infinite order, called vertex operators. Let $X(z) = X(z;\boldsymbol{x},\boldsymbol{y},\partial_{\boldsymbol{x}},\partial_{\boldsymbol{y}})$ and $\bar{X}(z) = \bar{X}(z;\boldsymbol{x},\boldsymbol{y},\partial_{\boldsymbol{x}},\partial_{\boldsymbol{y}})$ be the following linear differential operators:

$$X(z) = e^{\xi(\boldsymbol{x}-2\widetilde{\partial}_{\boldsymbol{y}},z)}e^{-2\xi(\widetilde{\partial}_{\boldsymbol{x}},z^{-1})} \quad \bar{X}(z) = e^{\xi(\boldsymbol{y}-2\widetilde{\partial}_{\boldsymbol{x}},z)}e^{-2\xi(\widetilde{\partial}_{\boldsymbol{y}},z^{-1})} \tag{3.1}$$

with a non-zero complex number z. With regards the notation used here, see the previous section. In physics, the operators of these types are called *vertex operators*

We define differential operators $X_n = X_n(\boldsymbol{x},\boldsymbol{y},\partial_{\boldsymbol{x}},\partial_{\boldsymbol{y}})$, $\bar{X}_n = \bar{X}_n(z;\boldsymbol{x},\boldsymbol{y},\partial_{\boldsymbol{x}},\partial_{\boldsymbol{y}})$ ($n \in \mathbf{Z}$) by expansions

$$X(z) = \sum_{n\in\mathbf{Z}} X_n z^n \quad \bar{X}(z) = \sum_{n\in\mathbf{Z}} \bar{X}_n z^n$$

or equivalently in terms of the elementary Q-functions (see (2.1))

$$X_n = \sum_{i\ge 0} q_{n+i}(\boldsymbol{x}-2\widetilde{\partial}_{\boldsymbol{y}})q_i(-2\widetilde{\partial}_{\boldsymbol{x}}) \quad \bar{X}_n = \sum_{i\ge 0} q_{n+i}(\boldsymbol{y}-2\widetilde{\partial}_{\boldsymbol{x}})q_i(-2\widetilde{\partial}_{\boldsymbol{y}})\,.$$

These operators have the following important property.

PROPOSITION 3.1. *Let $\lambda = (\lambda_1,\ldots,\lambda_{2r})$, $\mu = (\mu_1,\ldots,\mu_{2s})$ be arbitrary strict partitions. Then we have a formula*

$$Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = X_{\lambda_1}\cdots X_{\lambda_{2r}}\bar{X}_{\mu_1}\cdots\bar{X}_{\mu_{2s}}\cdot 1\,. \tag{3.2}$$

PROOF. Applying Lemma A.1 in the appendix, one easily obtain the equality

$$G_2(\boldsymbol{x},\boldsymbol{y};\underline{z},\underline{w}) = X(z_1)\cdots X(z_{2r})\bar{X}(w_1)\cdots\bar{X}(w_{2s})\cdot 1\,.$$

Equating the coefficient of $\underline{z}^{\lambda}\underline{w}^{\mu} = z_1^{\lambda_1}\cdots z_{2r}^{\lambda_{2r}}w_1^{\mu_1}\cdots w_{2s}^{\mu_{2s}}$ in both sides, we obtain the desirous formula by Lemma 2.12. □

LEMMA 3.2. *The differential operators X_n, $\bar{X}_n$ ($n \in \mathbf{Z}$) satisfy the relations*

$$[X_m, X_n]_+ = [\bar{X}_m, \bar{X}_n]_+ = 2(-1)^m \delta_{m+n,0} \quad [X_m, \bar{X}_n] = 0 \tag{3.3}$$

where $[A, B]_+ \stackrel{\text{def}}{=} AB + BA$ *and* $[A, B] \stackrel{\text{def}}{=} AB - BA$. *In particular*, $X_m^2 = \bar{X}_m^2 = \delta_{m,0}$.

PROOF. The relations are written equivalently as

$$[X(z), X(w)]_+ = [\bar{X}(z), \bar{X}(w)]_+ = 2\delta(-w/z) \quad [X(z), \bar{X}(w)] = 0$$

where $\delta(z) = \sum_{n\in\mathbf{Z}} z^n = 1/(1-z) + z^{-1}/(1-z^{-1})$ denotes the formal δ-function. These relations can be checked by using Lemma A.1 and $\delta(w/z)f(z,w) = \delta(w/z)f(z,z)$ that holds for any Laurent series $f(z,w)$ of z and w. □

3.2. Bilinear identities. Consider the bilinear relations for an unknown function $\tau = \tau(\boldsymbol{x}, \boldsymbol{y})$:

$$\sum_{n\in\mathbf{Z}} (-1)^n X_n\tau \otimes X_{-n}\tau = \sum_{n\in\mathbf{Z}} (-1)^n \bar{X}_n\tau \otimes \bar{X}_{-n}\tau = \tau \otimes \tau \tag{3.4}$$

or equivalently in term of the vertex operators:

$$\oint X(z)\tau \otimes X(-z)\tau \,\frac{dz}{2\pi i z} = \oint \bar{X}(z)\tau \otimes \bar{X}(-z)\tau \,\frac{dz}{2\pi i z} = \tau \otimes \tau \tag{3.5}$$

where the contour integral means an algebraic operation extracting the coefficient of z^0 for the (formal) Laurent series, i.e.,

$$\oint z^n \frac{dz}{2\pi i z} = \delta_{n,0}\,.$$

Hereafter we call (3.4) or (3.5) *bilinear identities*.

Note that

$$X_0 \cdot 1 = \bar{X}_0 \cdot 1 = 1 \quad X_n \cdot 1 = \bar{X}_n \cdot 1 = 0 \quad n < 0$$

which follows from $X(z) \cdot 1 = e^{\xi(\boldsymbol{x},z)}$ and $\bar{X}(z) \cdot 1 = e^{\xi(\boldsymbol{y},z)}$. Therefore any constant satisfies the bilinear identities.

The bilinear identities can be converted to an infinite number of Hirota bilinear equations for τ. Firstly the bilinear identities are equivalent to the following equations:

$$\oint e^{\xi(\boldsymbol{x}'-\boldsymbol{x},z)} \tau(\boldsymbol{x}' - 2[z^{-1}], \boldsymbol{y}' - 2[z])\tau(\boldsymbol{x} + 2[z^{-1}], \boldsymbol{y} + 2[z]) \frac{dz}{2\pi i z} = \tau(\boldsymbol{x}', \boldsymbol{y}')\tau(\boldsymbol{x}, \boldsymbol{y}) \tag{3.6a}$$

$$\oint e^{\xi(\boldsymbol{y}'-\boldsymbol{y},z)} \tau(\boldsymbol{x}' - 2[z], \boldsymbol{y}' - 2[z^{-1}])\tau(\boldsymbol{x} + 2[z], \boldsymbol{y} + 2[z^{-1}]) \frac{dz}{2\pi i z} = \tau(\boldsymbol{x}', \boldsymbol{y}')\tau(\boldsymbol{x}, \boldsymbol{y}) \tag{3.6b}$$

where $\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{x}, \boldsymbol{y}$ are arbitrary and $[z]$ stands for $(z, z^3/3, z^5/5, \dots)$. To rewrite these further, we here recall the definition of Hirota differentials [5].

DEFINITION 3.3. Let $P(D)$ be arbitrary polynomial (possibly formal power series) in $D = (D_{x_1}, D_{x_3}, D_{x_5}, \dots, D_{y_1}, D_{y_3}, D_{y_5}, \dots)$ (symbols of Hirota differentials). The Hirota bilinear equation $P(D)f \cdot g = 0$ for functions $f = f(\boldsymbol{x}, \boldsymbol{y})$ and $g = g(\boldsymbol{x}, \boldsymbol{y})$ is defined by setting

$$P(D)f \cdot g = P(\partial) f(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b}) g(\boldsymbol{x} - \boldsymbol{a}, \boldsymbol{y} - \boldsymbol{b})\big|_{\boldsymbol{a}=\boldsymbol{b}=0}$$

where $\partial = (\partial_{a_1}, \partial_{a_3}, \partial_{a_5}, \dots, \partial_{b_1}, \partial_{b_3}, \partial_{b_5}, \dots)$. For instance,

$$D_{x_1} f \cdot g = \partial_{x_1} f \cdot g - f \partial_{x_1} g \quad D_{x_1}^2 f \cdot g = \partial_{x_1}^2 f \cdot g - \partial_{x_1} f \cdot \partial_{x_1} g + f \partial_{x_1}^2 g\,.$$

We have the following proposition.

