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Spacelike Helicoidal Surfaces with Constant Mean Curvature in Minkowski 3-space

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1. Introduction.

Surfaces with constant mean curvature in Minkowski 3-space L^3 have been studied in a lot of fields. For example, there are researches in relational to harmonic mappings between hyperbolic 2-spaces (cf. [1], [2], [5]) and affine differential geometry (cf. [7]).

In this paper we study surfaces of revolution and spacelike helicoidal surfaces in L^3.

A surface in L^3 is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric. Moreover, a timelike surface in L^3 is a surface which inherits a non-degenerate indefinite metric from the standard metric in L^3.

It is well-known that C. Delaunay [6] solved the differential equation of surfaces of revolution under constancy of the mean curvature and gave a method of geometric constructions for such surfaces. For the proof, he first obtained a parametrization of an evolute of the generating curve. By making use of this parametrization, he found a representation formula for the generating curve. Therefore these solutions hold only on some intervals on which the evolute can be defined. More generally, K. Kenmotsu [11] gave a representation formula for surfaces of revolution with prescribed mean curvature in Euclidean 3-space.

Spacelike maximal surfaces of revolution in L^3 were studied by O. Kobayashi [12] and L. McNertney [15]. On the other hand, timelike minimal surfaces of revolution in L^3 were classified by L. McNertney (cf. [20]). Furthermore McNertney gave one parameter deformations for various catenoids and helicoids. Moreover, the timelike minimal surfaces have been a subject of several wide interests [14], [16], [17], [18] and [21].

J. Hano and K. Nomizu [9] classified the spacelike surfaces of revolution in L^3 that have constant mean curvature and proved that the profile curve of a surface of revolution with nonzero constant mean curvature in L^3 can be described as the locus of focus when a quadratic curve is rolled along the axis of revolution. However, they did not give a representation formula for spacelike surfaces in L^3.

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T. Ishihara and F. Hara [10] studied Minkowski versions of the work of K. Kenmotsu. They gave a representation formula for surfaces of revolution with prescribed mean curvature in L^3. However, there were some mistakes in their paper. In particular, there were not a case of timelike surface of revolution with spacelike axis in L^3. Thus they have not been finished the classification.

About helicoidal surfaces in Euclidean 3-space, M. P. do Carmo and M. Dajczer [8] proved that, by using a result of E. Bour [4], there exists a two-parameter family of helicoidal surfaces isometric to a given helicoidal surface. By making use of this parametrization, they found a representation formula for helicoidal surfaces with constant mean curvature. Furthermore they proved that the associated family of Delaunay surfaces is made up by helicoidal surfaces of constant mean curvature.

Moreover, helicoidal surfaces were treated by T. Sasai [19]. He gave another parametrization for helicoidal surfaces. By using his parametrization, he studied limiting surfaces of helicoidal surfaces.

The purpose of this note is to restudy surfaces of revolution in L^3 to complete a representation formula and consider spacelike helicoidal surfaces with constant mean curvature H in L^3.

More precisely we shall solve the differential equation which describes the mean curvature by an elementary method. Solutions are represented explicitly by generalized integrals which involve the mean curvature. Therefore, for a given continuous function H and constant H_0, we can construct surfaces of revolution admitting H as the mean curvature and spacelike helicoidal surfaces admitting H_0 as the mean curvature.

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2. Preliminaries.

We begin with fixing our terminology and notation (cf. [3], [13]). Let $L^3 = (\mathbf{R}^3, \tilde{g})$ denote Minkowski 3-space with flat Lorentzian metric $\tilde{g}$ of signature $(+, +, -)$. In terms of canonical coordinates (x, y, z) of $\mathbf{R}^3$, the metric $\tilde{g}$, denoted also by $\langle \cdot, \cdot \rangle$, can be expressed as $\tilde{g} = dx^2 + dy^2 - dz^2$. Let M^2 be a connected smooth 2-manifold, and $X : M^2 \to L^3$ be a smooth nondegenerate immersion. Then M^2 or X is said to be *spacelike* (resp. *timelike*) if the pulled back metric $g = X_\delta{}^*\tilde{g}$ of the Lorentzian metric $\tilde{g}$ via X_δ is a positive definite metric (resp. an indefinite metric) on M^2.

It will sometimes be expedient to use the notation M^2_δ or X_δ, $\delta = \pm 1$, to denote a nondegenerate surface in L^3, where $\delta = 1$ (resp. $\delta = -1$) means that M^2_δ is a spacelike (resp. timelike) surface or X_δ is a spacelike (resp. timelike).

Let us denote D and ∇ the Levi-Civita connection of L^3 and M^2_δ respectively. Then the *Gauss-Weingarten formulas* are given as follows:

$$D_X Y = \nabla_X Y + h(X, Y)N\,, \quad D_X N = -AX$$

for all vector fields X, Y on M. Here is N is a unit normal vector field to M^2_δ such that $\langle N, N\rangle = -\delta$, h is the second fundamental form of M^2_δ in L^3 and A is shape operator of M^2_δ relative to N. The Gaussian curvature K of M^2_δ is given by

$$K = -\delta \det A\,.$$

The mean curvature H of M^2_δ is defined by

$$H = -\frac{\delta}{2}\operatorname{tr} A\,.$$

We shall define a local Lorentzian frame field (e_1, e_2, e_3) adapted to M^2_δ in L^3 in the following manner. Let (e_1, e_2) be an orthonormal tangent frame field on M^2_δ compatible to the orientation of M^2_δ. Then we define e_3 by $e_3 = e_1 \times e_2$. Here the exterior product $v \times w$ of two vectors $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3) \in L^3$ is defined by

$$v \times w = -(v_3w_2 - v_2w_3,\, v_1w_3 - v_3w_1,\, v_1w_2 - v_2w_1)\,.$$

With respect to a Lorentian frame field adapted to M^2_δ in L^3, the second fundamental h with respect to the unit normal $N = e_3$ is expressed by the matrix $(h_{ij})_{1\le i,j\le 2}$:

$$h_{ij} = -\delta\langle D_{e_i}e_j, e_3\rangle\,.$$

The Gaussian curvature and the mean curvature of M^2_δ are given in terms of h_{ij} as follows

$$K = h_{12}h_{21} - h_{11}h_{22}\,, \qquad H = (\delta h_{11} + h_{22})/2\,.$$

3. Surfaces of revolution in L^3.

A surface in L^3 is called a *surface of revolution* with axis l if it is invariant under the action of the group of motions in L^3 which fix each point of the line l.

If the axis is timelike (resp. spacelike), we may assume without loss of generality that the axis is z-axis (resp. y-axis), since every timelike (resp. spacelike) unit vector can be transformed to $(0, 0, 1)$ (resp. $(0, 1, 0)$) by a Lorentzian transformation. Then the surface is expressed as follows:

$$\begin{aligned} X(s,\theta) &= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(s) \\ 0 \\ g(s) \end{pmatrix} \\ &= (f(s)\cos\theta,\, f(s)\sin\theta,\, g(s)) \end{aligned}$$

if the axis is timelike, and

$$\begin{aligned} X(s,\theta) &= \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix} \begin{pmatrix} 0 \\ g(s) \\ f(s) \end{pmatrix} \\ &= (f(s)\sinh\theta,\, g(s),\, f(s)\cosh\theta) \qquad \text{or} \end{aligned}$$

$$X(s,\theta)=\begin{pmatrix}\cosh\theta & 0 & \sinh\theta\\ 0 & 1 & 0\\ \sinh\theta & 0 & \cosh\theta\end{pmatrix}\begin{pmatrix}f(s)\\ g(s)\\ 0\end{pmatrix}$$
$$=(f(s)\cosh\theta,\,g(s),\,f(s)\sinh\theta)$$

if the axis is spacelike.

If the axis is lightlike, we may assume without loss of generality that it is $\mathbf{R}\cdot(1,0,1)$. Note that the subgroup of the Lorentzian group which fixes $(1,0,1)$ is

$$\left\{\begin{pmatrix}1-\frac{t^2}{2} & t & \frac{t^2}{2}\\ -t & 1 & t\\ -\frac{t^2}{2} & t & 1+\frac{t^2}{2}\end{pmatrix};\ t\in\mathbf{R}\right\}.$$

Hence, the surface can be written as

$$X(s,t)=\begin{pmatrix}1-\frac{t^2}{2} & t & \frac{t^2}{2}\\ -t & 1 & t\\ -\frac{t^2}{2} & t & 1+\frac{t^2}{2}\end{pmatrix}\begin{pmatrix}h(s)+s\\ 0\\ h(s)-s\end{pmatrix}$$
$$=(h(s)+s-t^2s,\,-2st,\,h(s)-s-t^2s).$$

Let $(f(s),g(s))$, $s\in I$, be any C^2-curve which is parametrized by the arc length and I is an open interval of real numbers including zero.

It will be expedient to use the notation $(\cdot,\cdot)_t$ (resp. $(\cdot,\cdot)_s$ or $(\cdot,\cdot)_l$) for formulas where $(\cdot,\cdot)_t$ (resp. $(\cdot,\cdot)_s$ or $(\cdot,\cdot)_l$) means that the axis of surfaces is timelike (resp. spacelike or lightlike).

