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Decompositions of Measures on Compact Abelian Groups

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Abstract. It is shown that the set of finite regular Borel measures with natural spectra for a compact abelian group $\mathfrak{G}$ is closed under addition if and only if $\mathfrak{G}$ is discrete. If G is a non-discrete locally compact abelian group, then there exists a finite regular Borel measure with natural spectrum such that the corresponding multiplication operator on $L^1(G)$ is not decomposable.

Let G be a locally compact abelian group and $\hat{G}$ the dual group of G. We denote by $M(G)$ the measure algebra of all bounded regular Borel measures on G. The subalgebra $M_0(G)$ consists of measures $\mu \in M(G)$ whose Fourier-Stieltjes transforms $\hat{\mu}$ vanishes at infinity on $\hat{G}$. We say that $\mu \in M(G)$ has a natural spectrum if the spectrum $\mathrm{sp}(\mu)$ coincides with the closure $\overline{\hat{\mu}(\hat{G})}$ of range of $\hat{\mu}$. The set of $\mu \in M(G)$ with a natural spectrum is denoted by $NS(G)$. Williamson [12] proved that $NS(G)$ is a proper subset of $M(G)$ if G is non-discrete. Rudin [9] and Valopoulos [11] proved that $NS(G) \cap M_0(G)$ is a proper subset of $M_0(G)$ for $G = \mathbf{R}$ and an arbitrary non-discrete G, respectively. Let $M_{00}(G)$ be the radical of $L^1(G)$, that is, $M_{00}(G)$ consists of those $\mu \in M(G)$ whose Gelfand transfrom vanishes on $\Phi_{M(G)} \setminus \hat{G}$, where $\Phi_{M(G)}$ denotes the maximal ideal space of $M(G)$. Thus we see that $M_{00}(G) \subset NS(G) \cap M_0(G)$. Let $M_d(G)$ be the subalgebra of $M(G)$ which consists of disdrete measures in $M(G)$. Let $DM(G)$ be the set of all $\mu \in M(G)$ such that the corresponding multiplier T_μ defined on $L^1(G)$ by $T_\mu f = f * \mu$ is decomposable. Given a Banach space X, a bounded linear operator T on X is called *decomposable* if for every open covering $\{U, V\}$ of the complex plane $\mathbf{C}$, there exist T-invariant closed linear subspaces X_U and X_V of X such that $\sigma(T|X_U) \subset U$, $\sigma(T \mid X_V) \subset V$ and $X_U + X_V = X$, where $\sigma(\cdot)$ denotes the spectrum of an operator. Albrecht [1, Theorem 3.1] proved that $DM(G)$ is a closed subalgebra of $M(G)$ which contains $M_{00}(G)$ and $M_d(G)$ (cf. [7, Theorem 2.5]). Zafran [13, Example 3.2] showed that on an I-group G there exist measures $\mu, \nu \in NS(G)$ such that $\mu + \nu \in M(G) \setminus NS(G)$. We call G an I-group if every neighborhood of 0 contains an element of infinite order. Thus we see that $NS(G)$ is not closed under addition if G is an I-group. As is pointed out by Albrecht [1], at least one of T_μ and T_ν is not decomposable.

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On the other hand, Laursen and Neumann [7, Theorem 4.5] proved, as a generalization of the results in [1] and [13] that

$$NS(\mathfrak{G}) \cap M_0(\mathfrak{G}) = M_{00}(\mathfrak{G}) = DM(\mathfrak{G}) \cap M_0(\mathfrak{G})$$

if $\mathfrak{G}$ is a *compact* abelian group. Thus $NS(\mathfrak{G}) \cap M_0(\mathfrak{G})$ is closed under addition. In particular, decomposability of corresponding multiplier on $L^1(\mathfrak{G})$ characterizes measures in $M_0(\mathfrak{G})$ with natural spectra. In [4] we proved that if G is a non-compact locally compact abelian group, then

$$NS(G) + L^1(G) = M(G)$$

and

$$NS(G) \cap M_0(G) + L^1(G) = M_0(G),$$

so that $NS(G)$ and $NS(G) \cap M_0(G)$ are not closed under addition if G is non-discrete and non-compact. It follows that there exists a measure $\mu \in NS(G) \cap M_0(G)$ of which corresponding multiplier T_μ is not decomposable, which is not the case for compact abelian groups.