PROPOSITION 3.4. *The bilinear identities are represented as the following system of Hirota bilinear equations*:

$$\sum_{n,m\geq 0} q_n(2\boldsymbol{a}) q_{n+m}(-2\widetilde{D}_{\boldsymbol{x}})\, q_m(-2\widetilde{D}_{\boldsymbol{y}}) e^{\langle \boldsymbol{a}, D_{\boldsymbol{x}}\rangle} e^{\langle \boldsymbol{b}, D_{\boldsymbol{y}}\rangle} \tau \cdot \tau = e^{\langle \boldsymbol{a}, D_{\boldsymbol{x}}\rangle} e^{\langle \boldsymbol{b}, D_{\boldsymbol{y}}\rangle} \tau \cdot \tau \tag{3.7a}$$

$$\sum_{n,m\geq 0} q_n(2\boldsymbol{b}) q_m(-2\widetilde{D}_{\boldsymbol{x}})\, q_{n+m}(-2\widetilde{D}_{\boldsymbol{y}})\, e^{\langle \boldsymbol{a}, D_{\boldsymbol{x}}\rangle} e^{\langle \boldsymbol{b}, D_{\boldsymbol{y}}\rangle} \tau \cdot \tau = e^{\langle \boldsymbol{a}, D_{\boldsymbol{x}}\rangle} e^{\langle \boldsymbol{b}, D_{\boldsymbol{y}}\rangle} \tau \cdot \tau \tag{3.7b}$$

where $\boldsymbol{a} = (a_1, a_3, a_5, \dots)$, $\boldsymbol{b} = (b_1, b_3, b_5, \dots)$ *are newly introduced variables, and*

$$\widetilde{D}_{\boldsymbol{x}} = (D_{x_1}, D_{x_3}/3, D_{x_5}/5, \dots) \quad \langle \boldsymbol{a}, D_{\boldsymbol{x}}\rangle = \sum_{n\geq 1} a_{2n-1} D_{x_{2n-1}}\,.$$

PROOF. If we substitute $\boldsymbol{x}' \mapsto \boldsymbol{x} + \boldsymbol{a}$, $\boldsymbol{x} \mapsto \boldsymbol{x} - \boldsymbol{a}$, $\boldsymbol{y}' \mapsto \boldsymbol{y} + \boldsymbol{b}$, $\boldsymbol{y} \mapsto \boldsymbol{y} - \boldsymbol{b}$ into (3.6a), then

$$\begin{aligned}\oint e^{\xi(2\boldsymbol{a},z)} &= \tau(\boldsymbol{x} + \boldsymbol{a} - 2[z^{-1}], \boldsymbol{y} + \boldsymbol{b} - 2[z])\tau(\boldsymbol{x} - \boldsymbol{a} + 2[z^{-1}], \boldsymbol{y} - \boldsymbol{b} + 2[z])\,\frac{dz}{2\pi i z} \\ &= \tau(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})\, \tau(\boldsymbol{x} - \boldsymbol{a}, \boldsymbol{y} - \boldsymbol{b})\,.\end{aligned}$$

By virtue of a calculus on Hirota differentials (cf. [9], Ch.14), this equation can be rewritten into

$$\oint e^{\xi(2\boldsymbol{a},z)} e^{-2\xi(\widetilde{D}_{\boldsymbol{x}}, z^{-1})} e^{-2\xi(\widetilde{D}_{\boldsymbol{y}}, z)} e^{\langle \boldsymbol{a}, D_{\boldsymbol{x}}\rangle} e^{\langle \boldsymbol{b}, D_{\boldsymbol{y}}\rangle} \tau \cdot \tau \frac{dz}{2\pi i z} = e^{\langle \boldsymbol{a}, D_{\boldsymbol{x}}\rangle} e^{\langle \boldsymbol{b}, D_{\boldsymbol{y}}\rangle} \tau \cdot \tau\,.$$

Let us express the first three exponentials on the left hand side as

$$\Big(\sum_{n\in\mathbf{Z}} q_n(2\boldsymbol{a}) z^n\Big)\Big(\sum_{n\in\mathbf{Z}} q_n\big(-2\widetilde{D}_{\boldsymbol{x}}\big) z^{-n}\Big)\Big(\sum_{n\in\mathbf{Z}} q_n\big(-2\widetilde{D}_{\boldsymbol{y}}\big) z^n\Big)$$

by using (2.1). Then by picking out the coefficient of z^0 for the last bilinear equation, we obtain the first equation (3.7a). We can derive the second equation (3.7b), starting from (3.6b) by the same calculation. □

Eq. (3.7) includes two copies of the bilinear BKP hierarchy in each sectors $\boldsymbol{x}$, $\boldsymbol{y}$. Indeed if τ is independent of $\boldsymbol{y}$, then (3.7a) reduces to the bilinear form of the BKP hierarchy [2]

$$\sum_{n\geq 1} q_n(2\boldsymbol{a})q_n(-2\widetilde{D}_{\boldsymbol{x}})\, e^{\langle \boldsymbol{a}, D_x\rangle}\tau\cdot\tau = 0 \tag{3.8a}$$

while (3.7b) turns to a trivial identity $e^{\langle \boldsymbol{a}, D_x\rangle}\tau\cdot\tau = e^{\langle \boldsymbol{a}, D_x\rangle}\tau\cdot\tau$. Similarly, if τ is independent of $\boldsymbol{x}$, then (3.7) reduces to the bilinear form of the BKP hierarchy with $\boldsymbol{y}$-flows:

$$\sum_{n\geq 1} q_n(2\boldsymbol{b})q_n(-2\widetilde{D}_{\boldsymbol{y}})\, e^{\langle \boldsymbol{b}, D_y\rangle}\tau\cdot\tau = 0\,. \tag{3.8b}$$

By expanding (3.7) into a multiple Taylor series with respect to the variables $(\boldsymbol{a}, \boldsymbol{b}) = (a_1, a_3, a_5, \dots, b_1, b_3, b_5, \dots)$, we can obtain several Hirota bilinear equations from each coefficients of $a_1^{r_1}a_3^{r_3}a_5^{r_5}\cdots b_1^{s_1}b_3^{s_3}b_5^{s_5}\cdots$. For example, from the coefficient of $\boldsymbol{a}^0\boldsymbol{b}^0 = a_1^0a_3^0a_5^0\cdots b_1^0b_3^0b_5^0\cdots$ in (3.7a), one obtains

$$\sum_{m\geq 1} q_m(-2\widetilde{D}_{\boldsymbol{x}})q_m(-2\widetilde{D}_{\boldsymbol{y}})\tau\cdot\tau = 0$$

which is a Hirota bilinear equation of infinite order. All the equations obtained from (3.7) in this way are in fact differential equations of infinite order, as in the case of the UC hierarchy [18].

DEFINITION 3.5. A whole system of the Hirota bilinear equations included in (3.7) is called the *UC hierarchy of B-type* or the *BUC hierarchy*.

3.3. The BUC hierarchy and generalized Q-functions. We present a class of polynomial solutions of the BUC hierarchy. First we prove the following transformation property for the BUC hierarchy.

LEMMA 3.6. *If* $\tau \in \mathbf{C}[\boldsymbol{x}, \boldsymbol{y}]$ *is a solution of the BUC hierarchy, then so are* $X_k\tau$ *and* $\bar{X}_k\tau$ *for any* $k \in \mathbf{Z}$.

PROOF. We express the bilinear identities (3.4) compactly as

$$\Omega_{\mathcal{B}}(\tau\otimes\tau) = \bar{\Omega}_{\mathcal{B}}(\tau\otimes\tau) = \tau\otimes\tau\,.$$

by defining $\Omega_{\mathcal{B}} \overset{\text{def}}{=} \sum_{n\in\mathbf{Z}}(-1)^n X_n\otimes X_{-n}$ and $\bar{\Omega}_{\mathcal{B}} \overset{\text{def}}{=} \sum_{n\in\mathbf{Z}}(-1)^n \bar{X}_n\otimes \bar{X}_{-n}$. The proof of the lemma is done by showing that $\Omega_{\mathcal{B}}$ and $\bar{\Omega}_{\mathcal{B}}$ commute with $X_k\otimes X_k$ and $\bar{X}_k\otimes\bar{X}_k$ for all $k\in\mathbf{Z}$. By virtue of Lemma 3.2, we obtain

$$\begin{aligned}\Omega_{\mathcal{B}}\circ(X_k\otimes X_k) &= \sum_{n\in\mathbf{Z}}(-1)^n(-X_kX_n + 2(-1)^n\delta_{k+n,0})\otimes(-X_kX_{-n} + 2(-1)^n\delta_{k-n,0})\\ &= (X_k\otimes X_k)\circ\Omega_{\mathcal{B}} - 2(X_k^2\otimes 1 + 1\otimes X_k^2) + 4\sum_{n\in\mathbf{Z}}(-1)^n\delta_{k+n,0}\otimes\delta_{k-n,0}\end{aligned}$$

$$= (X_k \otimes X_k) \circ \Omega_{\mathcal{B}}$$

where the last equality is due to $X_k^2 = \delta_{k,0}$. Similarly we have $[\bar{\Omega}_{\mathcal{B}}, \bar{X}_k \otimes \bar{X}_k] = 0$. Also we have $[\Omega_{\mathcal{B}}, \bar{X}_k \otimes \bar{X}_k] = [\bar{\Omega}_{\mathcal{B}}, X_k \otimes X_k] = 0$ by the commutative relation in Lemma 3.2. Hence the proof is completed. □

Since $\tau \equiv 1$ is a solution of the BUC hierarchy, Lemma 3.6 together with Proposition 3.1 leads to the following theorem.

THEOREM 3.7. *The generalized Q-function for any pair of strict partitions gives a solution of the BUC hierarchy.*

REMARK 3.8. Recall that the generalized Q-function not depending on $\boldsymbol{y}$ reduces to the Schur Q-function. Thus we can recover from Theorem 3.7 the result by Y. You [19] that the Schur Q-functions are solutions of the BKP hierarchy (3.8a).

3.4. (M, N)**-soliton solutions.** The BUC hierarchy has another important class of exact solutions called (M, N)-soliton solutions. Note that Theorem 3.9 and Proposition 3.11 given below will be proved in the appendix.

Define linear differential operators $\Gamma^{\pm}(z, w) = \Gamma^{\pm}(z, w; \boldsymbol{x}, \boldsymbol{y}, \partial_{\boldsymbol{x}}, \partial_{\boldsymbol{y}})$ by

$$\begin{aligned} \Gamma^+(z, w) &= e^{\xi(\boldsymbol{x}-2\widetilde{\partial}_{\boldsymbol{y}}, z)} e^{\xi(\boldsymbol{x}-2\widetilde{\partial}_{\boldsymbol{y}}, w)} e^{-2\xi(\widetilde{\partial}_{\boldsymbol{x}}, z^{-1})} e^{-2\xi(\widetilde{\partial}_{\boldsymbol{x}}, w^{-1})} \\ \Gamma^-(z, w) &= e^{\xi(\boldsymbol{y}-2\widetilde{\partial}_{\boldsymbol{x}}, z)} e^{\xi(\boldsymbol{y}-2\widetilde{\partial}_{\boldsymbol{x}}, w)} e^{-2\xi(\widetilde{\partial}_{\boldsymbol{y}}, z^{-1})} e^{-2\xi(\widetilde{\partial}_{\boldsymbol{y}}, w^{-1})} \quad (z, w \in \mathbf{C}^{\times}) \end{aligned} \tag{3.9}$$

which are also called vertex operators. These vertex operators have a close connection to a certain infinite dimensional Lie algebra (see Section 5.2). Notice that

$$X(z)X(w) = \frac{1 - w/z}{1 + w/z} \Gamma^+(z, w) \quad \bar{X}(z)\bar{X}(w) = \frac{1 - w/z}{1 + w/z} \Gamma^-(z, w) \tag{3.10}$$

where we assume $|z| > |w|$.