We deal with a surface M_δ of revolution in L^3 with non lightlike axis l and profile curve in Minkowski plane defined by

$(3.1)_t$ $\quad X_\delta(s,\theta)=(f(s)\cos\theta,\,f(s)\sin\theta,\,g(s))\,,\quad s\in I\,,\ 0\le\theta\le 2\pi\,,$

where

$(3.2)_t$ $\quad f'(s)^2-g'(s)^2=\delta\,,\qquad$ or

$(3.1)_s$ $\quad X_\delta(s,\theta)=(f(s)\sinh\theta,\,g(s),\,f(s)\cosh\theta)\,,\quad s\in I\,,\ \theta\in\mathbf{R},$

where

$(3.2)_s$ $\quad g'(s)^2-f'(s)^2=\delta\,,$

according as the axis l is timelike or spacelike.

We set $e_3=X_s\times(X_\theta/f)$ then the first and the second fundamental forms of M_δ are

$\delta ds^2+f(s)^2d\theta^2$ and $\delta\{f'(s)g''(s)-f''(s)g'(s)\}ds^2+\delta f(s)g'(s)d\theta^2\,,\qquad$ or

$\delta ds^2+f(s)^2d\theta^2$ and $\delta\{f'(s)g''(s)-f''(s)g'(s)\}ds^2-\delta f(s)g'(s)d\theta^2\,,$

respectively. By the regularity of the surface we may assume $f(s) > 0$ on I. The mean curvature $H(s)$, by definition, satisfies

$$(3.3)_t \quad 2H(s)f(s) - \delta g'(s) - (f'(s)g''(s) - f''(s)g'(s))f(s) = 0, \quad s \in I, \quad \text{or}$$

$$(3.3)_s \quad 2H(s)f(s) + \delta g'(s) - (f'(s)g''(s) - f''(s)g'(s))f(s) = 0, \quad s \in I.$$

Multiplying (3.3) by $g'(s)$ and making use of (3.2), we get

$$(3.4)_t \quad 2H(s)f(s)g'(s) - \delta(f(s)f'(s))' + 1 = 0, \quad \text{or}$$

$$(3.4)_s \quad 2H(s)f(s)g'(s) + \delta(f(s)f'(s))' + 1 = 0.$$

Similarly we have

$$(3.5)_t \quad 2H(s)f(s)f'(s) - \delta(f(s)g'(s))' = 0, \quad \text{or}$$

$$(3.5)_s \quad 2H(s)f(s)f'(s) + \delta(f(s)g'(s))' = 0.$$

Combining (3.4) and (3.5), we obtain the first order linear differential equations

$$(3.6)_t \quad \begin{cases} \delta Z_1'(s) - 2H(s)Z_1(s) - 1 = 0, & s \in I \\ \delta Z_2'(s) + 2H(s)Z_2(s) - 1 = 0, & s \in I, \end{cases}$$

$$(3.6)_s \quad \begin{cases} \delta Z_1'(s) + 2H(s)Z_1(s) + 1 = 0, & s \in I \\ \delta Z_2'(s) - 2H(s)Z_2(s) + 1 = 0, & s \in I, \end{cases}$$

where we put

$$(3.7) \quad \begin{cases} Z_1(s) = f(s)(f'(s) + g'(s)) \\ Z_2(s) = f(s)(f'(s) - g'(s)). \end{cases}$$

They can easily be solved. General solutions of (3.6) are

$$(3.8)_t \quad \begin{cases} Z_1(s) = \exp\left(2\delta \int_{s_0}^{s} H(s)ds\right)\left\{\delta \int_{s_1}^{s} \exp\left(-2\delta \int_{s_0}^{s} H(s)ds\right)ds + C_1\right\} \\ Z_2(s) = \exp\left(-2\delta \int_{s_0}^{s} H(s)ds\right)\left\{\delta \int_{s_1}^{s} \exp\left(2\delta \int_{s_0}^{s} H(s)ds\right)ds + C_2\right\}, \end{cases} \quad \text{or}$$

$$(3.8)_s \quad \begin{cases} Z_1(s) = \exp\left(-2\delta \int_{s_0}^{s} H(s)ds\right)\left\{-\delta \int_{s_1}^{s} \exp\left(2\delta \int_{s_0}^{s} H(s)ds\right)ds + C_1\right\} \\ Z_2(s) = \exp\left(2\delta \int_{s_0}^{s} H(s)ds\right)\left\{-\delta \int_{s_1}^{s} \exp\left(-2\delta \int_{s_0}^{s} H(s)ds\right)ds + C_2\right\}, \end{cases}$$

where C_1 and C_2 are constants. It is convenient to introduce the following functions:

$$(3.9) \quad \begin{cases} F(s) = \int_{s_1}^{s} \exp\left(-2\delta \int_{s_0}^{s} H(s)ds\right)ds \\ G(s) = \int_{s_1}^{s} \exp\left(2\delta \int_{s_0}^{s} H(s)ds\right)ds. \end{cases}$$

Then (3.8) is represented by

$$(3.10)_t \qquad \begin{cases} Z_1(s) = G'(s)(\delta F(s) + C_1) \\ Z_2(s) = F'(s)(\delta G(s) + C_2), \end{cases} \qquad \text{or}$$

$$(3.10)_s \qquad \begin{cases} Z_1(s) = F'(s)(-\delta G(s) + C_1) \\ Z_2(s) = G'(s)(-\delta F(s) + C_2). \end{cases}$$

By (3.2) and (3.7), we obtain

$$(3.11)_t \qquad f(s)^2 = \delta Z_1(s) Z_2(s), \qquad \text{or}$$

$$(3.11)_s \qquad f(s)^2 = -\delta Z_1(s) Z_2(s).$$

From (3.9) it follows that

$$(3.12)_t \qquad f(s)^2 = \delta(\delta F(s) + C_1)(\delta G(s) + C_2), \text{or}$$

$$(3.12)_s \qquad f(s)^2 = -\delta(-\delta G(s) + C_1)(-\delta F(s) + C_2).$$

Noting that

$$(3.13)_t \qquad \delta F(s)G(s) + C_2 F(s) + C_1 G(s) + \delta C_1 C_2 > 0, \quad s \in I, \qquad \text{or}$$

$$(3.13)_s \qquad -\delta F(s)G(s) + C_1 F(s) + C_2 G(s) - \delta C_1 C_2 > 0, \quad s \in I,$$

we see that the function $f(s)$ can be written as

$$(3.14)_t \qquad f(s) = (\delta F(s)G(s) + C_2 F(s) + C_1 G(s) + \delta C_1 C_2)^{1/2}, \quad s \in I, \qquad \text{or}$$

$$(3.14)_s \qquad f(s) = (-\delta F(s)G(s) + C_1 F(s) + C_2 G(s) - \delta C_1 C_2)^{1/2}, \quad s \in I.$$

On the other hand, since we have $Z_1(s) - Z_2(s) = 2f(s)g'(s)$, we obtain $g'(s) = (Z_1(s) - Z_2(s))/(2f(s))$. Hence,

$$(3.15)_t \qquad g(s) = \int_{s_2}^{s} \frac{G'(s)(\delta F(s) + C_1) - F'(s)(\delta G(s) + C_2)}{2f(s)} ds, \qquad \text{or}$$

$$(3.15)_s \qquad g(s) = \int_{s_2}^{s} \frac{F'(s)(-\delta G(s) + C_1) - G'(s)(-\delta F(s) + C_2)}{2f(s)} ds.$$

Next we deal with a timelike surface of revolution in L^3 with spacelike axis and profile curve in Euclidean plane defined by

$$(3.16) \qquad X(s, \theta) = (f(s)\cosh\theta, g(s), f(s)\sinh\theta), \quad s \in I, \ \theta \in \mathbf{R},$$

where

$$(3.17) \qquad g'(s)^2 + f'(s)^2 = 1.$$

We set $e_3 = X_s \times (X_\theta / f)$ then the first and the second fundamental forms are

$$ds^2 - f(s)^2 d\theta^2 \quad \text{and} \quad (f''(s)g'(s) - f'(s)g''(s))ds^2 - f(s)g'(s)d\theta^2.$$

By the regularity of the surface we may assume $f(s) > 0$ on I. The mean curvature $H(s)$, satisfies

$$(3.18) \qquad 2H(s)f(s) - g'(s) - (f'(s)g''(s) - f''(s)g'(s))f(s) = 0, \quad s \in I.$$

Multiplying (3.18) by $g'(s)$ and making use of (3.17), we get

$$2H(s)f(s)g'(s) + (f(s)f'(s))' - 1 = 0. \tag{3.19}$$

Similarly we have

$$2H(s)f(s)f'(s) - (f(s)g'(s))' = 0. \tag{3.20}$$

Combining (3.19) and (3.20), we obtain the first ordere complex linear differential equations

$$Z_1'(s) - 2iH(s)Z(s) - 1 = 0, \quad s \in I, \tag{3.21}$$

where we put

$$Z(s) = f(s)(f'(s) + ig'(s)).$$

They can easily be solved. The general solution of (3.21) is

$$Z(s) = \exp\left(2i \int_0^s H(s)ds\right)\left\{\int_0^s \exp\left(-2i\int_0^s H(s)ds\right)ds + C\right\}.$$

where C is a complex constant. Introducing the following functions:

$$\begin{cases} F(s) = \displaystyle\int_0^s \sin\left(2\int_0^s H(s)ds\right)ds \\ G(s) = \displaystyle\int_0^s \cos\left(2\int_0^s H(s)ds\right)ds. \end{cases} \tag{3.22}$$

Then the general solution (3.21) is represented by

$$Z(s) = \{(F(s) - c_1) + i(G(s) + c_2)\}(F'(s) - iG'(s)), \tag{3.23}$$

where we put $iC = -c_1 + ic_2$. Since we have $|Z(s)|^2 = f(s)^2$, we obtain for some constant C,

$$f(s)^2 = (F(s) - c_1)^2 + (G(s) + c_2)^2, \quad s \in I.$$

Noting that

$$(F(s) - c_1)^2 + (G(s) + c_2)^2 > 0, \quad s \in I, \tag{3.24}$$

we see that the function $f(s)$ can be written as

$$f(s) = \{(F(s) - c_1)^2 + (G(s) + c_2)^2\}^{1/2}, \quad s \in I. \tag{3.25}$$

Combining (3.23), (3.25) and $Z(s) - \overline{Z(s)} = 2if(s)g'(s)$, we get

$$g'(s) = \frac{(G(s) + c_2)F'(s) - (F(s) - c_1)G'(s)}{\{(F(s) - c_1)^2 + (G(s) + c_2)^2\}^{1/2}}.$$

Hence, we obtain

$$g(s) = \int_0^s \frac{(G(s) + c_2)F'(s) - (F(s) - c_1)G'(s)}{\{(F(s) - c_1)^2 + (G(s) + c_2)^2\}^{1/2}}ds + c_3, \tag{3.26}$$

where c_1, c_2 and c_3 are any integral constants.