The natural question occurs: for a non-discrete compact abelian group $\mathfrak{G}$, is the set $NS(\mathfrak{G})$ closed under addition?

First of all we claim that $NS(\mathfrak{G}) + L^1(\mathfrak{G})$ is a proper subset of $M(\mathfrak{G})$ for non-discrete compact abelian groups $\mathfrak{G}$. Suppose that $NS(\mathfrak{G}) + L^1(\mathfrak{G}) = M(\mathfrak{G})$. Let μ be an independent power Hermitian probability measure in $M(\mathfrak{G})$. Such a measure exists by [10, Theorems 5.2.6, 5.3.2]. Then by a theorem of Bailey, Broun and Moran [2, Theorem 1] we have $\mathrm{sp}(\mu) = \{z \in \mathbf{C} : |z| \leq 1\}$ (cf [14, Lemma 1.4]). Then there exist $\nu \in NS(\mathfrak{G})$ and $f \in L^1(\mathfrak{G})$ such that $\nu = \mu + f$. Since $\hat{\mathfrak{G}}$ is discrete, a closed set $\overline{\hat{f}(\hat{\mathfrak{G}})}$ is at most countable, where each point except 0 is isolated, and $\hat{\mu}(\hat{\mathfrak{G}}) \subset \mathbf{R}$, we see that the imarginary part $\mathfrak{I}\hat{\nu}(\hat{\mathfrak{G}})$ is at most countable. Let $\Phi_{M(\mathfrak{G})}$ be a maximal ideal space of $M(\mathfrak{G})$ and $\check{\sigma}$ denote the Gelfand transform of $\sigma \in M(\mathfrak{G})$. Since $\mathrm{sp}(\mu) = \check{\mu}(\Phi_{M(\mathfrak{G})})$ and $\hat{\mathfrak{G}} \subset \Phi_{M(\mathfrak{G})}$, we have

$$\{z \in \mathbf{C} : |z| \leq 1, \mathfrak{I}z \neq 0\} \subset \hat{\mu}(\Phi_{M(\mathfrak{G})} \setminus \hat{\mathfrak{G}}).$$

We also have $\underline{\check{f}(\Phi_{M(\mathfrak{G})} \setminus \hat{\mathfrak{G}})} = \{0\}$. Therefore every real number x with $0 \underline{<} |x| \leq 1$ is contained in $\mathfrak{I}\hat{\nu}(\Phi_{M(\mathfrak{G})} \setminus \hat{\mathfrak{G}})$, which is a contradiction sinse $\nu \in NS(G)$ and $\mathfrak{I}\hat{\nu}(\hat{\mathfrak{G}})$ is at most countable. Thus we see that $NS(\mathfrak{G}) + L^1(\mathfrak{G})$ is a proper subset of $M(\mathfrak{G})$.

Next we show that the equaliy

$$NS(\mathfrak{G}) + NS(\mathfrak{G}) + M_d(\mathfrak{G}) = M(\mathfrak{G})$$

for every compact abelian group $\mathfrak{G}$. It follows by this equation and a theorem of Williamson that $NS(\mathfrak{G})$ is not closed under addition for every non-discrete compact abelian group $\mathfrak{G}$.

THEOREM 1. *Let $\mathfrak{G}$ be a compact abelian group. Then we have $M(\mathfrak{G}) = NS(\mathfrak{G}) + NS(\mathfrak{G}) + M_d(\mathfrak{G})$.*

PROOF. Suppose that $\mathfrak{G}$ is discrete. Then $M(\mathfrak{G}) = L^1(\mathfrak{G})$, so $M(\mathfrak{G}) = NS(\mathfrak{G})$ and the conclusion holds. We shall give a proof for non-discrete $\mathfrak{G}$.