Let M, N be arbitrary non-negative integers. We define the function

$$\tau_{M,N}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = \prod_{i=-M}^{-1} e^{c_i \Gamma^-(a_i^{-1}, b_i^{-1})} \prod_{i=1}^{N} e^{c_i \Gamma^+(a_i, b_i)} \cdot 1 \tag{3.11}$$

with a_i, b_i, c_i $(i \in \{-M, \dots, -1, 1, \dots, N\})$ being complex constants such that $a_i \neq -a_j$, $a_i \neq -b_j$, $b_i \neq -b_j$ for $i < j$. Then we have

THEOREM 3.9. *The function $\tau_{M,N}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a solution of the BUC hierarchy, which is called the (M, N)-soliton solution.*

By using Lemma A.2 in the appendix, we can rewrite this solution as follows.

COROLLARY 3.10. *The (M, N)-soliton solution can be written as*

$$\tau_{M,N}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = \sum_{K \subset I_- \cup I_+} \Bigg(\prod_{\substack{i<j \\ i,j \in K}} A_{ij} \Bigg) \exp\Bigg(\sum_{i \in K} \eta_i \Bigg) \tag{3.12}$$

where $I_- = \{-M, \dots, -1\}$, $I_+ = \{1, \dots, N\}$ denote indexing sets and we have denoted

$$A_{i,j} = \frac{(a_i - a_j)(a_i - b_j)(b_i - a_j)(b_i - b_j)}{(a_i + a_j)(a_i + b_j)(b_i + a_j)(b_i + b_j)} \quad i, j \in I_- \cup I_+;$$

$$\eta_i = \begin{cases} \xi(\boldsymbol{y}, a_i^{-1}) + \xi(\boldsymbol{y}, b_i^{-1}) + \log c_i & i \in I_- \\ \xi(\boldsymbol{x}, a_i) + \xi(\boldsymbol{x}, b_i) + \log c_i & i \in I_+ \,. \end{cases}$$

The (M, N)-soliton solution can be expressible in terms of a Pfaffian described as follows. For each $i, j \in I_- \cup I_+$, we define the 2×2-matrix $S_{i,j}$ of the form

$$S_{i,j} = \begin{bmatrix} \frac{a_i - a_j}{a_i + a_j} e^{\widehat{\xi}_i(\boldsymbol{a}) + \widehat{\xi}_j(\boldsymbol{a})} & \frac{a_i - b_j}{a_i + b_j} e^{\widehat{\xi}_i(\boldsymbol{a}) + \widehat{\xi}_j(\boldsymbol{b})} \\ \frac{b_i - a_j}{b_i + a_j} e^{\widehat{\xi}_i(\boldsymbol{b}) + \widehat{\xi}_j(\boldsymbol{a})} & \frac{b_i - b_j}{b_i + b_j} e^{\widehat{\xi}_i(\boldsymbol{b}) + \widehat{\xi}_j(\boldsymbol{b})} \end{bmatrix} \tag{3.13}$$

where the exponential factors are defined by

$$\widehat{\xi}_i(\boldsymbol{a}) = \begin{cases} \xi(\boldsymbol{y}, a_i^{-1}) + \alpha_i^0 & i \in I_- \\ \xi(\boldsymbol{x}, a_i) + \alpha_i^0 & i \in I_+ \end{cases} \qquad \widehat{\xi}_i(\boldsymbol{b}) = \begin{cases} \xi(\boldsymbol{y}, b_i^{-1}) + \beta_i^0 & i \in I_- \\ \xi(\boldsymbol{x}, b_i) + \beta_i^0 & i \in I_+ \end{cases}$$

with α_i^0, β_i^0 $(i \in I_- \cup I_+)$ being arbitrary complex constants such that

$$\alpha_i^0 + \beta_i^0 = \log \frac{a_i + b_i}{a_i - b_i} + \log c_i \quad \text{for all } i \in I_- \cup I_+ \,. \tag{3.14}$$

We define the following $2(M + N) \times 2(M + N)$-matrix:

$$S = \left(S_{i,j}\right)_{i,j \in I_- \cup I_+}$$

which is a skew-symmetric matrix, because $S_{i,j} = -(S_{j,i})^{\mathrm{T}}$. Then we have the following proposition (see Appendix B for the proof).

PROPOSITION 3.11. *The (M, N)-soliton solution can be represented as a Pfaffian*

$$\tau_{M,N}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = \mathrm{Pf}\,[J + S] \tag{3.15}$$

where J denotes the $2(M + N) \times 2(M + N)$-matrix $\sum_{i=1}^{M+N} (E_{2i-1,2i} - E_{2i,2i-1})$ with $E_{i,j}$ being a matrix unit with a 1 on the (i, j)-th entry and zeros elsewhere.

EXAMPLE 3.12. As an example, we have the $(1, 1)$-soliton solution:

$$\tau_{1,1}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = 1 + e^{\eta_{-1}} + e^{\eta_1} + \frac{(a_{-1} - a_1)(b_{-1} - b_1)(a_{-1} - b_1)(b_{-1} - a_1)}{(a_{-1} + a_1)(b_{-1} + b_1)(a_{-1} + b_1)(b_{-1} + a_1)} e^{\eta_{-1} + \eta_1} \,.$$

Notice that $\eta_i = \widehat{\xi}_i(\boldsymbol{a})+\widehat{\xi}_i(\boldsymbol{b})-\alpha_i^0-\beta_i^0+\log c_i$ $(i \in I_- \cup I_+)$, so that $e^{\eta_i} = \frac{a_i-b_i}{a_i+b_i}\, e^{\widehat{\xi}_i(\boldsymbol{a})+\widehat{\xi}_i(\boldsymbol{b})}$. The $(1,1)$-soliton solution can thus be written as follows:

$$1+\frac{a_{-1}-b_{-1}}{a_{-1}+b_{-1}}e^{\widehat{\xi}_{-1}(\boldsymbol{a})+\widehat{\xi}_{-1}(\boldsymbol{b})}+\frac{a_1-b_1}{a_1+b_1}e^{\widehat{\xi}_1(\boldsymbol{a})+\widehat{\xi}_1(\boldsymbol{b})}$$
$$+\frac{(a_{-1}-a_1)(b_{-1}-b_1)(a_{-1}-b_1)(b_{-1}-a_1)(a_{-1}-b_{-1})(a_1-b_1)}{(a_{-1}+a_1)(b_{-1}+b_1)(a_{-1}+b_1)(b_{-1}+a_1)(a_{-1}+b_{-1})(a_1+b_1)}\,e^{\widehat{\xi}_{-1}(\boldsymbol{a})+\widehat{\xi}_{-1}(\boldsymbol{b})+\widehat{\xi}_1(\boldsymbol{a})+\widehat{\xi}_1(\boldsymbol{b})}$$

which coincides with an expansion of $\mathrm{Pf}\,[J+S]$ with $M=N=1$.

4. The boson-fermion correspondence

This section deals with a new kind of the (neutral) boson-fermion correspondence. The results of this section will be used in next section.

4.1. Neutral fermions and fermionic Fock space

DEFINITION 4.1. Let $\mathcal{A}$ be an associative algebra over $\mathbf{C}$ generated by $\phi_m, \bar{\phi}_m$ $(m \in \mathbf{Z})$ with anti-commutative and commutative relations

$$[\phi_m,\phi_n]_+ = [\bar{\phi}_m,\bar{\phi}_n]_+ = (-1)^m\delta_{m+n,0} \quad [\phi_m,\bar{\phi}_n]=0 \tag{4.1}$$

where $[A,B]_+ \overset{\mathrm{def}}{=} AB+BA$ and $[A,B] \overset{\mathrm{def}}{=} AB-BA$. The generators $\phi_m, \bar{\phi}_m$ $(m \in \mathbf{Z})$ are referred to as *neutral fermions*. In particular, $\phi_0^2 = \bar{\phi}_0^2 = 1/2$.

Note that the above definition of neutral fermions differs to $\phi_m, \widehat{\phi}_m$ in [7, 19] defined from "charged free fermions".

The algebra $\mathcal{A}$ has a standard representation in the so-called fermionic Fock space.