Finally if the axis l is lightlike, we deal with a surface M_δ of revolution with axis l in L^3 defined by

$$X_\delta(s,t) = (h(s) + s - t^2 s,\ -2st,\ h(s) - s - t^2 s), \quad s \in I, \ t \in \mathbf{R}, \tag{3.27}$$

where

$$\delta h'(s) > 0.$$

We set $e_3 = (1/(2\sqrt{\delta h'}))X_s \times (1/(2s))X_t$ then the first and the second fundamental forms of M_δ are

$$4h'(s)ds^2 + 4s^2dt^2 \quad \text{and} \quad -\delta\left\{\frac{h''(s)}{\sqrt{\delta h'(s)}}ds^2 - \frac{2s}{\sqrt{\delta h'(s)}}dt^2\right\},$$

respectively. The mean curvature $H(s)$, satisfies

$$8sH(s)h'(s)\sqrt{\delta h'(s)} + \delta(h''(s)s - 2h'(s)) = 0. \tag{3.28}$$

Multiplying (3.28) by $-(2h'(s)\sqrt{\delta h'(s)})^{-1}$, it becomes

$$-4sH(s) + \delta(s/\sqrt{\delta h'(s)})' = 0.$$

Hence, if we assume $H(s) \equiv 0$, $s \in I$, then we get

$$h(s) = \delta(as^3 + b), \quad a > 0. \tag{3.29}$$

If we assume $H(s) \neq 0$, then we get

$$h'(s) = \delta\left(\frac{\delta s}{\int_{s_0}^s 4sH(s)ds}\right)^2.$$

Thus the function $h(s)$ can be written as

$$h(s) = \delta\int_{s_1}^s \left(\frac{\delta s}{\int_{s_0}^s 4H(s)ds}\right)^2 ds, \tag{3.30}$$

where $H(s) \neq 0$.

Therefore we have proved the following.

THEOREM 3.1 (surfaces of revolution in L^3). (i) *If the axis l is timelike (resp. spacelike), let $(f(s), g(s))$, $s \in I$, be the generating curve, parametrized by the arc length, of a surface of revolution whose mean curvature at the point $(f(s), 0, g(s))$ (resp. $(0, g(s), f(s))$) is given by $H(s)$. Then for some constants C_1 and C_2 we have $(f(s), g(s))$ satisfying* (3.14) *and* (3.15). *Conversely for any given continuous function $H(s)$, $s \in I$, we take C_1 and C_2 in such a way that they satisfy* (3.13). *Then we can construct a surface of revolution by* (3.1) *and the initial data are given by* (3.9), (3.14) *and* (3.15).

(ii) *If the axis l is spacelike, let $(f(s), g(s))$, $s \in I$, be the generating curve, parametrized by the arc length, of a timelike surface of revolution whose mean curvature at the point $(f(s), g(s), 0)$ is given by $H(s)$. Then for some constants c_1, c_2 and c_3 we have $(f(s), g(s))$ satisfying* (3.25) *and* (3.26). *Conversely for any given continuous function $H(s)$, $s \in I$, we* *take* c_1, c_2 *and* c_3 *in such a way that they satisfy* (3.24). *Then we can construct a surface of revolution by* (3.17) *and the initial data are given by* (3.22), (3.25) *and* (3.26).

(iii) *If the axis* l *is lightlike, let* $(h(s)+s, h(s)-s)$, $s \in I$, *be the generating curve of a surface of revolution whose mean curvature at the point* $(h(s)+s, 0, h(s)-s)$ *is given by* $H(s)$. *We assume that* $H(s) \neq 0$, *then we can construct a surface of revolution by* (3.27) *and the initial data are given by* (3.30). *If* $H \equiv 0$, *then we can construct a surface of revolution by* (3.27) *and the initial data are given by* (3.29).

4. Surfaces of revolution with constant mean curvature in L^3.

We assume that the mean curvature $H(s)$ is a constant function.

If the constant H is zero, by Theorem 3.1 (i), then we have $F(s) = s$ and $G(s) = s$ where $s_0 = 0$ and $s_1 = 0$, which gives

$$(4.1)_t \qquad \begin{cases} f(s) = \{\delta s^2 + 2As + \delta(A+B)(A-B)\}^{1/2} \\ g(s) = \displaystyle\int_{s_2}^{s} (B/f(s))ds\,, \end{cases} \qquad \text{or}$$

$$(4.1)_s \qquad \begin{cases} f(s) = \{-\delta s^2 + 2As - \delta(A+B)(A-B)\}^{1/2} \\ g(s) = \displaystyle\int_{s_2}^{s} (B/f(s))ds\,, \end{cases}$$

where A and B are any constants.

Therefore we get the following.

EXAMPLE 4.1 (Maximal surface with timelike axis). Set $\delta = 1$, $s_2 = 0$, $A = 0$ and $B = 0$ in $(3.1)_t$ and $(4.1)_t$. Then we have $f(s) = s$ and $g(s) = 0$. Hence,

$$X(s,\theta) = (s\cos\theta, s\sin\theta, 0)\,.$$

This surface is (x, y)-plane, i.e., a spacelike plane.

EXAMPLE 4.2 (Maximal surface with timelike axis). Set $\delta = 1$, $s_2 = a > 0$, $A = 0$ and $B = a$ in $(3.1)_t$ and $(4.1)_t$. Then we have $f(s) = \sqrt{s^2 - a^2}$ and $g(s) = a\cosh^{-1}(s/a) - \log a$. Hence,

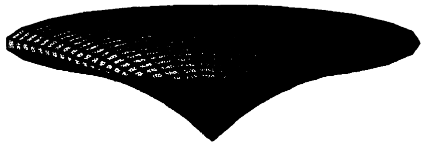
$$X(s,\theta) = \left(a\sinh\left(\frac{g(s)+\log a}{a}\right)\cos\theta,\ a\sinh\left(\frac{g(s)+\log a}{a}\right)\sin\theta,\ g(s)\right)\,.$$

This surface can be written as

$$x^2 + y^2 - a^2\sinh^2\left(\frac{z+\log a}{a}\right) = 0\,,$$

which shows that the surface is catenoid of the 1st kind (cf. Figure 1).

EXAMPLE 4.3 (Maximal surface with spacelike axis). Set $\delta = 1$, $s_2 = 0$, $A = 0$ and $B = a$ in $(3.1)_s$ and $(4.1)_s$. Then we have $f(s) = \sqrt{a^2 - s^2}$ and $g(s) = a\sin^{-1}(s/a)$. Put $g(s) = -a\alpha(s)$, then we have $f(s) = -a\cos(\alpha(s))$. Hence,



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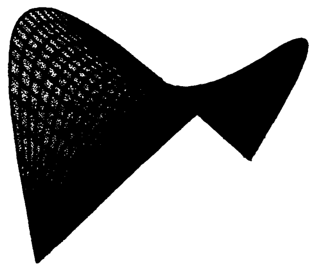
```
x1[a_][s_,t_]:={a Sqrt[s^2 - a^2] Cos[t],
  a Sqrt[s^2 - a^2] Sin[t],a ArcCosh[s/a] - Log[a]}
ParametricPlot3D[x1[1][s,t]//Evaluate,{s,1,4},
  {t,- Pi,Pi},PlotPoints->{24,24},
  ViewPoint->{-0.026,-3.299,0.751},
  Boxed->False,Axes->None]
```

FIGURE 1.



-Graphics3D-

```
x2[a_][s_,t_]:={a Sqrt[a^2 - s^2] Sinh[t],
  a ArcSin[s/a],a Sqrt[a^2 - s^2] Cosh[t]}
ParametricPlot3D[x2[1][s,t]//Evaluate,{s,-1,1},
  {t,-1,1},PlotPoints->{24,24},
  ViewPoint->{1.731,-2.775,0.867},
  Boxed->False,Axes->None]
```

FIGURE 2.

$$X(s,\theta) = (a\cos(\alpha(s))\sinh\theta,\ -a\alpha(s),\ a\cos(\alpha(s))\cosh\theta)\,.$$

This surface is catenoid of the 2nd kind (cf. Figure 2).

