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A Degenerate Neumann Problem for Quasilinear Elliptic Equations

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Abstract. The degenerate Neumann problem

$$\begin{cases} \sum_{i,j=1}^{n} a_{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u, Du) & \text{in } \Omega , \\ a(x)\dfrac{\partial u}{\partial \nu} + b(x)u = \varphi(x) & \text{on } \Gamma \end{cases}$$

is studied in the case where $a(x)$ and $b(x)$ are non-negative functions on Γ such that $a(x) + b(x) > 0$ on Γ. A classical existence and uniqueness theorem in the Hölder space $C^{2+\alpha}(\bar{\Omega})$ is proved under suitable regularity and structure conditions on the data.

1. Introduction and Main Theorem.

Let Ω be a bounded domain of Euclidean space $\mathbf{R}^n$, $n \geq 2$, with smooth boundary Γ and let $\nu(x)$ be the unit exterior normal to Γ. In this paper we study the following *quasilinear* elliptic boundary value problem:

$$\begin{cases} \sum_{i,j=1}^{n} a^{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u, Du) & \text{in } \Omega , \\ a(x)\dfrac{\partial u}{\partial \nu} + b(x)u = \varphi(x) & \text{on } \Gamma . \end{cases} \tag{1.1}$$

Here $a(x)$ and $b(x)$ are non-negative functions defined on Γ, and Du stands for the gradient $(\partial u/\partial x_1, \partial u/\partial x_2, \cdots, \partial u/\partial x_n)$ of u. Later on, we will denote by $C^{k+\alpha}(\bar{\Omega})$ the Hölder space of k-times continuously differentiable functions on the closure $\bar{\Omega} = \Omega \cup \Gamma$ whose k-th order derivatives are Hölder continuous with exponent α and also by $\|\cdot\|_{C^{k+\alpha}(\bar{\Omega})}$ its usual norm. The Sobolev space of k-times weakly differentiable functions in Ω whose derivatives up to order k belong to $L^p(\Omega)$ will be denoted as usual by $W^{k,p}(\Omega)$. The letter C stands for a generic positive constant depending only on known quantities but not on u, which may vary from a line into another.

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The linear problem (1.1) (i.e. $f(x, z, p) = \sum_i b^i(x)p_i + c(x)z$) has been well studied in the recent years by Taira [9] and [10] both in the frameworks of Hölder and Sobolev spaces. In the case where the function f is nonlinear in u but independent of Du (i.e. $f(x, z, p) = f(x, z)$), there is a similar result due to Taira-Umezu [12] where a global static bifurcation theory is elaborated. We should also note the recent paper Taira [11] where the homogeneous problem (1.1) ($\varphi \equiv 0$) with divergence form linear elliptic operator has been studied by means of the super-subsolution method. The interest to the problems of type (1.1) is prompted by their importance in probability theory and stochastic processes, as well as in Riemannian geometry. Thus the second-order differential operator in the problem (1.1) is called a diffusion operator describing analytically a strong Markov process with continuous paths in the state space Ω (see [2], [10]) while the two terms $a(x)(\partial u/\partial \nu)$ and $b(x)u$ of the boundary condition correspond to reflection and absorption phenomena on Γ, respectively. On the other hand, the problem (1.1) with $f(x, z, p) = f(x)z^{(n+2)/(n-2)}$, $n \geq 3$, is related to the so-called Yamabe problem which is a basic problem in Riemannian geometry (see [3], [6], [7]).

In this paper the data of the problem (1.1) will be subject to the following conditions:

Uniform ellipticity condition: There exists a positive constant a_0 such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2 \text{ for all } x \in \bar{\Omega},\ \xi \in \mathbf{R}^n,\ a^{ij}(x) = a^{ji}(x). \tag{1.2}$$

Regularity conditions:

$$\begin{cases} a^{ij} \in C^\infty(\bar{\Omega}),\ f(x, z, p) \in C^\alpha(\bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n), \quad 0 < \alpha < 1, \\ f(x, z, p) \text{ is continuously differentiable with respect to } z \text{ and } p. \end{cases} \tag{1.3}$$

Monotonicity condition: There exists a positive constant f_0 such that

$$\frac{\partial f}{\partial z}(x, z, p) \geq f_0 \text{ for all } (x, z, p) \in \bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n. \tag{1.4}$$

Quadratic gradient growth condition: There exists a positive and non-decreasing function $f_1(t)$ such that

$$|f(x, z, p)| \leq f_1(|z|)(1 + |p|^2) \text{ for all } (x, z, p) \in \bar{\Omega} \times \mathbf{R} \times \mathbf{R}^n. \tag{1.5}$$