DEFINITION 4.2. The *fermionic Fock space* is a vector space $\mathcal{F} = \mathcal{A}|0\rangle = \{a|0\rangle \mid a \in \mathcal{A}\}$ with a vacuum vector $|0\rangle$ satisfying the relations

$$\phi_n|0\rangle = \bar{\phi}_n|0\rangle = 0 \quad \text{for} \quad n<0\,. \tag{4.2}$$

A representation of $\mathcal{A}$ on $\mathcal{F}$, by left multiplication, is called the *Fock representation*. Note that $\mathcal{F}$ is an infinite dimensional vector space with a basis

$$\{\phi_{m_1}\cdots\phi_{m_r}\bar{\phi}_{n_1}\cdots\bar{\phi}_{n_s}|0\rangle \mid m_1>\cdots>m_r\ge 0,\ n_1>\cdots>n_s\ge 0\}\,.$$

Similarly we define a representation of $\mathcal{A}$ on the vector space $\mathcal{F}^* = \langle 0|\mathcal{A} = \{\langle 0|a \mid a \in \mathcal{A}\}$ by right multiplication, where the (dual) vacuum vector $\langle 0|$ satisfies the relations

$$\langle 0|\phi_n = \langle 0|\bar{\phi}_n = 0 \quad \text{for}\ n>0\,. \tag{4.3}$$

Two representations of $\mathcal{A}$ above are dual to each other with respect to the canonical pairing between $\mathcal{F}^*$ and $\mathcal{F}$, namely

DEFINITION 4.3. There exists a unique bilinear form $\mathcal{F}^* \times \mathcal{F} \to \mathbf{C}$:

$$(\langle 0|a,\ b|0\rangle) \mapsto \langle 0|a \cdot b|0\rangle \stackrel{\text{def}}{=} \langle ab\rangle$$

such that $\langle 1\rangle = 1$, $\langle \phi_0 \rangle = \langle \bar{\phi}_0 \rangle = \langle \phi_0 \bar{\phi}_0 \rangle = 0$. The quantity $\langle a\rangle$ ($a \in \mathcal{A}$) is said to be the (*vacuum*) *expectation value* of a.

EXAMPLE 4.4. The expectation values for quadratic elements are calculated as

$$\langle \phi_m \phi_n\rangle = \langle \bar{\phi}_m \bar{\phi}_n\rangle = \begin{cases} 0 & n < 0 \\ \delta_{m,0}/2 & n = 0 \\ (-1)^m \delta_{m+n,0} & n > 0 \end{cases}$$

and $\langle \phi_m \bar{\phi}_n\rangle = 0$ for all $m, n \in \mathbf{Z}$.

EXAMPLE 4.5 (Wick's theorem). Let w_i $(1 \le i \le r)$ be elements from $\mathcal{A}$ expressed by linear combinations of ϕ_m and $\bar{\phi}_m$ ($m \in \mathbf{Z}$). The *Wick's theorem* gives a simple way to compute the expectation value (cf. [8]):

$$\langle w_1 w_2 \cdots w_r\rangle = \sum{}' \operatorname{sgn} \begin{pmatrix} 1 & \cdots & r \\ i_1 & \cdots & i_r \end{pmatrix} \langle w_{i_1} w_{i_2}\rangle \cdots \langle w_{i_{r-1}} w_{i_r}\rangle \quad (r\text{: even})$$

and $\langle w_1 w_2 \cdots w_r\rangle = 0$ for odd r. Here $\sum'$ denotes the summation over all permutations $(i_1, \ldots, i_r)$ of $(1, \ldots, r)$ satisfying $i_1 < i_3 < \cdots < i_{2r-1}$ and $i_1 < i_2, i_3 < i_4, \ldots, i_{2r-1} < i_{2r}$.

We decompose the fermionic Fock space $\mathcal{F}$ as a direct sum

$$\mathcal{F} = \bigoplus_{i,j=0,1} \mathcal{F}_{i,j}\,. \tag{4.4}$$

Here each element of $\mathcal{F}$ belongs to the space $\mathcal{F}_{i,j}$ $(i, j = 0, 1)$, according to whether it is spanned by monomial vectors of the form $\phi_{m_1} \cdots \phi_{m_r} \bar{\phi}_{n_1} \cdots \bar{\phi}_{n_s}|0\rangle$ with $r \equiv i$, $s \equiv j \bmod 2$.

We further decompose each $\mathcal{F}_{i,j}$ into subspaces $\mathcal{F}_{i,j}^{(d_1,d_2)}$ with fixed degree (d_1, d_2):

$$\mathcal{F}_{i,j} = \bigoplus_{d_1,d_2 \ge 0} \mathcal{F}_{i,j}^{(d_1,d_2)} \tag{4.5}$$

by letting for $|\nu\rangle = \phi_{m_1} \cdots \phi_{m_r} \bar{\phi}_{n_1} \cdots \bar{\phi}_{n_s}|0\rangle$,

$$\deg(|\nu\rangle) = (d_1, d_2) \quad \text{if} \ \sum m_i = d_1 \ \ \sum n_i = d_2\,.$$

Note that $\deg(|\nu\rangle)$ is always a pair of non-negative integers.

The following lemma will be used later.

LEMMA 4.6. *The dimension of* $\mathcal{F}_{i,j}^{(d_1,d_2)}$ *is equal to* $\bar{P}(d_1)\bar{P}(d_2)$, *where* $\bar{P}(n)$ *denotes the number of the strict partition for the natural number* n, *i.e.*,

$$\prod_{n\geq 1}\frac{1}{1-z^{2n-1}} = \sum_{m_1,m_3,m_5,\cdots\geq 0} z^{m_1+3m_3+5m_5+\cdots} = \sum_{n\in\mathbf{Z}}\bar{P}(n)z^n .$$

4.2. Boson-fermion correspondence. The Fock representation has an explicit realization in the polynomial algebra with infinite variables called the bosonic Fock space.

DEFINITION 4.7. Let $\mathbf{C}[\boldsymbol{x}, \boldsymbol{y}, q, \bar{q}\ ]$ be a polynomial algebra with variables $\boldsymbol{x}$, $\boldsymbol{y}$, q,$\bar{q}$, and $\mathcal{T}$ an ideal of $\mathbf{C}[\boldsymbol{x}, \boldsymbol{y}, q, \bar{q}\ ]$ generated by elements of the form $q^2 - 1/2$, $\bar{q}^{\,2} - 1/2$. The *bosonic Fock space* is defined by

$$\mathcal{B} = \mathbf{C}[\boldsymbol{x}, \boldsymbol{y}, q, \bar{q}\ ]/\mathcal{T}$$

which has a decomposition $\mathcal{B} = \bigoplus_{i,j=0,1}\mathcal{B}_{i,j}$, where $\mathcal{B}_{i,j} = \mathbf{C}[\boldsymbol{x}, \boldsymbol{y}]q^i\bar{q}^{\,j}$.

The boson-fermion correspondence states that the fermionic Fock space can be identified with the bosonic Fock space.

THEOREM 4.8. *There exists a vector space isomorphism* $\sigma : \mathcal{F} \cong \mathcal{B}$.

The proof of this theorem will be given in Section 4.3.

A concrete form of σ can be constructed in the following way. For each $m \in 2\mathbf{Z}+1$, we put

$$H_m = \frac{1}{2}\sum_{j\in\mathbf{Z}}(-1)^{j+1}\phi_j\phi_{-j-m} \quad \bar{H}_m = \frac{1}{2}\sum_{j\in\mathbf{Z}}(-1)^{j+1}\bar{\phi}_j\bar{\phi}_{-j-m} .$$

It is straightforward to check that

$$H_m|0\rangle = \bar{H}_m|0\rangle = 0 \quad \text{if } m > 0 ; \tag{4.6}$$

$$[H_m, \phi_n] = \phi_{n-m} \quad [\bar{H}_m, \bar{\phi}_n] = \bar{\phi}_{n-m} \tag{4.7}$$

and also that

$$[H_m, H_n] = [\bar{H}_m, \bar{H}_n] = \frac{m}{2}\delta_{m+n,0}\cdot 1 . \tag{4.8}$$

We now introduce the operator called “Hamiltonian" with variables $(\boldsymbol{x}, \boldsymbol{y})$:

$$H(\boldsymbol{x}, \boldsymbol{y}) = \sum_{l\in\mathbf{N}_{\mathrm{odd}}}\left\{\left(x_l - \frac{2}{l}\frac{\partial}{\partial y_l}\right)H_l + \left(y_l - \frac{2}{l}\frac{\partial}{\partial x_l}\right)\bar{H}_l\right\}.$$

Multiplication of $H(\boldsymbol{x}, \boldsymbol{y})$ on $\mathcal{F}$, as well as $e^{H(\boldsymbol{x},\boldsymbol{y})}$, is well-defined, so that we can define a linear map $\sigma : \mathcal{F} \to \mathcal{B}$ by

$$\sigma(|\nu\rangle) = \sum_{i,j=0,1} 2^{i+j}\, q^i\, \bar{q}^j \langle \phi_0^i\, \bar{\phi}_0^j\, e^{H(\boldsymbol{x},\boldsymbol{y})}|\nu\rangle \cdot 1 \tag{4.9}$$

which yields the isomorphism of Theorem 4.8. An image of σ will be described by means of the generalized Q-functions (see (4.13)).

The following lemma is easily verified.