EXAMPLE 4.4 (Timelike minimal surface with timelike axis). Set $\delta = -1$, $s_2 = 0$, $A = 0$ and $B = a > 0$ in $(3.1)_t$ and $(4.1)_t$. Then we have $f(s) = \sqrt{a^2 - s^2}$ and $g(s) = a\sin^{-1}(s/a)$. Hence,

$$X(s,\theta) = (\sqrt{a^2-s^2}\cos\theta,\ \sqrt{a^2-s^2}\sin\theta,\ a\sin^{-1}(s/a))\,.$$

This surface can be written as

$$x^2 + y^2 - a^2\cos^2(z/a) = 0\,.$$

This surface is catenoid of the 3rd kind (cf. Figure 3).

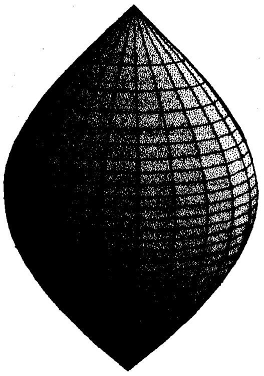
EXAMPLE 4.5 (Timelike minimal surface with spacelike axis). Set $\delta = -1$, $s_2 = 0$, $A = 0$ and $B = 0$ in $(3.1)_s$ and $(4.1)_s$. Then we have $f(s) = s$ and $g(s) = 0$. Hence,

$$X(s,\theta) = (s\sinh\theta, 0, s\cosh\theta)\,.$$

This surface is a timelike plane.

EXAMPLE 4.6 (Timelike minimal surface with spacelike axis). Set $\delta = -1$, $s_2 = a > 0$, $A = 0$ and $B = a > 0$ in $(3.1)_s$ and $(4.1)_s$. Then we have $F(s) = s$, $G(s) = s$, $f(s) = \sqrt{s^2 - a^2}$ and $g(s) = a\cosh^{-1}(s/a) - \log a$. Hence,

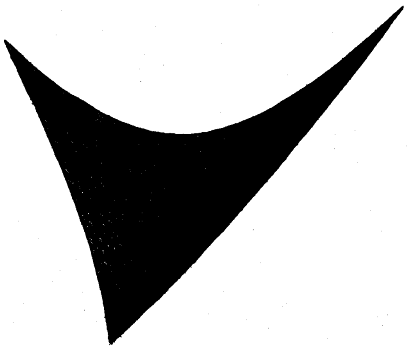
$$X(s,\theta) = \left(a\sinh\left(\frac{g(s)+\log a}{a}\right)\sinh\theta,\ g(s),\ a\sinh\left(\frac{g(s)+\log a}{a}\right)\cosh\theta\right)\,.$$



```
x3[a_][s_,t_]:={Sqrt[a^2 - s^2] Cos[t],
  Sqrt[a^2 - s^2] Sin[t],a ArcSin[s/a]}
ParametricPlot3D[x3[1][s,t]//Evaluate,
  {s,-1,1},{t,- Pi,Pi},PlotPoints->{24,24},
  ViewPoint->{1.731,-2.775,0.867},
  Boxed->False,Axes->None]
```

-Graphics3D-

FIGURE 3.



```
x4[a_][s_,t_]:={a Sqrt[s^2 - a^2] Sinh[t],
  a ArcCosh[s/a] - Log[a],a Sqrt[s^2 - a^2]Cosh[t]}
ParametricPlot3D[x4[1][s,t]//Evaluate,
  {s,1,4},{t,-1,1},PlotPoints->{24,24},
  ViewPoint->{2.136,-2.623,-0.080},
  Boxed->False,Axes->None]
```

-Graphics3D-

FIGURE 4.

This surface can be written as

$$x^2 - z^2 + a^2 \sinh^2\left(\frac{y + \log a}{a}\right) = 0 .$$

This surface is catenoid of the 4th kind (cf. Figure 4).

If the constant is non-zero, by Theorem 3.1 (i), then we have $F(s) = (\exp(-2H\delta s) - 1)/(-2H\delta)$ and $G(s) = (\exp(2H\delta s) - 1)/(2H\delta)$ where $H(s) = H \neq 0$, $s_0 = 0$ and $s_1 = 0$, which gives

$$(4.2)_t \qquad \begin{cases} f(s) = \{\delta(\sinh(H\delta s)/H)^2 - B\cosh(2H\delta s)/(H\delta) \\ \qquad\quad +A\sinh(2H\delta s)/(H\delta) - A/(2H\delta) + \delta(A+B)(A-B)\}^{1/2} \\ g(s) = \displaystyle\int_{s_2}^{s} \frac{2\sinh^2(H\delta s)/H + 2A\sinh(2H\delta s) + 2B\cosh(2H\delta s)}{2f(s)} ds , \end{cases} \qquad \text{or}$$

$$(4.2)_s \qquad \begin{cases} f(s) = \{-\delta(\sinh(H\delta s)/H)^2 - A\cosh(2H\delta s)/(H\delta) \\ \qquad\quad +B\sinh(2H\delta s)/(H\delta) - A/(2H\delta) - \delta(A+B)(A-B)\}^{1/2} \\ g(s) = \displaystyle\int_{s_2}^{s} \frac{2\sinh^2(H\delta s)/H - 2A\sinh(2H\delta s) + 2B\cosh(2H\delta s)}{2f(s)} ds , \end{cases}$$

where A and B are any constants.

Therefore we get the following.

EXAMPLE 4.7 (Spacelike surface with timelike axis). Set $\delta = 1$, $s_2 = 0$, $A = 0$ and $B = 0$ in $(3.1)_t$ and $(4.2)_t$. Then we have $f(s) = (\sinh Hs)/|H|$ and $g(s) = (-1 + \cosh Hs)/H$. Hence,

$$X(s, \theta) = \left(\frac{1}{|H|} \sinh Hs \cos\theta,\ \frac{1}{|H|} \sinh Hs \sin\theta,\ \frac{1}{H}(-1 + \cosh Hs) \right).$$

This surface can be written as

$$x^2 + y^2 - \left(z + \frac{1}{H} \right)^2 = -\frac{1}{H^2}, \quad \frac{|H|}{H} z \geq 0,$$

which shows that the surface is a connected component of hyperbolid of two sheets. Furthermore X is a totally umbilic imbedding of a hyperbolic 2-space.

EXAMPLE 4.8 (Spacelike surface with spacelike axis). Set $\delta = 1$, $s_2 = 0$, $A = 0$ and $B = -1/(2H)$ in $(3.1)_s$ and $(4.2)_s$. Then we have $f(s) = 1/(2|H|)$ and $g(s) = -(|H|/H)s$. Hence,

$$X(s, \theta) = \left(\frac{1}{2|H|} \sinh\theta,\ -\frac{|H|}{H} s,\ \frac{1}{2|H|} \cosh\theta \right).$$

This surface can be written as

$$z^2 - x^2 = \frac{1}{4H^2}, \quad z > 0,$$

which shows that the surface is the spacelike hyperbolic cylinder.

EXAMPLE 4.9 (Timelike surface with timelike axis). Set $\delta = -1$, $s_2 = 0$, $A = 0$ and $B = -1/(2H)$ in $(3.1)_t$ and $(4.2)_t$. Then we have $f(s) = 1/(2|H|)$ and $g(s) = -(|H|/H)s$. Hence,

$$X(s, \theta) = \left(\frac{1}{2|H|} \cos\theta,\ \frac{1}{2|H|} \sin\theta,\ -\frac{|H|}{H} s \right).$$

This surface can be written as

$$x^2 + y^2 = \frac{1}{4H^2},$$

which shows that the surface is the timelike circular cylinder.

EXAMPLE 4.10 (Timelike surface with spacelike axis). Set $\delta = -1$, $s_2 = 0$, $A = 0$ and $B = 0$ in $(3.1)_s$ and $(4.2)_s$. Then we have $f(s) = (\sinh Hs)/|H|$ and $g(s) = (-1 + \cosh Hs)/H$. Hence,

$$X(s, \theta) = \left(\frac{1}{|H|} \sinh Hs \sinh\theta,\ \frac{1}{H}(-1 + \cosh Hs),\ \frac{1}{|H|} \sinh Hs \cosh\theta \right).$$

This surface can be written as

$$x^2 + \left(y + \frac{1}{H} \right)^2 - z^2 = \frac{1}{H^2}, \quad \frac{|H|}{H} y \geq 0.$$

This surface is drawn in Figure 5. This surface is a hyperbololid of one sheet. Namely X is a totally umbilic imbedding of a Lorentz sphere into L^3.

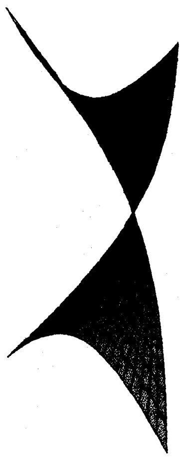
If the constant H is zero, by Theorem 3.1 (ii), then we get the following.

EXAMPLE 4.11 (Timelike minimal surface with spacelike axis). Set $c_1 \neq 0$ in (3.16), (3.22), (3.25) and (3.26). Then we have,

$$X(s,\theta) = \left(\{c_1^2 + (s+c_2)^2\}^{1/2}\cosh\theta,\right.$$
$$\left.\int_0^s c_1\{c_1^2 + (s+c_2)^2\}^{-1/2}ds + c_3,\ \{c_1^2 + (s+c_2)^2\}^{1/2}\sinh\theta\right).$$

This surface is catenoid of the 5th kind (cf. Figure 6).