We denote by $\mathfrak{G}_d$ the group $\mathfrak{G}$ with the discrete topology. Then the dual group $\widehat{\mathfrak{G}_d}$ is the Bohr compactification of $\hat{\mathfrak{G}}$. Let

$$\beta : \hat{\mathfrak{G}} \to \widehat{\mathfrak{G}_d}$$

be defined by

$$\beta(\gamma)(x) = \gamma(x)\,, \quad \gamma \in \hat{\mathfrak{G}}\,, \quad x \in \mathfrak{G}_d\,.$$

Then β is a continuous isomorphism and $\beta(\hat{\mathfrak{G}})$ is dense in $\widehat{\mathfrak{G}_d}$ [10, Section 1.8]. The Fourier transform of $g \in L^1(\mathfrak{G}_d)$ is denoted by $\tilde{g}$. For $g \in L^1(\mathfrak{G}_d)$ and a Borel set E, put

$$\mu_g(E) = \int \chi_E(x) g(x) dx\,,$$

where dx denotes the normalized Harr measure on $\mathfrak{G}_d$. Then we have $\mu_g(E) = \sum_{x \in E} g(x)$, and $\mu_g \in M_d(\mathfrak{G})$. Hence we have that

$$\widehat{\mu_g}(\gamma) = \sum_{x \in \mathfrak{G}} \gamma(-x) g(x)$$

for every $\gamma \in \hat{\mathfrak{G}}$. We also have that

$$\tilde{g}(\beta(\gamma)) = \int \beta(\gamma)(-x) g(x) dx = \sum \beta(\gamma)(-x) g(x)\,,$$

henceforce

$$\widehat{\mu_g}(\gamma) = \tilde{g}(\beta(\gamma))$$

for every $\gamma \in \hat{\mathfrak{G}}$. Let U_0 and U_1 be a pair of non-empty open sets with disjoint closures of $\widehat{\mathfrak{G}_d}$. Since $L^1(\mathfrak{G}_d)$ is a regular Banach algebra and since $\widehat{\mathfrak{G}_d}$ is compact, there exists $f \in L^1(\mathfrak{G}_d)$ such that

$$\tilde{f}(\rho) = \begin{cases} 0, & \rho \in \overline{U_0} \\ 1, & \rho \in \overline{U_1}\,. \end{cases}$$

Since $\mathfrak{G}$ is non-discrete, $\mathfrak{G}_d$ is infinite, hence $\widehat{\mathfrak{G}_d}$ is non-discrete. By [5, Theorem 41.5, Theorem 41.13] there exists a Helson set $K_0 \subset U_0$ (resp. $K_1 \subset U_1$) which is homeomorphic with Cantor's ternary set H. Let π_0 (resp. π_1) be a homeomorphism from K_0 (resp. K_1) onto H. Let c be the restriction to H of Cantor's function defined on the unit interval I. Then $c(H) = I$. Let p be a continuous function defined on I onto the closed unit disk $\Delta = \{z \in \mathbf{C} : |z| \leq 1\}$. Then $p \circ c \circ \pi_0$ (resp. $p \circ c \circ \pi_1$) is a continuous function on K_0 (resp. K_1). Since K_0 (resp. K_1) is a Helson set and since $L^1(\mathfrak{G}_d)$ is regular, there exists $g_0 \in L^1(\mathfrak{G}_d)$ (resp. $g_1 \in L^1(\mathfrak{G}_d)$) such that $\tilde{g}_0(K_0) = \Delta$ (resp. $\tilde{g}_1(K_1) = \Delta$) and $\tilde{g}_0 = 0$ (resp. $\tilde{g}_1 = 0$) on $\widehat{\mathfrak{G}_d} \setminus U_0$ (resp. $\widehat{\mathfrak{G}_d} \setminus U_1$).

Let $\mu \in M(\mathfrak{G})$. Put $\mu_0 = \mu * \mu_f$ and $\mu_1 = \mu - \mu_0$. We denote the spectral radius of μ_0 (resp. μ_1) by r_0 (resp. r_1). Put $\nu_0 = \mu_0 + r_0 \mu_{g_0}$, $\nu_1 = \mu_1 + r_1 \mu_{g_1}$ and $\nu_2 = -r_0 \mu_{g_0} - r_1 \mu_{g_1}$. Then we have a decomposition of $\mu : \mu = \nu_0 + \nu_1 + \nu_2$ and $\nu_2 \in M_d(\mathfrak{G})$. We show that $\nu_0 \in NS(\mathfrak{G})$. In the same way we see that $\nu_1 \in NS(\mathfrak{G})$.