Our final condition concerns the behavior of the functions a and b on Γ:

$$\begin{cases} a(x),\ b(x) \in C^\infty(\Gamma), \\ a(x) \geq 0,\ b(x) \geq 0,\ a(x) + b(x) > 0 \text{ for all } x \in \Gamma. \end{cases} \tag{1.6}$$

It should be noted that the condition (1.6) allows the problem (1.1) to include both the purely Dirichlet ($a(x) \equiv 0$) and Neumann ($b(x) \equiv 0$) boundary conditions as particular cases. What is the important feature, however, of the condition (1.6) is that the problem (1.1) becomes a singular boundary value problem from an analytical point of view. This is due to the fact that, having a first order pseudo-differential operator T on Γ, the so-called Shapiro-Lopatinskii complementary condition is violated at the points $x \in \Gamma$ where $a(x) = 0$. In fact, the main difficulty of the problem (1.1) comes from the fact that the operator T is not of principal type (see [9]). Amann-Crandall [1] studied the non-degenerate case; more precisely they assume that the boundary Γ is the *disjoint* union of the two closed subsets $\Gamma_0 = \{x \in \Gamma : a(x) = 0\}$

and $\Gamma_1 = \{x \in \Gamma : a(x) > 0\}$, each of which is an $(n-1)$-dimensional compact smooth manifold. On the other hand, the intuitive meaning of the requirement $a(x) + b(x) > 0$ on Γ is that, for the diffusion process described by the problem (1.1), either the reflection phenomenon or the absorption phenomenon occurs at each point of the boundary Γ (see [10]).

The main purpose of the present paper is to extend the above cited results by Taira [11] and Taira-Umezu [12] to the non-homogeneous problem (1.1) allowing quadratic nonlinearity in f with respect to the gradient Du of the unknown function u. We prove an existence and uniqueness theorem for the problem (1.1) in the Hölder space $C^{2+\alpha}(\bar{\Omega})$. This is carried out by utilizing the Leray-Schauder fixed point theorem which reduces the solvability of the problem (1.1) to the establishment of an *a priori* estimate in $C^{1+\alpha}(\bar{\Omega})$ for all solutions to a family related to the problem (1.1). The deriving of the desired *a priori* estimate is a two-step process consisting of successive bounds on $\|u\|_{C(\bar{\Omega})}$ and $\|Du\|_{C^\alpha(\bar{\Omega})}$. The estimate of $\|u\|_{C(\bar{\Omega})}$ follows, as usual, by using the maximum principle. As it concerns the *a priori* bound for $\|Du\|_{C^\alpha(\bar{\Omega})}$, after reducing it to an estimate for $\|Du\|_{W^{1,p}(\Omega)}$ with $p = n/(1-\alpha)$ (recall the Sobolev imbedding $W^{1,p}(\Omega) \hookrightarrow C^\alpha(\bar{\Omega})$), we apply a $W^{2,p}(\Omega)$-*a priori* bound for the solutions to the problem (1.1) derived by Taira [11]. A very important role in this procedure is played by the conditions (1.4) and (1.5), as well as by the results of Taira [10] on the isomorphic properties in Hölder and Sobolev spaces of the linear operators appearing in the problem (1.1).

Following Taira [9] and [10], we introduce the next interpolation Banach space

$$C_*^{1+\alpha}(\Gamma) = \{\varphi = a(x)\varphi_1 + b(x)\varphi_2 : \varphi_1 \in C^{1+\alpha}(\Gamma), \varphi_2 \in C^{2+\alpha}(\Gamma)\},$$

equipped with the norm

$$\|\varphi\|_{C_*^{1+\alpha}(\Gamma)} = \inf\{\|\varphi_1\|_{C^{1+\alpha}(\Gamma)} + \|\varphi_2\|_{C^{2+\alpha}(\Gamma)} : \varphi = a(x)\varphi_1 + b(x)\varphi_2\}.$$

Now our main theorem can be stated as follows:

THEOREM 1.1. *Suppose that the conditions* (1.2) *through* (1.6) *are fulfilled. Then the problem* (1.1) *admits a unique classical solution* $u \in C^{2+\alpha}(\bar{\Omega})$ *for each* $\varphi \in C_*^{1+\alpha}(\Gamma)$.

For Theorem 1.1, we give a simple example of the function $f(x, z, p)$:

EXAMPLE 1.2. $f(x, z, p) = z \pm |p|^2$. In this case one may take $f_0 = 1$ and $f_1(t) = 1 + t$.