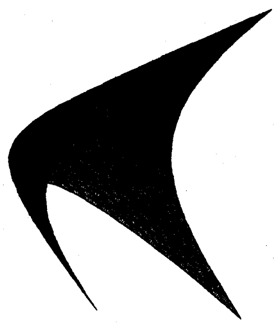
If the constant H is non-zero, by Theorem 3.1 (ii), then we get the following.



```
x5[H_][s_,t_]:={(Sinh[H s] Sinh[t])/(Abs[H]),
  (-1 + Cosh[H s])/H,(Sinh[H s] Cosh[t])/(Abs[H])}
ParametricPlot3D[x5[-1][s,t]//Evaluate,
  {s,-1,1},{t,-1,1},PlotPoints->{24,24},
  ViewPoint->{2.748,-1.774,0.867},
  Boxed->False,Axes->None]
```

-Graphics3D-

FIGURE 5.



```
x6[C1_,C2_,C3_][s_,t_]:={
  Sqrt[C1^2 + (s + C2)^2] Cosh[t],
  Integrate[C1/Sqrt[C1^2 + (u + C2)^2],
  {u, 0, s}] + C3,
  Sqrt[C1^2 + (s + C2)^2] Sinh[t]}
ParametricPlot3D[x6[1,1,0][s,t]//Evaluate,
  {s,-4,4},{t,-1,1},PlotPoints->{24,24},
  ViewPoint->{1.300,-2.400,2.000},
  Boxed->False,Axes->None]
```

-Graphics3D-

FIGURE 6.

EXAMPLE 4.12 (Timelike surface with spacelike axis). Set $H(s) = H \neq 0$ in (3.16), (3.22), (3.25) and (3.26). Then we have,

$$X(s,\theta) = \left(\frac{1}{2|H|}\{1 + B^2 + 2B\sin 2Hs\}^{1/2}\cosh\theta, \right.$$
$$\left. \int_0^s \frac{1 + B\sin 2Hs}{\{1 + B^2 + 2B\sin 2Hs\}^{1/2}} ds,\ \frac{1}{2|H|}\{1 + B^2 + 2B\sin 2Hs\}^{1/2}\sinh\theta \right),$$

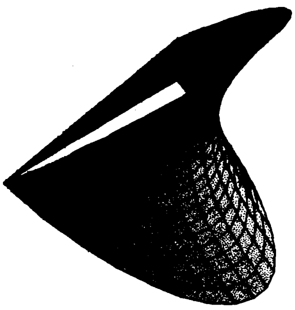
where B is any constant (cf. Figure 7).

If the constant H is zero, by Theorem 3.1 (iii), then we get the following.

EXAMPLE 4.13 (Maximal surface with lightlike axis). Set $\delta = 1$ in (3.27) and (3.29). Then we have

$$X(s,t) = (as^3 + b + s - t^2 s,\ -2st,\ as^3 + b - s - t^2 s), \quad a > 0.$$

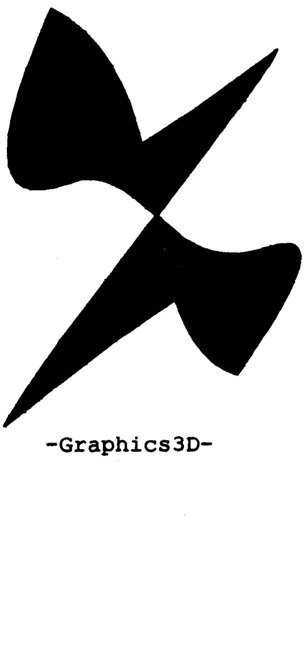
This surface is an Enneper's surface of the 2nd kind (cf. Figure 8). Note that the conjugate of Enneper's surface of the 2nd kind is just, after an affine change of coordinates,



-Graphics3D-

```
x7[H_,B_][s_,t_]:={
  Sqrt[1 + B^2 + 2B Sin[2 H s]]/(2Abs[H]) Cosh[t],
  Integrate[
  (1 + B Sin[2 H u])/Sqrt[1 + B^2 + 2B Sin[2 H u]],
  {u, 0, s}],
  Sqrt[1 + B^2 + 2B Sin[2 H s]]/(2Abs[H]) Sinh[t]}
ParametricPlot3D[x7[1,1][s,t]//Evaluate,
  {s,-2,2},{t,-1,1},PlotPoints->{24,24},
  ViewPoint->{1.300,-2.400,2.000},
  Boxed->False,Axes->None]
```

FIGURE 7.



-Graphics3D-

```
x8[a_,b_][s_,t_]:={a s^3 + b + s- s t^2,-2 s t,
  a s^3 +b - s - s t^2}
ParametricPlot3D[x8[1/3,0][s,t]//Evaluate,
  {s,-1.4,1.4},{t,-1.4,1.4},PlotPoints->{24,24},
  ViewPoint->{1.612,-2.975,0.040},
  Boxed->False,Axes->None]
```

FIGURE 8.

Cayley's ruled surface of 3rd degree studied in affine differential geometry (cf. [7]).

EXAMPLE 4.14 (Timelike minimal surface with lightlike axis). Set $\delta = -1$ in (3.27) and (3.29). Then we have

$$X(s,t) = (-as^3 - b + s - t^2 s,\ -2st,\ -as^3 - b - s - t^2 s)\,, \quad a > 0\,.$$

This surface is an Enneper's surface of the 3rd kind (cf. Figure 9).

If the constant H is non-zero, by Theorem 3.1 (iii), then we have

$$h(s) = \delta\{1/(4H^2 a) - 1/(4H^2 s)\}\,, \tag{4.3}$$

where $H(s) = H \neq 0$, $s_0 = 0$ and $s_1 = a \neq 0$.

Therefore we get the following.

EXAMPLE 4.15 (Spacelike surface with lightlike axis). Set $\delta = 1$ in (3.27) and (4.3). Then we have $h(s) = 1/(4H^2 a) - 1/(4H^2 s)$. Hence,

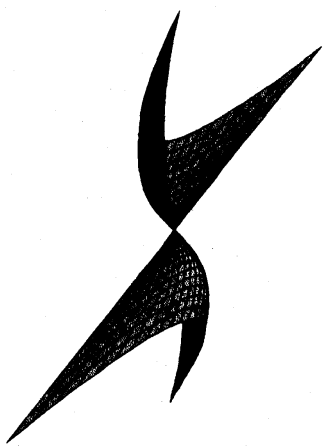
$$X(s,t) = \left(\frac{1}{4H^2 a} - \frac{1}{4H^2 s} + s - t^2 s,\ -2st,\ \frac{1}{4H^2 a} - \frac{1}{4H^2 s} - s - t^2 s\right)\,.$$

This surface is drawn in Figures 10, 11 and 12.

EXAMPLE 4.16 (Timelike surface with lightlike axis). Set $\delta = -1$ in (3.27) and (4.3). Then we have $h(s) = -1/(4H^2 a) + 1/(4H^2 s)$. Hence,

$$X(s,t) = \left(\frac{1}{4H^2 s} - \frac{1}{4H^2 a} + s - t^2 s,\ -2st,\ \frac{1}{4H^2 s} - \frac{1}{4H^2 a} - s - t^2 s\right)\,.$$

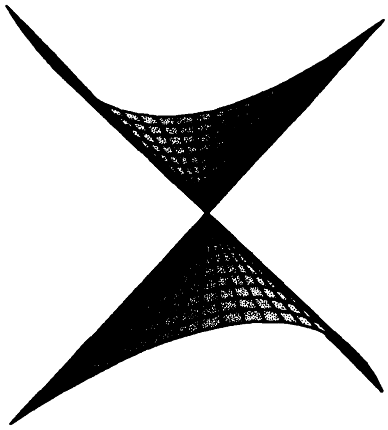
This surface is drawn in Figures 13, 14 and 15.



```
x9[a_,b_][s_,t_]:={-a s^3 - b + s - s t^2,-2 s t,
  -a s^3 -b - s -s t^2}
ParametricPlot3D[x9[1/3,0][s,t]//Evaluate,
  {s,-1.4,1.4},{t,-1.4,1.4},PlotPoints->{24,24},
  ViewPoint->{1.612,-2.975,0.040},
  Boxed->False,Axes->None]
```

```
-Graphics3D-
```

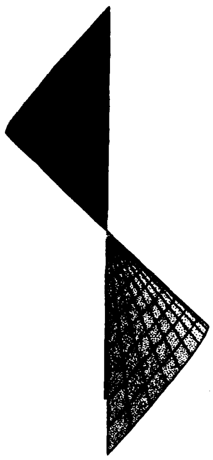
FIGURE 9.



```
x10[H_,a_][s_,t_]:=
  {1/(4 a H^2) - 1/(4 s H^2) + s - s t^2,-2 s t,
  1/(4 a H^2) - 1/(4 s H^2) - s - s t^2}
ParametricPlot3D[x10[-100,1/10][s,t]//Evaluate,
  {s,-1,1},{t,-1,1},PlotPoints->{20,20},
  ViewPoint->{3.047,-1.463,0.159},
  Boxed->False,Axes->None]
```

-Graphics3D-

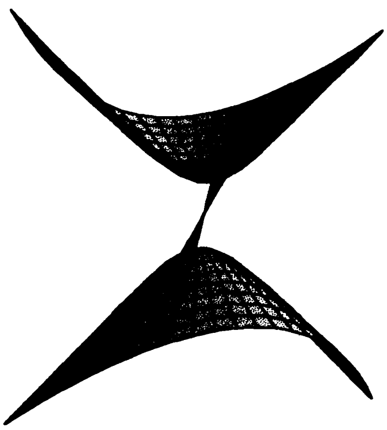
FIGURE 10 (front view).



```
ParametricPlot3D[x10[-100,1/10][s,t]//Evaluate,
  {s,-1,1},{t,-1,1},PlotPoints->{20,20},
  ViewPoint->{-0.027,-3.384,-0.000},
  Boxed->False,Axes->None]
```