LEMMA 4.9. *We have the following formulae for* $m = 1, 3, 5, \ldots$

$$\begin{aligned} \sigma H_m \sigma^{-1} &= \frac{\partial}{\partial x_m} \quad \sigma H_{-m}\sigma^{-1} = \frac{m}{2}x_m - \frac{\partial}{\partial y_m} \\ \sigma \bar{H}_m \sigma^{-1} &= \frac{\partial}{\partial y_m} \quad \sigma \bar{H}_{-m}\sigma^{-1} = \frac{m}{2}y_m - \frac{\partial}{\partial x_m}\,. \end{aligned} \tag{4.10}$$

REMARK 4.10. If we put $\alpha_m = \frac{2}{m}(H_{-m} + \bar{H}_m)$, $\bar{\alpha}_m = \frac{2}{m}(\bar{H}_{-m} + H_m)$ for $m = 1, 3, 5, \ldots$, then we have the following equality:

$$\sigma(\alpha_{m_1}\cdots\alpha_{m_r}\bar{\alpha}_{n_1}\cdots\bar{\alpha}_{n_s}\phi_0^i\bar{\phi}_0^j|0\rangle) = x_{m_1}\cdots x_{m_r}y_{n_1}\cdots y_{n_s}q^i\bar{q}^j\,. \tag{4.11}$$

4.3. Verification of Theorem 4.8. The proof is completely parallel to that of Theorem 2.1 in [18]. We count the degree of each variables as $\deg x_n = n$ and $\deg y_n = -n$. Fix non-negative integers M and N. Let $\mathbf{C}^{M,N}[\boldsymbol{x}, \boldsymbol{y}]$ be a linear span of the following set of polynomials belonging to $\mathbf{C}[\boldsymbol{x}, \boldsymbol{y}]$:

$$\left\{ x_{m_1}\cdots x_{m_r}y_{n_1}\cdots y_{n_s} \,\middle|\, \sum m_i \le M, \sum n_i \le N, \sum m_i - \sum n_i = M - N \right\}\,.$$

The dimension of $\mathbf{C}^{M,N}[\boldsymbol{x}, \boldsymbol{y}]$ can be readily calculated by means of (strict) partition numbers:

$$\dim \mathbf{C}^{M,N}[\boldsymbol{x}, \boldsymbol{y}] = \sum_{\substack{0\le d_1\le M,\ 0\le d_2\le N \\ d_1-d_2=M-N}} \bar{P}(d_1)\bar{P}(d_2) = \sum_{k\ge 0} \bar{P}(M-k)\bar{P}(N-k)\,.$$

We note that

$$\mathbf{C}[\boldsymbol{x}, \boldsymbol{y}] = \varinjlim \sum_{\text{finite}} \mathbf{C}^{M,N}[\boldsymbol{x}, \boldsymbol{y}]\,.$$

Define the subspace $\mathcal{F}_{i,j}^{M,N}$ of the fermionic Fock space as follows:

$$\mathcal{F}_{i,j}^{M,N} = \bigoplus_{\substack{0\le d_1\le M,\ 0\le d_2\le N \\ d_1-d_2=M-N}} \mathcal{F}_{i,j}^{(d_1,d_2)}\,.$$

Then by Lemma 4.6, we have

$$\dim \mathcal{F}_{i,j}^{M,N} = \sum_{k\ge 0} \bar{P}(M-k)\bar{P}(N-k) \quad \mathcal{F}_{i,j} = \varinjlim \sum_{\text{finite}} \mathcal{F}_{i,j}^{M,N}\,.$$

To prove the theorem, it suffices to verify that the map σ gives a bijection between $\mathcal{F}_{i,j}^{M,N}$ and $\mathcal{B}_{i,j}^{M,N} \stackrel{\text{def}}{=} \mathbf{C}^{M,N}[\boldsymbol{x}, \boldsymbol{y}]q^i\bar{q}^j \subset \mathcal{B}_{i,j}$. Since the dimension of $\mathcal{B}_{i,j}^{M,N}$ is equal to that of $\mathcal{F}_{i,j}^{M,N}$, we have to check that σ is surjective from $\mathcal{F}_{i,j}^{M,N}$ to $\mathcal{B}_{i,j}^{M,N}$. Take any basis element $x_{m_1}\dots x_{m_r}y_{n_1}\dots y_{n_s}q^i\bar{q}^j$ from $\mathcal{B}_{i,j}^{M,N}$, i.e.,

$$\sum m_i \le M \quad \sum n_i \le N \quad \sum m_i - \sum n_i = M-N\,.$$

By (4.11), we have $\sigma(\alpha_{m_1}\cdots\alpha_{m_r}\bar{\alpha}_{n_1}\cdots\bar{\alpha}_{n_s}\phi_0^i\bar{\phi}_0^j|0\rangle) = x_{m_1}\dots x_{m_r}y_{n_1}\dots y_{n_s}q^i\bar{q}^j$. We here notice that H_m (resp. $\bar{H}_m$) is a linear map of degree $(-m,0)$ (resp. $(0,-m)$), i.e., maps $\mathcal{F}_{i,j}^{(d_1,d_2)}$ into $\mathcal{F}_{i,j}^{(d_1-m,d_2)}$ (resp. $\mathcal{F}_{i,j}^{(d_1,d_2-m)}$). From this fact, we see that $\alpha_{m_1}\cdots\alpha_{m_r}\bar{\alpha}_{n_1}\cdots\bar{\alpha}_{n_s}\phi_0^i\bar{\phi}_0^j|0\rangle$ belongs to $\mathcal{F}_{i,j}^{M,N}$. This means that σ is surjective from $\mathcal{F}_{i,j}^{M,N}$ to $\mathcal{B}_{i,j}^{M,N}$. The proof is done.

4.4. Realization of the neutral fermions. Having established the boson-fermion correspondence, we now describe the action of $\mathcal{A}$ on the bosonic Fock space. For our purpose, let us introduce the generating sums of neutral fermions

$$\phi(z)=\sum_{n\in\mathbf{Z}}\phi_n z^n \qquad \bar{\phi}(z)=\sum_{n\in\mathbf{Z}}\bar{\phi}_n z^n\,.$$

THEOREM 4.11. *Let $|\nu\rangle\in\mathcal{F}$. We have the following correspondence of operators*:

$$\sigma(\phi(z)|\nu\rangle) = qX(z)\sigma(|\nu\rangle) \qquad \sigma(\bar{\phi}(z)|\nu\rangle) = \bar{q}\bar{X}(z)\sigma(|\nu\rangle). \tag{4.12}$$

PROOF. We put $\chi(z)=e^{2\xi(\widetilde{\partial}_y,z)}\sigma\phi(z)\sigma^{-1}$. With the help of Lemma 4.9, we can show that

$$[x_n,\chi(z)] = \frac{2}{n}z^{-n}\chi(z) \quad \left[\frac{\partial}{\partial x_n},\chi(z)\right] = z^n\chi(z) \quad [y_n,\chi(z)] = \left[\frac{\partial}{\partial y_n},\chi(z)\right] = 0\,.$$

By virtue of a calculus on the vertex operators (see [9], Lemma 14.5), $\chi(z)$ can be uniquely represented as the form $\chi(z)=c\,e^{\xi(\boldsymbol{x},z)}e^{-2\xi(\widetilde{\partial}_x,z^{-1})}$ for some constant c. To determine c, we have to notice that $e^{H(\boldsymbol{x},\boldsymbol{y})}\phi(z) = e^{\xi(\boldsymbol{x}-2\widetilde{\partial}_y,z)}\phi(z)e^{H(\boldsymbol{x},\boldsymbol{y})}$ which follows from (4.7). Since $\sigma^{-1}(1)=|0\rangle$, $e^{H(\boldsymbol{x},\boldsymbol{y})}|0\rangle=|0\rangle$ and $\langle\phi_0\,\phi(z)\rangle = 1/2$, we have

$$\begin{aligned}\chi(z)\cdot 1 &= e^{2\xi(\widetilde{\partial}_y,z)}\sigma(\phi(z)|0\rangle) = 2qe^{2\xi(\widetilde{\partial}_y,z)}\langle\phi_0 e^{H(\boldsymbol{x},\boldsymbol{y})}\phi(z)\rangle\cdot 1\\ &= 2qe^{2\xi(\widetilde{\partial}_y,z)}\langle\phi_0 e^{\xi(\boldsymbol{x}-2\widetilde{\partial}_y,z)}\phi(z)e^{H(\boldsymbol{x},\boldsymbol{y})}\rangle\cdot 1 = 2qe^{\xi(\boldsymbol{x},z)}\langle\phi_0\,\phi(z)\rangle = qe^{\xi(\boldsymbol{x},z)}\end{aligned}$$

which together with $\chi(z)\cdot 1 = c\,e^{\xi(\boldsymbol{x},z)}$ yields $c=q$. This means $\sigma\phi(z)\sigma^{-1} = e^{-2\xi(\widetilde{\partial}_y,z)}\chi(z) = qX(z)$, as required. We can derive the second equality of (4.12) by the similar calculation. □

REMARK 4.12. The above theorem implies the following equalities:

$$X(z)\langle 0|\phi_0^i\,\bar{\phi}_0^j\,e^{H(\boldsymbol{x},\boldsymbol{y})}|\nu\rangle = \langle 0|\phi_0^{i-1}\,\bar{\phi}_0^j\,e^{H(\boldsymbol{x},\boldsymbol{y})}\phi(z)|\nu\rangle$$
$$\bar{X}(z)\langle 0|\phi_0^i\,\bar{\phi}_0^j\,e^{H(\boldsymbol{x},\boldsymbol{y})}|\nu\rangle = \langle 0|\phi_0^i\,\bar{\phi}_0^{j-1}\,e^{H(\boldsymbol{x},\boldsymbol{y})}\bar{\phi}(z)|\nu\rangle$$

for $i, j \in \mathbf{Z}$ and $|\nu\rangle \in \mathcal{F}$.

As an application of this theorem, we can describe the image of σ by means of the generalized Q-functions. Let $|\nu\rangle = \phi_{\lambda_1} \cdots \phi_{\lambda_r} \bar{\phi}_{\mu_1} \cdots \bar{\phi}_{\mu_s} |0\rangle$ where $\lambda_1 > \cdots > \lambda_r \geq 0$ and $\mu_1 > \cdots > \mu_s \geq 0$. By virtue of Theorem 4.11, the image of $|\nu\rangle$ can be written as

$$\sigma(|\nu\rangle) = q^r \bar{q}^s X_{\lambda_1} \cdots X_{\lambda_r} \bar{X}_{\mu_1} \cdots \bar{X}_{\mu_s} \cdot 1$$

here notice that $\sigma(|0\rangle) = 1$. By Proposition 2.4, we have the following corollary.

COROLLARY 4.13. *We have for* $\lambda_1 > \cdots > \lambda_r \geq 0$ *and* $\mu_1 > \cdots > \mu_s \geq 0$

$$\sigma(|\nu\rangle) = q^r \bar{q}^s \, Q_{[(\lambda_1,\ldots,\lambda_r),(\mu_1,\ldots,\mu_s)]}(\boldsymbol{x}, \boldsymbol{y}). \tag{4.13}$$

Notice that $\{\phi_{\lambda_1} \cdots \phi_{\lambda_{2r}} \bar{\phi}_{\mu_1} \cdots \bar{\phi}_{\mu_{2s}} |0\rangle \mid \lambda_1 > \cdots > \lambda_{2r} \geq 0, \mu_1 > \cdots > \mu_{2s} \geq 0\}$ is a linear basis of $\mathcal{F}_{0,0}$, and $\mathcal{F}_{0,0}$ is isomorphic to $\mathcal{B}_{0,0} = \mathbf{C}[\boldsymbol{x}, \boldsymbol{y}]$ by Theorem 4.8. The formula (4.13) further leads to the following result.