Theorem 1.1 will be extended to the *integro-differential operator* case in the forthcoming paper Palagachev-Popivanov-Taira [8].

2. Proof of Main Theorem.

As it was mentioned above, the main theorem, Theorem 1.1 will be proved by making use of the Leray-Schauder fixed point theorem. For this purpose, we need to establish an *a priori* estimate for the $C^{1+\alpha}(\bar{\Omega})$-norm of each solution $u \in C^{2+\alpha}(\bar{\Omega})$ to the problem (1.1).

Let us start with the following comparison principle for quasilinear operators:

LEMMA 2.1. *Suppose that the conditions* (1.2) *and* (1.6) *are fulfilled and that* $f(x, z, p)$ *is increasing in* z *for each* $(x, p) \in \Omega \times \mathbf{R}^n$ *and is differentiable with respect to* p *for each* $(x, z) \in \Omega \times \mathbf{R}$. *Let* $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *satisfy the conditions*

$$\begin{cases} \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} - f(x, u, Du) \geq \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} - f(x, v, Dv) & \text{in } \Omega\,, \\ a(x)\frac{\partial u}{\partial \nu} + b(x)u \leq a(x)\frac{\partial v}{\partial \nu} + b(x)v & \text{on } \Gamma\,. \end{cases}$$

Then it follows that $u \leq v$ *on* $\bar{\Omega}$.

PROOF. Let $w = u - v$, and suppose to the contrary that the set

$$\Omega^+ = \{x \in \Omega : w(x) > 0\} = \{x \in \Omega : u(x) > v(x)\}$$

is non-empty. Then it follows that

$$\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j} + f(x, u, Dv) - f(x, u, Du) \geq f(x, u, Dv) - f(x, v, Dv) > 0 \quad \text{in } \Omega^+\,,$$

since $f(x, z, p)$ increases with respect to the second argument z. Thus, by letting

$$b^i(x) = -\int_0^1 \frac{\partial f}{\partial p_i}(x, u(x), tDw(x) + Dv(x))dt\,,$$

we obtain that

$$\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x)\frac{\partial w}{\partial x_i} > 0 \quad \text{in } \Omega^+\,.$$

If x_0 is a point of $\bar{\Omega}$ such that $w(x_0) = \max_{\bar{\Omega}} w(x) > 0$, then it follows from an application of the strong interior maximum principle (cf. [5, Theorem 3.5]) that

$$x_0 \in \partial\Omega^+ \cap \Gamma\,.$$

Thus we have, by the boundary point lemma (cf. [5, Lemma 3.4]),

$$\frac{\partial w}{\partial \nu}(x_0) > 0\,.$$

However it follows from the condition (1.6) that

$$Bw(x_0) = a(x_0)\frac{\partial w}{\partial \nu}(x_0) + b(x_0)w(x_0) > 0\,.$$

This contradicts the boundary condition $Bw(x_0) = Bu(x_0) - Bv(x_0) \leq 0$.

Therefore we have proved that the set Ω^+ is empty, and the statement follows. □

2.1. A priori estimate for $\|u\|_{C(\bar{\Omega})}$. As the first step in obtaining the desired *a priori* estimate, we will consider the homogeneous case. Namely, let $u \in C^{2+\alpha}(\bar{\Omega})$ be a solution to the problem

$$\left\{\begin{array}{ll} \sum_{i,j=1}^{n} a^{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j} = f(x,u,Du) & \text{in } \Omega\,, \\ a(x)\dfrac{\partial u}{\partial \nu} + b(x)u = 0 & \text{on } \Gamma\,. \end{array}\right. \tag{2.1}$$

Then we have the following estimate:

LEMMA 2.2. *Suppose that the conditions* (1.2), (1.3), (1.4) *and* (1.6) *are fulfilled, and let* $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *be a solution to the problem* (2.1). *Then we have the estimate*

$$\|u\|_{C(\bar{\Omega})} = \max_{\bar{\Omega}} |u(x)| \le \frac{\max_{\bar{\Omega}} |f(x,0,0)|}{f_0}\,. \tag{2.2}$$