-Graphics3D-

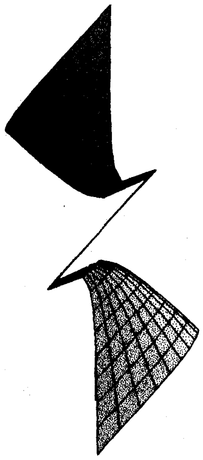
FIGURE 10 (side view).



```
ParametricPlot3D[x10[-3,1/4][s,t]//Evaluate,
  {s,-1,1},{t,-1,1},PlotPoints->{20,20},
  ViewPoint->{3.047,-1.463,0.159},
  Boxed->False,Axes->None]
```

-Graphics3D-

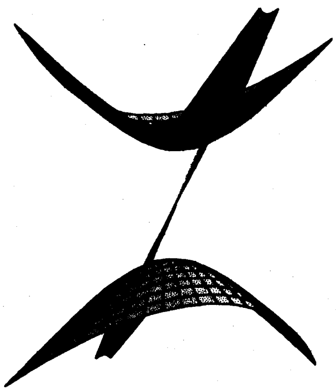
FIGURE 11 (front view).



```
ParametricPlot3D[x10[-3,1/4][s,t]//Evaluate,
  {s,-1,1},{t,-1,1},PlotPoints->{20,20},
  ViewPoint->{-0.027,-3.384,-0.000},
  Boxed->False,Axes->None]
```

-Graphics3D-

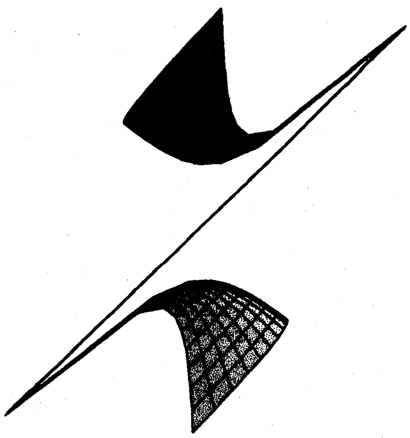
FIGURE 11 (side view).



```
ParametricPlot3D[x10[-3/2,1/4][s,t]//Evaluate,
  {s,-1,1},{t,-1,1},PlotPoints->{20,20},
  ViewPoint->{3.047,-1.463,0.159},
  Boxed->False,Axes->None]
```

-Graphics3D-

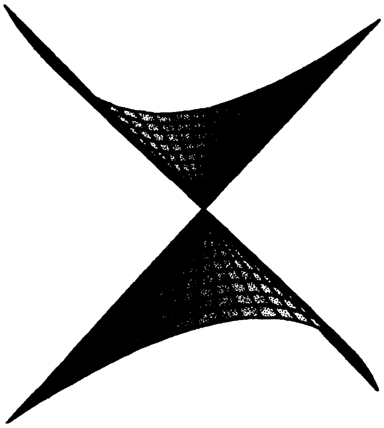
FIGURE 12 (front view).



```
ParametricPlot3D[x10[-3/2,1/4][s,t]//Evaluate,
  {s,-1,1},{t,-1,1},PlotPoints->{20,20},
  ViewPoint->{-0.027,-3.384,-0.000},
  Boxed->False,Axes->None]
```

-Graphics3D-

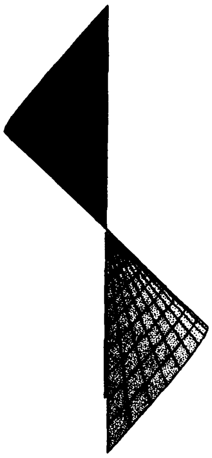
FIGURE 12 (side view).



```
x11[H_,a_][s_,t_]:=
  {1/(4 s H^2) - 1/(4 a H^2) + s - s t^2,-2 s t,
  1/(4 s H^2) - 1/(4 a H^2) - s -s t^2}
ParametricPlot3D[x11[-100,1/10][s,t]//Evaluate,
  {s,-1,1},{t,-1,1},PlotPoints->{20,20},
  ViewPoint->{3.047,-1.463,0.159},
  Boxed->False,Axes->None]
```

-Graphics3D-

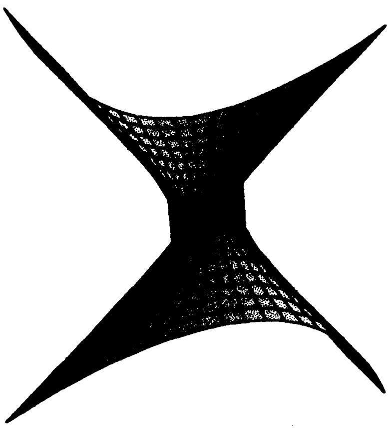
FIGURE 13 (front view).



```
ParametricPlot3D[x11[-100,1/10][s,t]//Evaluate,
  {s,-1,1},{t,-1,1},PlotPoints->{20,20},
  ViewPoint->{-0.027,-3.384,-0.000},
  Boxed->False,Axes->None]
```

-Graphics3D-

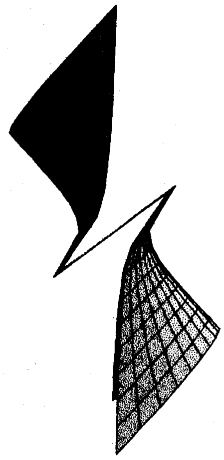
FIGURE 13 (side view).



```
ParametricPlot3D[x11[-3,1/4][s,t]//Evaluate,
  {s,-1,1},{t,-1,1},PlotPoints->{20,20},
  ViewPoint->{3.047,-1.463,0.159},
  Boxed->False,Axes->None]
```

-Graphics3D-

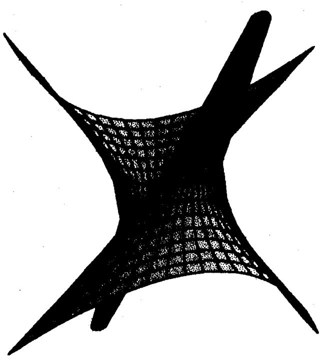
FIGURE 14 (front view).



```
ParametricPlot3D[x11[-3,1/4][s,t]//Evaluate,
  {s,-1,1},{t,-1,1},PlotPoints->{20,20},
  ViewPoint->{-0.027,-3.384,-0.000},
  Boxed->False,Axes->None]
```

-Graphics3D-

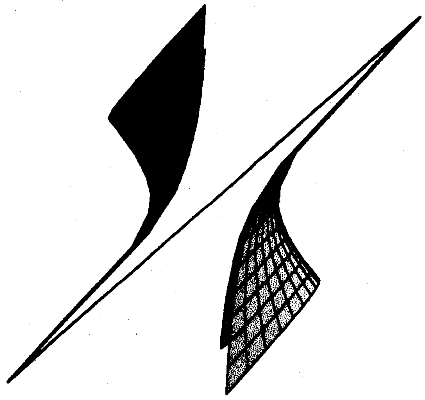
FIGURE 14 (side view).



```
ParametricPlot3D[x11[-3/2,1/4][s,t]//Evaluate,
  {s,-1,1},{t,-1,1},PlotPoints->{20,20},
  ViewPoint->{3.047,-1.463,0.159},
  Boxed->False,Axes->None]
```

-Graphics3D-

FIGURE 15 (front view).



```
ParametricPlot3D[x11[-3/2,1/4][s,t]//Evaluate,
  {s,-1,1},{t,-1,1},PlotPoints->{20,20},
  ViewPoint->{-0.027,-3.384,-0.000},
  Boxed->False,Axes->None]
```

-Graphics3D-

FIGURE 15 (side view).

5. Spacelike helicoidal surfaces in L^3.

In Euclidean 3-space, a *helicoidal motion* is defined as a one-parameter subgroup of the Euclidean motion group which is a rotation together with a translation in the axis of rotation. However, in Minkowski 3-space, it is easy to see that a rotation around lightlike axis together with a translation in the direction of the axis is again a rotation around a lightlike axis. In Minkowski 3-space helicoidal motions are defined as non-trivial one-parameter subgroup of Minkowski motion group. The helicoidal motions are classified and explicitly given as follows [7]:

$$g_t\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\cos t & -\sin t & 0\\ \sin t & \cos t & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}+ht\begin{pmatrix}0\\0\\1\end{pmatrix},\quad t\in\mathbf{R},$$

$$g_t\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\cosh t & 0 & \sinh t\\ 0 & 1 & 0\\ \sinh t & 0 & \cosh t\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}+ht\begin{pmatrix}0\\1\\0\end{pmatrix},\quad t\in\mathbf{R},\qquad\text{or}$$

$$g_t\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1-\frac{t^2}{2} & t & \frac{t^2}{2}\\ -t & 1 & t\\ -\frac{t^2}{2} & t & 1+\frac{t^2}{2}\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}+h\begin{pmatrix}\frac{t^3}{3}-t\\ t^2\\ \frac{t^3}{3}+t\end{pmatrix},\quad t\in\mathbf{R},$$

according as the axis l is timelike, spacelike or lightlike, is called a *helicoidal motion g_t with the axis l and pitch h*. A *helicoidal surface* with the axis l and pitch h is a surface that is invariant by g_t, for all t. When $h=0$, they reduce to surfaces of revolution.