COROLLARY 4.14. *A whole set of the generalized Q-functions forms a linear basis of the polynomial algebra* $\mathbf{C}[\boldsymbol{x}, \boldsymbol{y}]$.

5. The BUC hierarchy from representation theory

In this section, we define the Lie algebra $\mathfrak{go}_{2\infty}$ and discuss the relation to the BUC hierarchy.

5.1. The Lie algebras $\overline{\mathfrak{go}}_\infty$ and $\mathfrak{go}_\infty$. We first recall an infinite rank matrix Lie algebra $\mathfrak{go}_\infty$ [2, 13]. Consider the set of infinite complex matrices $a = (a_{ij})$ satisfying the condition:

$$a_{ij} = 0 \quad \text{for} \quad |i - j| \gg 0 \tag{5.1}$$

i.e., all non-zero entries of the matrix are within a finite distance from the main diagonal. The set of infinite matrices satisfying this condition forms a Lie algebra, which we denote by $\overline{\mathfrak{gl}}_\infty$.

The Lie algebra $\overline{\mathfrak{gl}}_\infty$ has a one dimensional central extension $\mathfrak{gl}_\infty = \overline{\mathfrak{gl}}_\infty \oplus \mathbf{C}c_A$ with the following bracket relation:

$$[a \oplus \alpha c_A,\, b \oplus \beta c_A] = (ab - ba) \oplus \mu(a, b)c_A$$

for $a, b \in \overline{\mathfrak{gl}}_\infty$ and $\alpha, \beta \in \mathbf{C}$. Here μ denotes a 2-cocycle on $\overline{\mathfrak{gl}}_\infty$ defined by

$$\mu(e_{i,j}, e_{k,l}) = \delta_{i,l}\delta_{j,k}(\theta(j) - \theta(i)) \tag{5.2}$$

where $e_{i,j}$ denotes an infinite matrix unit with a 1 on (i, j)-th entry and zeros elsewhere, and θ is defined by $\theta(i) = 1$ for $i \geq 0$ and $\theta(i) = 0$ for $i < 0$.

Define

$$\overline{\mathfrak{go}}_\infty = \{a = (a_{ij}) \in \overline{\mathfrak{gl}}_\infty \mid a_{ij} = (-1)^{i+j+1} a_{-j,-i}\} \tag{5.3}$$

which forms a Lie subalgebra of $\overline{\mathfrak{gl}}_\infty$. The 2-cocycle μ defined above is also defined on $\overline{\mathfrak{go}}_\infty$, so that a central extension $\mathfrak{go}_\infty=\overline{\mathfrak{go}}_\infty \oplus \mathbf{C}c_B$ can be defined by setting the bracket relation:

$$[a \oplus \alpha c_B,\ b \oplus \beta c_B]=(ab-ba) \oplus \frac{1}{2}\,\mu(a,b)c_B$$

for $a, b \in \overline{\mathfrak{go}}_\infty$ and $\alpha, \beta \in \mathbf{C}$.

LEMMA 5.1 ([13, 19]). *We have a representation π_0 of $\mathfrak{go}_\infty$ on the fermionic Fock space $\mathcal{F}$ defined by*

$$\pi_0(f_{i,j}) =: \phi_i\phi_{-j} : \quad \pi_0(c_B) = 1$$

where $f_{i,j}=(-1)^j e_{i,j} - (-1)^i e_{-j,-i} = (-1)^{i+j+1} f_{j,i}$ and $: \phi_m\phi_n : = \phi_m\phi_n - \langle\phi_m\phi_n\rangle$ is a normal product. Note that any matrix belonging to $\overline{\mathfrak{go}}_\infty$ can be expressed as a linear combination of $f_{i,j}$ $(i < j)$.

5.2. The Lie algebra $\mathfrak{go}_{2\infty}$. In order to discuss the BUC hierarchy from the Lie algebraic viewpoint, we consider a direct sum of $\overline{\mathfrak{go}}_\infty$, which forms a Lie algebra with a canonically defined bracket. Let us define the one dimensional central extension $\mathfrak{go}_{2\infty}=\overline{\mathfrak{go}}_\infty \oplus \overline{\mathfrak{go}}_\infty \oplus \mathbf{C}c_B$ by introducing the following bracket relation:

$$[a \oplus \bar{a} \oplus \alpha c_B,\ b \oplus \bar{b} \oplus \beta c_B]=(ab-ba) \oplus (\bar{a}\bar{b}-\bar{b}\bar{a}) \oplus \frac{1}{2}\,\big(\mu(a,b)+\mu(\bar{a},\bar{b})\big)\,c_B$$

for $a, b, \bar{a}, \bar{b} \in \overline{\mathfrak{go}}_\infty$ and $\alpha, \beta \in \mathbf{C}$. Then we have the following lemma.

LEMMA 5.2. *We can define a representation π of $\mathfrak{go}_{2\infty}$ on the fermionic Fock space $\mathcal{F}$ by*

$$\pi(F_{i,j}) =: \phi_i\phi_{-j} : \quad \pi(\bar{F}_{i,j}) =: \bar{\phi}_i\bar{\phi}_{-j} : \quad \pi(c_B) = 1$$

where $F_{i,j}=f_{i,j} \oplus \mathbf{0}$, $\bar{F}_{i,j}=\mathbf{0} \oplus f_{i,j} \in \overline{\mathfrak{go}}_\infty \oplus \overline{\mathfrak{go}}_\infty$, and $: :$ denotes the normal product notation defined as before.

In terms of the generating series $f(z,w)=\sum_{i,j\in\mathbf{Z}} F_{i,j} z^i w^{-j}$ and $\bar{f}(z,w)=\sum_{i,j\in\mathbf{Z}} \bar{F}_{i,j} z^i w^{-j}$, the above lemma reads

$$\pi(f(z,w)) =: \phi(z)\phi(w) : \quad \pi(\bar{f}(z,w)) =: \bar{\phi}(z)\bar{\phi}(w) : .$$

Notice that the representation π is invariant with respect to the decomposition $\mathcal{F} = \oplus\mathcal{F}_{i,j}$. In particular, $\mathcal{F}_{0,0}$ is an invariant subspace mapped to $\mathcal{B}_{0,0} = \mathbf{C}[\boldsymbol{x}, \boldsymbol{y}]$ via the isomorphism σ. We note

$$\langle\phi(z)\phi(w)\rangle = \langle\bar{\phi}(z)\bar{\phi}(w)\rangle = \frac{1}{2}\frac{1-w/z}{1+w/z} = 1 + 2\sum_{k\geq 1}(-w/z)^k .$$

Combining this, (3.10) and (4.12), we obtain the following proposition.

PROPOSITION 5.3. *The Lie algebra* $\mathfrak{go}_{2\infty}$ *acts on* $\mathbf{C}[\boldsymbol{x}, \boldsymbol{y}]$ *by the following formulae*:

$$\begin{aligned} \sigma(:\phi(z)\phi(w):)\sigma^{-1} &= \frac{1}{2}\frac{1-w/z}{1+w/z}\left(\Gamma^{+}(z,w)-1\right) \\ \sigma(:\bar{\phi}(z)\bar{\phi}(w):)\sigma^{-1} &= \frac{1}{2}\frac{1-w/z}{1+w/z}\left(\Gamma^{-}(z,w)-1\right) \end{aligned} \tag{5.4}$$

where $\Gamma^{\pm}(z,w)$ *are the vertex operators defined in* (3.9) *and we assume* $|z| > |w|$.

5.3. Bilinear identities and τ**-function.** We now give a Lie algebraic description of the BUC hierarchy. We regard the Lie algebra $\mathfrak{go}_{2\infty}$ as

$$\mathfrak{go}_{2\infty} \cong \left\{ X = \sum_{i,j\in\mathbf{Z}} \left(a_{ij} :\phi_i\phi_{-j}: + \bar{a}_{ij} :\bar{\phi}_i\bar{\phi}_{-j}:\right) + c \;\middle|\; (a_{ij}),\, (\bar{a}_{ij}) \text{ satisfy (5.1) and } c \in \mathbf{C} \right\}.$$

We define the (formal) Lie group $\boldsymbol{G}$ corresponding to $\mathfrak{go}_{2\infty}$:

$$\boldsymbol{G} = \{e^{X_1}\cdots e^{X_k} | X_i \in \mathfrak{go}_{2\infty} : \text{locally nilpotent}\}. \tag{5.5}$$

Of particular importance is a $\boldsymbol{G}$-orbit space of the vacuum vector

$$\boldsymbol{G}|0\rangle = \{g|0\rangle \,|\, g \in \boldsymbol{G}\} \subset \mathcal{F}_{0,0}.$$

The following proposition can be verified in the same way as in the case of the BKP hierarchy (cf. [11, 19]).