PROOF. By letting

$$K = \frac{\max_{\bar{\Omega}} |f(x,0,0)|}{f_0}\,,$$

we obtain that

$$\begin{aligned} \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 K}{\partial X_i \partial x_j} - f(x,K,DK) &= -f(x,K,0) \\ &= -K\int_0^1 \frac{\partial f}{\partial z}(x,tK,0)dt - f(x,0,0) \\ &\le -Kf_0 - f(x,0,0) \\ &\le 0 \quad \text{for each } x \in \Omega\,, \end{aligned}$$

as a consequence of the condition (1.4). Hence it follows that

$$\begin{aligned} \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} - f(x,u,Du) &= 0 \\ &\ge \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 K}{\partial x_i \partial x_j} - f(x,K,DK) \quad \text{in } \Omega\,. \end{aligned}$$

On the other hand, we have

$$a(x)\frac{\partial u}{\partial \nu} + b(x)u = 0 \le b(x)K = a(x)\frac{\partial K}{\partial \nu} + b(x)K \quad \text{on } \Gamma\,.$$

Therefore it follows from an application of Lemma 2.1 that $u(x) \le K$ for each $x \in \bar{\Omega}$.

Repeating the above considerations with $u(x)$ replaced by $-u(x)$ and $f(x,z,p)$ replaced by $-f(x,-z,-p)$, respectively, we obtain that $-u(x) \le K$ for each $x \in \bar{\Omega}$.

Summing up, we have proved the estimate (2.2). □

2.2. A priori estimate for $[u]_{C^{1+\alpha}(\bar{\Omega})}$. After having the estimate (2.2), the desired bound on $\|u\|_{C^{1+\alpha}(\bar{\Omega})}$ will follow immediately if we have a uniform estimate for the Hölder seminorm

$$[u]_{C^{1+\alpha}(\bar{\Omega})} := [Du]_{C^{\alpha}(\bar{\Omega})} = \sup_{x,y\in\Omega} \frac{|Du(x) - Du(y)|}{|x-y|^{\alpha}}.$$

On the other hand, the Morrey lemma assures the imbedding of the Sobolev space $W^{2,p}(\Omega)$ into the Hölder space $C^{1+\alpha}(\bar{\Omega})$ with $p = n/(1-\alpha)$. Therefore the bound on $[Du]_{C^{\alpha}(\bar{\Omega})}$ becomes equivalent to a uniform (with respect to u) estimate of the Sobolev norm $\|u\|_{W^{2,p}(\Omega)}$ for each solution u to the problem (2.1). The last norm, however, is estimated in terms of $\|u\|_{C(\bar{\Omega})}$ as is shown in [11, Proposition 2.3]. More precisely, there exists a non-negative and increasing function $\gamma(t)$, depending only on known quantities, such that

$$\|u\|_{W^{2,p}(\Omega)} \leq \gamma(\|u\|_{C(\bar{\Omega})}) \tag{2.3}$$

for each solution $u \in W^{2,p}(\Omega)$ to the homogeneous problem (2.1).

In this way we have the following result:

THEOREM 2.3. *Suppose that the conditions* (1.2) *through* (1.6) *are fulfilled. Then there exists a positive constant C, independent of u, such that*

$$\|u\|_{C^{1+\alpha}(\bar{\Omega})} \leq C \tag{2.4}$$

for each solution $u \in C^{2+\alpha}(\bar{\Omega})$ *to the problem* (1.1) *with* $\varphi \in C_*^{1+\alpha}(\Gamma)$.

PROOF. The estimate (2.4) is an immediate consequence of Lemma 2.2, the Morrey lemma and the estimate (2.3) in the case where u solves the homogeneous problem (2.1).

To deal with the non-homogeneous problem (1.1), note that [10, Theorem 1.1] implies the existence of a unique solution $v \in C^{2+\alpha}(\bar{\Omega})$ to the linear problem

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} - v = 0 & \text{in } \Omega, \\ a(x)\dfrac{\partial v}{\partial \nu} + b(x)v = \varphi & \text{on } \Gamma, \end{cases}$$

and further the norm $\|v\|_{C^{2+\alpha}(\bar{\Omega})}$ depends continuously on the norm $\|\varphi\|_{C_*^{1+\alpha}(\Gamma)}$. Hence, if u is a solution to the problem (1.1), then the function $w = u - v$ solves the homogeneous problem

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j} = \bar{f}(x, w, Dw) & \text{in } \Omega, \\ a(x)\dfrac{\partial w}{\partial \nu} + b(x)w = 0 & \text{in } \Gamma, \end{cases}$$

with the nonlinear term $\bar{f}(x,z,p) = f(x, z + v(x), p + Dv(x)) - v(x)$. Moreover, the conditions (1.4) and (1.5) are fulfilled by the function $\bar{f}(x,z,p)$.