Let $f: M\to L^3$ be a spacelike immersion and let $U\subset M$ be an open set. Assume for the time being that the intersection of the image $f(U)$ with some plane $\Pi\subset L^3$ containing the line $l=z$-axis (resp. y-axis or the direction $(1,0,1)$) is a curve which is a graph $z=\lambda(\rho)$ (resp. $y=\lambda(\rho)$ or $(x+z)/2=\lambda(\rho)$) over the intersection of Π and xy-plane (resp. xz-plane or the lightlike plane spanned by $(1,0,1)$ and $(1,0,-1)$). If f is invariant by a helicoidal motion around the axis l with pitch h, the restriction $f|_U$ can be written as

$$(5.1)_t\qquad f(\rho,\varphi)=\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\rho\\0\\\lambda(\rho)\end{pmatrix}+h\varphi\begin{pmatrix}0\\0\\1\end{pmatrix},$$

where

$$(5.2)_t\qquad 1-\lambda'(\rho)^2>0,\quad \begin{vmatrix}1-\lambda'(\rho)^2 & -h\lambda'(\rho)\\ -h\lambda'(\rho) & \rho^2-h^2\end{vmatrix}>0,\quad \rho^2-h^2>0,$$

$$(5.1)_s\qquad f(\rho,\varphi)=\begin{pmatrix}\cosh\varphi & 0 & \sinh\varphi\\ 0 & 1 & 0\\ \sinh\varphi & 0 & \cosh\varphi\end{pmatrix}\begin{pmatrix}0\\\lambda(\rho)\\\rho\end{pmatrix}+h\varphi\begin{pmatrix}0\\1\\0\end{pmatrix},$$

where

$$(5.2)_s \qquad \lambda'(\rho)^2 - 1 > 0\,, \quad \begin{vmatrix} \lambda'(\rho)^2 - 1 & h\lambda'(\rho) \\ h\lambda'(\rho) & \rho^2 + h^2 \end{vmatrix} > 0\,, \quad \rho^2 + h^2 > 0\,, \qquad \text{or}$$

$$(5.1)_l \qquad f(\rho, \varphi) = \begin{pmatrix} 1 - \frac{\varphi^2}{2} & \varphi & \frac{\varphi^2}{2} \\ -\varphi & 1 & \varphi \\ -\frac{\varphi^2}{2} & \varphi & 1 + \frac{\varphi^2}{2} \end{pmatrix} \begin{pmatrix} \lambda(\rho) + \rho \\ 0 \\ \lambda(\rho) - \rho \end{pmatrix} + h \begin{pmatrix} \frac{\varphi^3}{3} - \varphi \\ \varphi^2 \\ \frac{\varphi^3}{3} + \varphi \end{pmatrix},$$

where

$$(5.2)_l \qquad \lambda'(\rho) > 0\,, \quad \begin{vmatrix} 4\lambda'(\rho) & -2h\lambda'(\rho) \\ -2h\lambda'(\rho) & 4\rho^2 \end{vmatrix} > 0\,, \quad \rho^2 > 0\,,$$

according as the axis l is timelike, spacelike or lightlike.

LEMMA 5.1. *Given a spacelike helicoidal surface of the forms* (5.1) *and* (5.2) *there exists a two-parameter family of spacelike helicoidal surfaces isometric to* (5.1) *and* (5.2).

PROOF. The first fundamental form $d\sigma^2$ of (5.1) can be written as

$$(5.3)_t \quad d\sigma^2 = (1 - \rho^2\lambda'(\rho)^2(\rho^2 - h_0^2)^{-1})d\rho^2 + (\rho^2 - h_0^2)(d\varphi - h_0\lambda'(\rho)(\rho^2 - h_0^2)^{-1}d\rho)^2\,,$$

$$(5.3)_s \quad d\sigma^2 = (-1 + \rho^2\lambda'(\rho)^2(\rho^2 + h_0^2)^{-1})d\rho^2 + (\rho^2 + h_0^2)(d\varphi + h_0\lambda'(\rho)(\rho^2 + h_0^2)^{-1}d\rho)^2\,,$$

or

$$(5.3)_l \quad d\sigma^2 = \lambda'(\rho)(4 - \lambda'(\rho)h_0^2\rho^{-2})d\rho^2 + 4\rho^2(d\varphi - h_0\lambda'(\rho)(2\rho^2)^{-1}d\rho)^2\,,$$

where the prime denotes the derivative in ρ and we have set, for definiteness, $h = h_0$ in (5.1)–(5.2). We introduce new parameters (s, t) in (5.1) by functions $s = s(\rho, \varphi)$, $t = t(\rho, \varphi)$ that satisfy

$$(5.4)_t \qquad \begin{cases} ds = (1 - \rho^2\lambda'(\rho)^2(\rho^2 - h_0^2)^{-1})^{1/2}d\rho \\ dt = d\varphi - h_0\lambda'(\rho)(\rho^2 - h_0^2)^{-1}d\rho\,, \end{cases}$$

$$(5.4)_s \qquad \begin{cases} ds = (-1 + \rho^2\lambda'(\rho)^2(\rho^2 + h_0^2)^{-1})^{1/2}d\rho \\ dt = d\varphi + h\lambda'(\rho)(\rho^2 + h_0^2)^{-1}d\rho\,, \qquad \text{or} \end{cases}$$

$$(5.4)_l \qquad \begin{cases} ds = (\lambda'(\rho)(4 - \lambda'(\rho)h_0^2\rho^{-2}))^{1/2}d\rho \\ dt = d\varphi - h_0\lambda'(\rho)(2\rho^2)^{-1}d\rho\,. \end{cases}$$

Notice that the Jacobian $\partial(s, t)/\partial(\rho, \varphi)$ is nonzero and that (s, t) is a natural parametrization on $U \subset M$. By setting

$$(5.5)_t \qquad U(s) = (\rho^2(s) - h_0^2)^{1/2}\,,$$

$$(5.5)_s \qquad U(s) = (\rho^2(s) + h_0^2)^{1/2}\,, \qquad \text{or}$$

$$(5.5)_l \qquad U(s) = 2\rho(s)\,.$$

we can write, in the natural parametrization, $d\sigma^2 = ds^2 + U^2(s)dt^2$.

We are now reduced to showing that, given a function $U = U(s)$, we can find functions ρ, λ and φ of s and t that satisfy

$$(5.6)_t \qquad ds^2 = d\rho^2 - \rho^2(\rho^2 - h^2)^{-1}d\lambda^2\,,$$

$$(5.7)_t \qquad Udt = \pm(\rho^2 - h^2)^{1/2}(d\varphi - h(\rho^2 - h^2)^{-1}d\lambda)\,,$$

$$(5.6)_s \qquad ds^2 = -d\rho^2 + \rho^2(\rho^2 + h^2)^{-1}d\lambda^2\,,$$

$$(5.7)_s \qquad Udt = \pm(\rho^2 + h^2)^{1/2}(d\varphi + h(\rho^2 + h^2)^{-1}d\lambda)\,, \qquad \text{or}$$

$$(5.6)_l \qquad ds^2 = (4d\rho - h^2\rho^{-2}d\lambda)d\lambda\,,$$

$$(5.7)_l \qquad Udt = \pm 2\rho(d\varphi - h(2\rho^2)^{-1}d\lambda)\,,$$

for an arbitrary constant h.

We first observe, from (5.6), that ρ and λ do not depend on t. Then, from (5.7), we obtain

$$(5.8)_t \qquad \begin{cases} \dfrac{\partial\varphi}{\partial s} = \dfrac{h}{\rho^2 - h^2}\dfrac{d\lambda}{ds} \\ \dfrac{\partial\varphi}{\partial t} = \pm U(\rho^2 - h^2)^{-1/2} = \pm 1/m\,, \end{cases}$$

$$(5.8)_s \qquad \begin{cases} \dfrac{\partial\varphi}{\partial s} = -\dfrac{h}{\rho^2 + h^2}\dfrac{d\lambda}{ds} \\ \dfrac{\partial\varphi}{\partial t} = \pm U(\rho^2 + h^2)^{-1/2} = \pm 1/m\,, \end{cases} \qquad \text{or}$$

$$(5.8)_l \qquad \begin{cases} \dfrac{\partial\varphi}{\partial s} = \dfrac{h}{2\rho^2}\dfrac{d\lambda}{ds} \\ \dfrac{\partial\varphi}{\partial t} = \pm U(2\rho)^{-1} = \pm 1/m\,, \end{cases}$$

where m is a constant.

Therefore we can write (5.7) as

$$(5.9)_t \qquad d\varphi = \pm m^{-1}dt + h(mU)^{-2}d\lambda\,,$$

$$(5.9)_s \qquad d\varphi = \pm m^{-1}dt - h(mU)^{-2}d\lambda\,, \qquad \text{or}$$

$$(5.9)_l \qquad d\varphi = \pm m^{-1}dt + 2h(mU)^{-2}d\lambda\,.$$

Now, from (5.8), it follows that

$$(5.10)_t \qquad \dot\rho = m^2U\dot U(m^2U^2 + h^2)^{-1/2}\,,$$

$$(5.10)_s \qquad \dot\rho = m^2U\dot U(m^2U^2 - h^2)^{-1/2}\,, \qquad \text{or}$$

$$(5.10)_l \qquad \dot\rho = \frac{1}{2}m\dot U\,,$$

where dot denotes the derivative in s.