PROPOSITION 5.4. *A non-zero* $|L\rangle \in \mathcal{F}_{0,0}$ *lies in* $\boldsymbol{G}|0\rangle$ *if and only if* $|L\rangle$ *satisfies the following quadratic relations on* $\mathcal{F}_{0,0} \otimes \mathcal{F}_{0,0}$:

$$\begin{cases} \displaystyle\sum_{n\in\mathbf{Z}} (-1)^n \phi_n|L\rangle \otimes \phi_{-n}|L\rangle = Q|L\rangle \otimes Q|L\rangle \\ \displaystyle\sum_{n\in\mathbf{Z}} (-1)^n \bar{\phi}_n|L\rangle \otimes \bar{\phi}_{-n}|L\rangle = \bar{Q}|L\rangle \otimes \bar{Q}|L\rangle. \end{cases} \tag{5.6}$$

Here Q *and* $\bar{Q}$ *are linear operators on* $\mathcal{F}$ *defined via* $\sigma Q\sigma^{-1} = q$ *and* $\sigma\bar{Q}\sigma^{-1} = \bar{q}$, *respectively.*

REMARK 5.5. The above operators Q and $\bar{Q}$ satisfy the following basic properties:

$$Q|0\rangle = \phi_0|0\rangle \qquad \bar{Q}|0\rangle = \bar{\phi}_0|0\rangle \qquad [Q, \bar{Q}] = 0;$$

$$[Q, \phi_m] = [Q, \bar{\phi}_m] = [\bar{Q}, \phi_m] = [\bar{Q}, \bar{\phi}_m] = 0 \qquad \text{for all } m \in \mathbf{Z}.$$

We are now in a position to state the following theorem.

THEOREM 5.6. *Let* $\tau \in \mathbf{C}[\boldsymbol{x}, \boldsymbol{y}]$. *Then* τ *satisfies the bilinear identities* (3.4) *if and only if there exists a* $g|0\rangle \in \boldsymbol{G}|0\rangle$ *such that*

$$\tau = \sigma(g|0\rangle) = \langle e^{H(\boldsymbol{x},\boldsymbol{y})} g\rangle \cdot 1. \tag{5.7}$$

PROOF. The bilinear identities (3.4) are equivalent to (5.6) by the correspondence (4.12). Therefore this theorem is a consequence of Lemma 5.4. □

We have thus shown that a $\boldsymbol{G}$-orbit space $\boldsymbol{G}|0\rangle$ can be identified with a space of polynomial solutions of the BUC hierarchy. In particular, (5.7) gives a general formula for the polynomial solutions. The solution expressed in this form should be traditionally referred to as a *τ-function* in soliton theory. For example, let $X = a\phi_1\phi_0 + b\bar{\phi}_3\bar{\phi}_2$ $(a, b \in \mathbf{C})$, then $e^X = 1 + a\phi_1\phi_0 + b\bar{\phi}_3\bar{\phi}_2 + ab\phi_1\phi_0\bar{\phi}_3\bar{\phi}_2$, and hence the corresponding τ-function is

$$\sigma(e^X|0\rangle) = 1 + \frac{a}{2}\,Q_{(1,0)}(\boldsymbol{x}) + \frac{b}{2}\,Q_{(3,2)}(\boldsymbol{y}) + \frac{ab}{4}\,Q_{[(1,0),(3,2)]}(\boldsymbol{x}, \boldsymbol{y})$$

which solves the BUC hierarchy. It should be remarked that the expectation value $\langle\,\cdot\,\rangle$ in (5.7) becomes in general a differential operator; this is a crucial difference to usual τ-functions for well-known KP-type hierarchies.

5.4. Relation to τ-function for the BKP hierarchy. Since our setting of the neutral fermion system (considered in Section 4) includes a single component neutral fermion system, we can consider the τ-function of the BKP hierarchy in our framework.

To introduce the τ-function for the BKP hierarchy, consider the Lie algebra $\mathfrak{go}_\infty$ defined in Section 5.1:

$$\mathfrak{go}_\infty \cong \left\{ Y = \sum_{i,j\in\mathbf{Z}} a_{ij} : \phi_i\phi_{-j} : +c \;\middle|\; (a_{ij}) \text{ satisfies (5.1) and } c \in \mathbf{C} \right\} \tag{5.8}$$

and define the corresponding Lie group $\boldsymbol{H} = \{e^{Y_1} \cdots e^{Y_k} \mid Y_i \in \pi_0(\mathfrak{go}_\infty) : \text{locally nilpotent}\}$.

DEFINITION 5.7. For any $h \in \boldsymbol{H}$, we define the τ-function of the BKP hierarchy as a special case of our τ-function:

$$\tau_{BKP}(\boldsymbol{x}; h) \overset{\text{def}}{=} \sigma(h|0\rangle)\,. \tag{5.9}$$

By noting that $[\bar{H}_n, h] = 0$ and $\bar{H}_n|0\rangle = 0$ for $n > 0$, we can rewrite it as follows:

$$\begin{aligned} \tau_{BKP}(\boldsymbol{x}; h) &= \langle e^{H(\boldsymbol{x},\boldsymbol{y})} h\rangle \cdot 1 \\ &= \langle e^{\sum\left(x_n - \frac{2}{n}\partial_{y_n}\right)H_n} h\rangle \cdot 1 = \langle e^{H_{BKP}(\boldsymbol{x})} h\rangle \end{aligned} \tag{5.10}$$

where $H_{BKP}(\boldsymbol{x}) \overset{\text{def}}{=} \sum_{l\geq 1} x_{2l-1} H_{2l-1}$. Since the right hand side does not depend on $\boldsymbol{y}$, the τ-function (5.10) gives a solution of the BKP hierarchy.

Before entering the following theorem, let us note that any element $g \in \boldsymbol{G}$ can be decomposed as a product $g = h_1\nu(h_2)$ for some $h_1, h_2 \in \boldsymbol{H}$, where ν denotes an automorphism of the algebra $\mathcal{A}$ defined by

$$\nu(\phi_m) = \bar{\phi}_m \quad \nu(\bar{\phi}_m) = \phi_m \quad (m \in \mathbf{Z})\,. \tag{5.11}$$

This decomposition of g is unique up to arbitrariness $(h_1, h_2) \mapsto (ch_1, c^{-1}h_2)$ for $c \in \mathbf{C}^\times$. Given such a decomposition,

$$\langle g\rangle = \langle h_1\nu(h_2)\rangle = \langle h_1\rangle\langle\nu(h_2)\rangle = \langle h_1\rangle\langle h_2\rangle$$

where the second equality follows by applying the Wick's theorem (see Example 4.5).

We are now in a position to prove the following theorem.

THEOREM 5.8. *$\tau(\boldsymbol{x}, \boldsymbol{y}) \in \mathbf{C}[\boldsymbol{x}, \boldsymbol{y}]$ is a solution of the BUC hierarchy if and only if there exist solutions $\tau_1(\boldsymbol{x}), \tau_2(\boldsymbol{x}) \in \mathbf{C}[\boldsymbol{x}]$ of the BKP hierarchy such that*

$$\tau(\boldsymbol{x}, \boldsymbol{y}) = \tau_1(\boldsymbol{x} - 2\widetilde{\partial}_{\boldsymbol{y}})\tau_2(\boldsymbol{y} - 2\widetilde{\partial}_{\boldsymbol{x}}) \cdot 1\,. \tag{5.12}$$

This theorem can be easily deduced from the following lemma.

LEMMA 5.9. *Let $g \in \boldsymbol{G}$ and decompose it as $g = h_1\nu(h_2)$ for $h_1, h_2 \in \boldsymbol{H}$. Then the corresponding τ-function $\tau = \sigma(g|0\rangle)$ can be represented as*

$$\tau = \tau_{BKP}(\boldsymbol{x} - 2\widetilde{\partial}_{\boldsymbol{y}}; h_1)\,\tau_{BKP}(\boldsymbol{y} - 2\widetilde{\partial}_{\boldsymbol{x}}; h_2) \cdot 1\,. \tag{5.13}$$

PROOF. The Hamiltonian operator $H(\boldsymbol{x}, \boldsymbol{y})$ breaks into

$$H(\boldsymbol{x}, \boldsymbol{y}) = H_{BKP}(\boldsymbol{x} - 2\widetilde{\partial}_{\boldsymbol{y}}) + \nu(H_{BKP}(\boldsymbol{y} - 2\widetilde{\partial}_{\boldsymbol{x}}))\,.$$

Notice that two terms on the right hand side commute with each other. Hence we have

$$\begin{aligned}\tau(\boldsymbol{x}, \boldsymbol{y}; g) &= \langle e^{H_{BKP}(\boldsymbol{x}-2\widetilde{\partial}_{\boldsymbol{y}})}e^{\nu(H_{BKP}(\boldsymbol{y}-2\widetilde{\partial}_{\boldsymbol{x}}))}h_1\,\nu(h_2)\rangle \cdot 1\\ &= \langle e^{H_{BKP}(\boldsymbol{x}-2\widetilde{\partial}_{\boldsymbol{y}})}h_1 e^{\nu(H_{BKP}(\boldsymbol{y}-2\widetilde{\partial}_{\boldsymbol{x}}))}\,\nu(h_2)\rangle \cdot 1\\ &= \langle e^{H_{BKP}(\boldsymbol{x}-2\widetilde{\partial}_{\boldsymbol{y}})}h_1\rangle\langle e^{\nu(H_{BKP}(\boldsymbol{y}-2\widetilde{\partial}_{\boldsymbol{x}}))}\,\nu(h_2)\rangle \cdot 1\\ &= \tau_{BKP}(\boldsymbol{x} - 2\widetilde{\partial}_{\boldsymbol{y}}; h_1)\,\tau_{BKP}(\boldsymbol{y} - 2\widetilde{\partial}_{\boldsymbol{x}}; h_2) \cdot 1\end{aligned}$$

as required. □

REMARK 5.10. Arbitrariness of the decomposition of $g \in \boldsymbol{G}$ induces the arbitrariness of the BKP τ-functions in (5.13) such as $(\tau_{BKP}(\boldsymbol{x}; h_1), \tau_{BKP}(\boldsymbol{x}; h_2)) \to (c\,\tau_{BKP}(\boldsymbol{x}; h_1), c^{-1}\tau_{BKP}(\boldsymbol{x}; h_2))$ for $c \in \mathbf{C}^\times$. Therefore we have a one-to-one correspondence:

$$\{\tau(\boldsymbol{x}, \boldsymbol{y}; g) \mid g \in \boldsymbol{G}\} \overset{1:1}{\longleftrightarrow} \{(\tau_{BKP}(\boldsymbol{x}; h_1), \tau_{BKP}(\boldsymbol{x}; h_2)) \mid h_1, h_2 \in \boldsymbol{H}\}/\mathbf{C}^\times\,.$$

6. Concluding remarks

We have defined the generalized Q-functions and introduced the system of differential equations, called the BUC hierarchy. There are some interesting problems concerning to the present work.