Therefore, the estimate (2.4) holds for the function w and so it is satisfied also by $u = v + w$ with a new positive constant C depending on $\|v\|_{C^{2+\alpha}(\bar{\Omega})}$, i.e., on $\|\varphi\|_{C_*^{1+\alpha}(\Gamma)}$, in addition. □

2.3. Proof of Theorem 1.1. The *uniqueness* assertion follows immediately from the comparison principle (Lemma 2.1).

In order to prove the *existence* part, we shall make use of the Leray-Schauder fixed point theorem (see [4, Theorem 5.4.14]; [5, Theorem 11.3]):

THEOREM 2.4. *Let $f(x,t)$ be a one-parameter family of compact operators defined on a Banach space X for $t \in [0,1]$, with $f(x,t)$ uniformly continuous in t for fixed $x \in X$. Furthermore suppose that* every *solution of $x = f(x,t)$ for each $t \in [0,1]$ is contained in the fixed open ball $\Sigma = \{x \in X : \|x\| < M\}$. Then, assuming $f(x,0) \equiv 0$, the operator $f(x,1)$ has a fixed point $x \in \Sigma$.*

Let $v \in C^{1+\alpha}(\bar{\Omega})$, and consider the linear problem

$$(2.5) \qquad \begin{cases} \sum_{i,j=1}^{n} a^{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j} = f(x, v, Dv) & \text{in } \Omega, \\ a(x)\dfrac{\partial u}{\partial \nu} + b(x)u = \varphi(x) & \text{on } \Gamma. \end{cases}$$

Then, in view of the condition (1.3) it follows that $f(x, v, Dv) \in C^{\alpha}(\bar{\Omega})$. Therefore [10, Theorem 1.1] asserts the unique classical solvability in the Hölder space $C^{2+\alpha}(\bar{\Omega})$ of the problem (2.5). Defining a nonlinear operator

$$\mathcal{H} : C^{1+\alpha}(\bar{\Omega}) \to C^{2+\alpha}(\bar{\Omega}) \underset{\text{compactly}}{\hookrightarrow\hookrightarrow} C^{1+\alpha}(\bar{\Omega})$$

by the formula $\mathcal{H}v = u$, it is an immediate consequence of the cited Taira's result that $\mathcal{H}$ is a continuous operator. Indeed, as shows [10, Theorem 1.1] the mapping

$$u \mapsto \left(\sum_{i,j=1}^{n} a^{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}, a\frac{\partial u}{\partial \nu} + bu \right)$$

is an algebraic and topological isomorphism of $C^{2+\alpha}(\bar{\Omega})$ onto $C^{\alpha}(\bar{\Omega}) \oplus C_*^{1+\alpha}(\Gamma)$ for $\alpha \in (0,1)$. This implies the continuity of $\mathcal{H}$ considered as an operator from $C^{1+\alpha}(\bar{\Omega})$ into $C^{2+\alpha}(\bar{\Omega})$. Furthermore, since the space $C^{2+\alpha}(\bar{\Omega})$ is compactly imbedded into the space $C^{1+\alpha}(\bar{\Omega})$, we derive immediately also the compactness of the mapping $\mathcal{H} : C^{1+\alpha}(\bar{\Omega}) \to C^{1+\alpha}(\bar{\Omega})$.

Now, for each $\rho \in [0,1]$, consider the equation $u = \rho\mathcal{H}u$, that is, the problem

$$(2.6) \qquad \begin{cases} \sum_{i,j=1}^{n} a^{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j} = \rho f(x, u, Du) & \text{in } \Omega, \\ a(x)\dfrac{\partial u}{\partial \nu} + b(x)u = \rho\varphi(x) & \text{on } \Gamma. \end{cases}$$

Then Theorem 2.3 assures the existence of a positive constant C, which depends only on the data of the problem (1.1) but not on u and ρ, such that

$$(2.7) \qquad \|u\|_{C^{1+\alpha}(\bar{\Omega})} \le C$$

for each solution $u \in C^{2+\alpha}(\bar{\Omega})$ to the problem (2.6).

In this way the properties of the operator $\mathcal{H}$ and the estimate (2.7) imply, by Theorem 2.4, the existence of a fixed point $u \in C^{1+\alpha}(\bar{\Omega})$ of the operator $\mathcal{H}$. The function u becomes a solution to the problem (1.1) in view of the definition of $\mathcal{H}$. Finally, the smoothing properties of $\mathcal{H}$ yield that $u = \mathcal{H}u \in C^{2+\alpha}(\bar{\Omega})$.

The proof of Theorem 1.1 is now complete. □

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