From (5.6) and (5.10) we obtain

$$(5.11)_t \qquad d\lambda = \pm(m^2U^2(m^2\dot U^2 - 1) - h^2)^{1/2}(m^2U^2 + h^2)^{-1}mUds\,,$$

$$(5.11)_s \qquad d\lambda = \pm(m^2U^2(m^2\dot{U}^2+1)-h^2)^{1/2}(m^2U^2-h^2)^{-1}mU\,ds\,, \qquad \text{or}$$

$$(5.11)_l \qquad d\lambda = \frac{m^3U^2\dot{U}^2 + 2mU\{m^4U^2\dot{U}^2/4-h^2\}^{1/2}}{4h^2}ds\,, \quad m^4U^2\dot{U}^2/4-h^2 \ge 0\,.$$

It follows from (5.8), (5.10) and (5.11) that the spacelike helicoidal surface (5.1), where ρ, φ and λ are given by

$$(5.12)_t \qquad \begin{cases} \rho = (m^2U^2+h^2)^{1/2}\,, \\ \varphi = \pm\dfrac{1}{m}\displaystyle\int dt \pm h\int \frac{(m^2U^2(m^2\dot{U}^2-1)-h^2)^{1/2}}{mU(m^2U^2+h^2)}ds\,, \\ \lambda = \pm\displaystyle\int \frac{(m^2U^2(m^2\dot{U}^2-1)-h^2)^{1/2}}{m^2U^2+h^2}mU\,ds\,, \end{cases}$$

$$(5.12)_s \qquad \begin{cases} \rho = (m^2U^2-h^2)^{1/2}\,, \\ \varphi = \pm\dfrac{1}{m}\displaystyle\int dt \mp h\int \frac{(m^2U^2(m^2\dot{U}^2+1)-h^2)^{1/2}}{mU(m^2U^2-h^2)}ds\,, \\ \lambda = \pm\displaystyle\int \frac{(m^2U^2(m^2\dot{U}^2+1)-h^2)^{1/2}}{m^2U^2-h^2}mU\,ds\,, \end{cases} \qquad \text{or}$$

$$(5.12)_l \qquad \begin{cases} \rho = \dfrac{1}{2}mU\,, \\ \varphi = \pm\dfrac{1}{m}\displaystyle\int dt + \int \frac{m^2U\dot{U}^2 \pm 2(m^4U^2\dot{U}^2/4-h^2)^{1/2}}{2mh}ds\,, \\ \lambda = \displaystyle\int \frac{m^3U^2\dot{U}^2 \pm 2mU\{m^4U^2\dot{U}^2/4-h^2\}^{1/2}}{4h^2}ds\,, \end{cases}$$

are all isometric with the first fundamental form given by $d\sigma^2 = ds^2 + U^2dt^2$. Thus there are essentially two parameters in the family described by (5.12). This proves Lemma 5.1.

REMARK 5.1. The family (5.12) contains the surface we started with for $m = 1$, $h = h_0$. In particular, Lemma 5.1 asserts the existence of a two-parameter family of spacelike helicoidal surfaces isometric to a given spacelike surface of revolution ($m = 1, h = 0$). Thus we have proved that the associated family of a spacelike surface of revolution with constant mean curvature is made up by spacelike helicoidal surfaces with constant mean curvature.

6. Spacelike helicoidal surfaces with constant mean curvature in L^3.

A spacelike surface of family (5.12) is determined by giving a function $U(s)$ and constant h. For convenience, let us denote it by $[U, m, h]$. In this section we shall study spacelike helicoidal surface $[U, m, h]$ of constant mean curvature with non lightlike axis.

LEMMA 6.1 *A spacelike helicoidal surface $[U, m, h]$ with non lightlike axis has constant mean curvature H if and only if $U(s)$ satisfies the equation*

$$(6.1)_t \qquad m^2U\ddot{U} + m^2\dot{U}^2 - 1 = 2H(m^2U^2(m^2\dot{U}^2-1)-h^2)^{1/2}\,, \qquad \text{or}$$

$(6.1)_s$ $$-m^2U\ddot{U} - m^2\dot{U}^2 - 1 = 2H(m^2U^2(m^2\dot{U}^2 + 1) - h^2)^{1/2}.$$

PROOF. We choose an orthonormal frame $\{e_1, e_2, e_3\}$ by setting $e_1 = f_s$, $e_2 = f_t/U$ and $e_3 = e_1 \times e_2$. Note that $(e_1, e_2, e_3) = 1$.

We obtain from (5.1) and (5.8) that

$(6.2)_t$ $$\begin{cases} (f_{tt}, f_t, f_s) = \mp\rho^2\dot{\lambda}m^{-3} \\ (f_{ts}, f_t, f_s) = (h\dot{\rho}^2 - \dot{\varphi}\lambda\rho^2)m^{-2} = hm^{-2}, \end{cases} \quad \text{or}$$

$(6.2)_s$ $$\begin{cases} (f_{tt}, f_t, f_s) = \pm\rho^2\dot{\lambda}m^{-3} \\ (f_{ts}, f_t, f_s) = (h\dot{\rho}^2 + \dot{\varphi}\lambda\rho^2)m^{-2} = -hm^{-2}, \end{cases}$$

where $(\cdot, \cdot, \cdot)$ denotes the determinant of the enclosed vectors and dots denote derivatives in s. It follows that

$(6.3)_t$ $$\begin{cases} h_{22} = \pm\rho^2\dot{\lambda}m^{-3}U^{-3} \\ h_{12} = h_{21} = -hm^{-2}U^{-2}, \end{cases} \quad \text{or}$$

$(6.3)_s$ $$\begin{cases} h_{22} = \mp\rho^2\dot{\lambda}m^{-3}U^{-3} \\ h_{12} = h_{21} = hm^{-2}U^{-2}. \end{cases}$$

Furthermore, since the Gaussian curvature K is easily seen to be $K = -\ddot{U}/U$, we obtain that

$(6.4)_t$ $$h_{11} = \pm(h^2m^{-1}U^{-1} + m^3U^2\ddot{U})(\rho^2\dot{\lambda})^{-1}, \quad \text{or}$$

$(6.4)_s$ $$h_{11} = \mp(h^2m^{-1}U^{-1} + m^3U^2\ddot{U})(\rho^2\dot{\lambda})^{-1}.$$

Finally, since $2H = h_{11} + h_{22}$, we obtain the equation (6.1), and this proves Lemma 6.1.

The equation (6.1) is easily integrated if we make the changes of variables

$(6.5)_t$ $$\begin{cases} x = mU \\ y = (x^2(\dot{x}^2 - 1) - h^2)^{1/2}, \end{cases} \quad \text{or}$$

$(6.5)_s$ $$\begin{cases} x = mU \\ y = (x^2(\dot{x}^2 + 1) - h^2)^{1/2}. \end{cases}$$

Then (6.1) becomes

$(6.6)_t$ $$\dot{y} = 2Hx\dot{x}, \quad \text{or}$$

$(6.6)_s$ $$\dot{y} = -2Hx\dot{x},$$

an integral of which is

$(6.7)_t$ $$y = Hx^2 + a, \quad \text{or}$$

$(6.7)_s$ $$y = -Hx^2 + a,$$

where a is a constant.

By assuming $H \neq 0$ and returning to the variable x, we obtain

$$(6.8)_t \qquad \dot{x}^2 = ((Hx^2+a)^2 + x^2 + h^2)/x^2\,, \qquad \text{or}$$

$$(6.8)_s \qquad \dot{x}^2 = ((Hx^2-a)^2 - x^2 + h^2)/x^2\,,$$

which can be integrated by $z = x^2$ to transform it into

$$(6.9)_t \qquad \left\{\left(Hz + \frac{1+2Ha}{2H}\right)^2 - \frac{(1+2Ha)^2}{4H^2} + a^2 + h^2\right\}^{-1/2} dz = 2ds\,, \qquad \text{or}$$

$$(6.9)_s \qquad \left\{\left(Hz - \frac{2Ha+1}{2H}\right)^2 - \frac{(2Ha+1)^2}{4H^2} + a^2 + h^2\right\}^{-1/2} dz = 2ds\,.$$

Since $z = m^2U^2$ we finally arrive at

$$(6.10)_t \qquad U(s)^2 = \pm\frac{\exp(2H(2s+b)) + (2Ha+1)^2/(4H^2) - a^2 - h^2}{2Hm^2\exp(H(2s+b))} - \frac{2Ha+1}{2H^2m^2}\,, \qquad \text{or}$$

$$(6.10)_s \qquad U(s)^2 = \pm\frac{\exp(2H(2s+b)) + (2Ha+1)^2/(4H^2) - a^2 - h^2}{2Hm^2\exp(H(2s+b))} + \frac{2Ha+1}{2H^2m^2}\,,$$

where b is a constant, that yields a two-parameter family of functions U_{ab} such that $[U_{ab}, m, h]$ is a spacelike helicoidal surface with constant mean curvature H.

Therefore we have proved the following.

THEOREM 6.1 (spacelike helicoidal surfaces in L^3 with none lightlike axis). *The spacelike helicoidal surfaces with non lightlike axis of the forms* (5.1) *and* (5.2) *that have constant mean curvature* $H \neq 0$ *constitute a four-parameter family with parameters* a, b, m *and* h *and are given in a natural parametrization if we replace* ρ, φ *and* λ *in* (5.1) *by their values obtained by* (5.12) *and* (6.10).

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