Firstly, for most of the integrable hierarchies of soliton equations, the equations are expressed as a system of compatible evolution equations of Lax type. For the BUC hierarchy (as well as the UC hierarchy), the author does not know how to express the hierarchy as a system of Lax type evolution equations.

Secondly it is known that the BKP hierarchy is a reduction of the KP hierarchy [2] (see also [7, 19]). This is simply viewed from that the solutions of the BKP hierarchy are Pfaffians, and their squares are determinants which give determinant solutions of the KP hierarchy. Since the solutions of the BUC hierarchy are Pfaffians as well, we might expect that the BUC hierarchy is understood as a reduction of the UC hierarchy.

Finally as mentioned in Introduction, the Schur Q-functions arise from the theory of the projective representations of the symmetric groups. It might be an intriguing issue on whether the generalized Q-functions have such a representation–theoretical meaning. We hope to discuss these subjects elsewhere.

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A. Verification of Theorem 3.9. We first prove an elementary lemma below, which we have used in the text in several times.

LEMMA A.1. *The following "operator identity" holds*:

$$e^{-2\xi(\widetilde{\partial}_{\boldsymbol{x}},z^{-1})}e^{\xi(\boldsymbol{x},w)} = \frac{1-w/z}{1+w/z}\,e^{\xi(\boldsymbol{x},w)}e^{-2\xi(\widetilde{\partial}_{\boldsymbol{x}},z^{-1})} \quad \text{where } |z|>|w|\,. \tag{A.1}$$

PROOF. We use the well-known formula [8] $e^Ae^Be^{-A}=e^{[A,B]}e^B$ that holds for arbitrary operators A and B such that $[A,B](=AB-BA)$ is a scalar. By direct calculation,

$$\begin{aligned}[-2\xi(\widetilde{\partial}_{\boldsymbol{x}},z^{-1}),\xi(\boldsymbol{x},w)] &= -2\sum_{l\in\mathbf{N}_{\mathrm{odd}}}\frac{1}{l}\left(\frac{w}{z}\right)^l \\ &= -\sum_{l\geq 1}\frac{1}{l}\left(\frac{w}{z}\right)^l+\sum_{l\geq 1}\frac{1}{l}\left(-\frac{w}{z}\right)^l = \log\left(\frac{1-w/z}{1+w/z}\right)\end{aligned}$$

which leads to the lemma. □

Let : : denote a normal product notation for differential operators defined by

$$:x_n\frac{\partial}{\partial x_n}:=:\frac{\partial}{\partial x_n}x_n:=x_n\frac{\partial}{\partial x_n}\,.$$

Namely, : : is the operation that rearranges the order of operators inside the colons such that all the differentials are right to multiplications. By Lemma A.1, one easily prove the following lemma.

LEMMA A.2. *The following formulae hold*:

$$\begin{aligned}\Gamma^{\pm}(a_i,b_i)\Gamma^{\pm}(a_j,b_j) &= A_{ij} : \Gamma^{\pm}(a_i,b_i)\Gamma^{\pm}(a_j,b_j) : \\ \Gamma^{-}(a_i^{-1},b_i^{-1})\Gamma^{+}(a_j,b_j) &= A_{ij} : \Gamma^{-}(a_i^{-1},b_i^{-1})\Gamma^{+}(a_j,b_j) :\end{aligned} \tag{A.2}$$

where

$$A_{ij} = \frac{(a_i-a_j)(a_i-b_j)(b_i-a_j)(b_i-b_j)}{(a_i+a_j)(a_i+b_j)(b_i+a_j)(b_i+b_j)}.$$

In particular, $\Gamma^{\pm}(a_i,b_i)^2 = 0$, *so that* $\exp c\Gamma^{\pm}(a_i,b_i) = 1 + c\Gamma^{\pm}(a_i,b_i)$ *for* $c \in \mathbf{C}^{\times}$.

We further prepare two lemmas below.

LEMMA A.3. *If* τ *is a solution of the BUC hierarchy*, *then* $\Gamma^{\pm}(a,b)\tau$ $(a,b \in \mathbf{C}^{\times})$ *are also solutions.*

PROOF. Notice that $\Omega_{\mathcal{B}} = \sum_{n\in\mathbf{Z}}(-1)^n X_n \otimes X_{-n}$ and $\bar{\Omega}_{\mathcal{B}} = \sum_{n\in\mathbf{Z}}(-1)^n \bar{X}_n \otimes \bar{X}_{-n}$ commute with $X(z)$ and $\bar{X}(z)$ (cf. the proof of Lemma 3.6). Therefore this lemma follows from (3.10). □

LEMMA A.4. *It holds that*

$$[\Omega_{\mathcal{B}}, 1\otimes\Gamma^{\pm}(a,b) + \Gamma^{\pm}(a,b)\otimes 1] = [\bar{\Omega}_{\mathcal{B}}, 1\otimes\Gamma^{\pm}(a,b) + \Gamma^{\pm}(a,b)\otimes 1] = 0 \tag{A.3}$$

where $\Omega_{\mathcal{B}} = \oint X(z)\otimes X(-z)\frac{dz}{2\pi iz}$ *and* $\bar{\Omega}_{\mathcal{B}} = \oint \bar{X}(z)\otimes \bar{X}(-z)\frac{dz}{2\pi iz}$.

PROOF. We have the equality (cf. the formula displayed in the proof of Lemma 3.2)

$$[X(a)X(b), X(z)] = 2X(a)\delta(-b/a) - 2X(b)\delta(-a/z)$$

and similarly for $\bar{X}$, from which the lemma follows by direct calculation. □

Suppose τ being a solution of the bilinear identities (3.4). We put $\widetilde{\tau} \overset{\text{def}}{=} (1+c\Gamma^{\pm}(a,b))\tau$. Then by using (3.4), Lemma A.3 and A.4,

$$\begin{aligned}\Omega_{\mathcal{B}}(\widetilde{\tau}\otimes\widetilde{\tau}) &= \Omega_{\mathcal{B}}(\tau\otimes\tau) + c\Omega_{\mathcal{B}}(\tau\otimes\Gamma^{\pm}(a,b)\tau + \Gamma^{\pm}(a,b)\tau\otimes\tau) \\ &\quad + c^2\Omega_{\mathcal{B}}\left(\Gamma^{\pm}(a,b)\tau\otimes\Gamma^{\pm}(a,b)\tau\right) \\ &= \tau\otimes\tau + c\left(\tau\otimes\Gamma^{\pm}(a,b)\tau + \Gamma^{\pm}(a,b)\tau\otimes\tau\right) + c^2\Gamma^{\pm}(a,b)\tau\otimes\Gamma^{\pm}(a,b)\tau \\ &= \widetilde{\tau}\otimes\widetilde{\tau}\end{aligned}$$

and similarly for $\bar{\Omega}_{\mathcal{B}}$. Hence $\widetilde{\tau}$ solves (3.4). Noticing that $\tau \equiv 1$ solves (3.4), we see that the (M,N)-soliton solution defined in (3.11) is indeed a solution of the bilinear identities. The theorem is proved.

B. Verification of Proposition 3.11. In order to see the equivalence between (3.11) and (3.15), we recall the following lemma ("Fredholm Pfaffian"), due to E. M. Rains.

LEMMA B.1 ([15]). *Let* $X = (X_{i,j})_{1\le i,j\le p}$ *be a skew-symmetric matrix of size* $2p \times 2p$, *where every* $X_{i,j}$ *is a* 2×2*-block. Let* J *be the* $2p \times 2p$*-matrix defined as in Proposition* 3.11. *Then*

$$\mathrm{Pf}\,[J + X] = \sum_{\substack{0\le r\le p\\ 1\le i_1<i_2<\cdots<i_r\le p}} \mathrm{Pf}\left[X^*_{\langle i_1,i_2,\dots,i_r\rangle}\right] \tag{B.1}$$

where $X^*_{\langle i_1,i_2,\dots,i_r\rangle}$ *denotes the skew-symmetric submatrix* $(X_{i,j})_{i,j=i_1,i_2,\dots,i_r}$ *of* X.

Applying Lemma B.1 to the case where $X = S$ yields the expansion

$$\mathrm{Pf}\,[J + S] = \sum_{K\subset I_-\cup I_+} \mathrm{Pf}\left[S^*_{\langle K\rangle}\right]$$

where K runs over all the subsets of $I_-\cup I_+$, and $S^*_{\langle K\rangle}$ denotes the skew-symmetric submatrix of S corresponding to K. By noting the formula $\prod_{i<j}\frac{z_i-z_j}{z_i+z_j} = \mathrm{Pf}\left[\frac{z_i-z_j}{z_i+z_j}\right]_{i,j}$, we can directly verify

$$\mathrm{Pf}[S^*_{\langle K\rangle}] = \prod_{i\in K}\frac{a_i-b_i}{a_i+b_i}\prod_{\substack{i<j\\ i,j\in K}}\frac{(a_i-a_j)(a_i-b_j)(b_i-a_j)(b_i-b_j)}{(a_i+a_j)(a_i+b_j)(b_i+a_j)(b_i+b_j)}\prod_{i\in K}e^{\widehat{\xi}_i(\boldsymbol{a})+\widehat{\xi}_i(\boldsymbol{b})}$$

for every subset K of $I_-\cup I_+$. Plugging this expression into the above expansion and noting the relation (3.14), we obtain the proposition.

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