Since $\beta(\beta^{-1}(\overline{U_0}))$ is dense in $\overline{U_0}$ and $\tilde{g}_0$ is continuous on $\widehat{\mathfrak{G}_d}$, we see that

$$\begin{aligned}\{z \in \mathbf{C} : |z| \le r_0\} &= r_0\tilde{g}_0(K_0)\\ &\subset r_0\tilde{g}_0(\overline{U_0}) \subset \overline{r_0\tilde{g}_0(\beta(\beta^{-1}(\overline{U_0})))} \subset \overline{r_0\widehat{\mu_{g_0}}(\hat{\mathfrak{G}})}\,.\end{aligned}$$

We have $(\mu_f * \mu_{g_0})\hat{}(\gamma) = 0$ for every $\gamma \in \hat{\mathfrak{G}}$, since

$$(\mu_f * \mu_{g_0})\hat{}(\gamma) = \tilde{f}(\beta(\gamma))\tilde{g}_0(\beta(\gamma))$$

and $\tilde{f}(\overline{U_0}) = 0$ and $\tilde{g}_0(U_0^c) = 0$. Henceforce $\mu_0 * \mu_{g_0} = 0$ since $\mu_0 = \mu * \mu_f$. We also see that $\widehat{\mu_0} = 0$ on $\beta^{-1}(\overline{U_0})$. Thus we have

$$r_0\tilde{g}_0(\beta(\beta^{-1}(\overline{U_0}))) = r_0\widehat{\mu_{g_0}}(\beta^{-1}(\overline{U_0})) = \widehat{\nu_0}(\beta^{-1}(\overline{U_0})) \subset \widehat{\nu_0}(\hat{\mathfrak{G}})\,,$$

so that

$$0 \in \overline{\widehat{\nu_0}(\hat{\mathfrak{G}})}\,.$$

Since $\mu_0 * \mu_{g_0} = 0$ we see that

$$\widehat{\nu_0}(\hat{\mathfrak{G}}) \subset \widehat{\mu_0}(\hat{\mathfrak{G}}) \cup (r_0\widehat{\mu_{g_0}}(\hat{\mathfrak{G}}))\,.$$

Since r_0 is the spectral radius of μ_0, we have

$$\widehat{\mu_0}(\hat{\mathfrak{G}}) \subset \{z \in \mathbf{C} : |z| \le r_0\}\,.$$

Henceforce

$$\widehat{\nu_0}(\hat{\mathfrak{G}}) \subset r_0\widehat{\mu_{g_0}}(\hat{\mathfrak{G}})\,.$$

Suppose that $\gamma \in \hat{\mathfrak{G}}$. If $\widehat{\mu_{g_0}}(\gamma) = 0$, then $r_0\widehat{\mu_{g_0}}(\gamma) = 0 \in \overline{\widehat{\nu_0}(\hat{\mathfrak{G}})}$. If $\widehat{\mu_{g_0}}(\gamma) \ne 0$, then $\widehat{\mu_0}(\gamma) = 0$ since $\mu_0 * \mu_{g_0} = 0$. Thus $r_0\widehat{\mu_{g_0}}(\gamma) = \widehat{\nu_0}(\gamma)$, therefore we see that

$$r_0\widehat{\mu_{g_0}}(\hat{\mathfrak{G}}) \subset \overline{\widehat{\nu_0}(\hat{\mathfrak{G}})}\,.$$

It follows that

$$\overline{\widehat{\nu_0}(\hat{\mathfrak{G}})} = \overline{r_0\widehat{\mu_{g_0}}(\hat{\mathfrak{G}})}\,.$$

Let $\Phi_{M(\mathfrak{G})}$ be the maximal ideal space of $M(\mathfrak{G})$. We denote the Gelfand transform of $\nu \in M(\mathfrak{G})$ by $\check{\nu}$. We may suppose that $\hat{\mathfrak{G}}$ is a subset of $\Phi_{M(\mathfrak{G})}$ and $\check{\nu} = \hat{\nu}$ on $\hat{\mathfrak{G}}$. Since $\overline{\widehat{\nu_0}(\hat{\mathfrak{G}})} \subset \check{\nu_0}(\Phi_{M(\mathfrak{G})})$, we have $0 \in \check{\nu_0}(\Phi_{M(\mathfrak{G})})$. Since $\mu_0 * \mu_{g_0} = 0$, we have that $\check{\mu_0}(p) = 0$ or $r_0\check{\mu}_{g_0}(p) = 0$ for every $p \in \Phi_{M(\mathfrak{G})}$, so

$$\check{\nu_0}(\Phi_{M(\mathfrak{G})}) \subset \check{\mu_0}(\Phi_{M(\mathfrak{G})}) \cup (r_0\check{\mu}_{g_0}(\Phi_{M(\mathfrak{G})}))\,.$$

Since

$$\check{\mu_0}(\Phi_{M(\mathfrak{G})}) \subset \{z \in \mathbf{C} : |z| \le r_0\} \subset \overline{r_0\widehat{\mu_{g_0}}(\hat{\mathfrak{G}})} \subset r_0\check{\mu}_{g_0}(\Phi_{M(\mathfrak{G})})$$

we see that

$$\check{\nu_0}(\Phi_{M(\mathfrak{G})}) \subset r_0\check{\mu}_{g_0}(\Phi_{M(\mathfrak{G})})\,.$$

Suppose that $p \in \Phi_{M(\mathfrak{G})}$. If $r_0\check{\mu}_{g_0}(p) = 0$, then $r_0\check{\mu}_{g_0}(p) = 0 \in \check{\nu_0}(\Phi_{M(\mathfrak{G})})$. If $r_0\check{\mu}_{g_0}(p) \ne 0$, then $\check{\mu_0}(p) = 0$. Thus

$$r_0\check{\mu}_{g_0}(p) = \check{\nu_0}(p) \in \check{\nu_0}(\Phi_{M(\mathfrak{G})})\,.$$

Therefore we have that

$$\check{\nu}_0(\Phi_{M(\mathfrak{G})}) = r_0\check{\mu}_{g_0}(\Phi_{M(\mathfrak{G})}) .$$

Since $\mu_{g_0} \in M_d(\mathfrak{G})$ and $M_d(\mathfrak{G}) \subset NS(\mathfrak{G})$ we see that

$$\overline{r_0\widehat{\mu_{g_0}}(\hat{\mathfrak{G}})} = r_0\check{\mu}_{g_0}(\Phi_{M(\mathfrak{G})}) .$$

It follows that

$$\overline{\widehat{\nu_0}(\hat{\mathfrak{G}})} = \check{\nu}_0(\Phi_{M(\mathfrak{G})}) ,$$

that is, $\nu_0 \in NS(\mathfrak{G})$. □

Note that a slight stronger version of Theorem 1 holds. Let $\mathfrak{G}_S$ be a locally compact abelian group induced by $\mathfrak{G}$ with a stronger topology than the original one. Then we may suppose that $L^1(\mathfrak{G}_S) \subset M(\mathfrak{G})$ and the dual group $\widehat{\mathfrak{G}_S}$ is contained in the Bohr compactification of $\hat{\mathfrak{G}}$ (cf. [6, p. 84]). Then, in a way similar to the above, we have that $M(\mathfrak{G}) = NS(\mathfrak{G}) + NS(\mathfrak{G}) + L^1(\mathfrak{G}_S)$. Theorem 1 corresponds to the case where $\mathfrak{G}_S$ is the discrete group.

COROLLARY 2. *Let G be a non-discrete locally compact abelian group. Then $NS(G)$ is not closed under addition.*

PROOF. If G is not compact, then by [4, Theorem 1] we see that

$$NS(G) + NS(G) = M(G)$$

since $L^1(G) \subset NS(G)$. If G is compact, then by Theorem 1 we see that

$$NS(G) + NS(G) + NS(G) = M(G)$$

since $M_d(G) \subset NS(G)$. It follows by a theorem of Williamson that $NS(G)$ is not closed under addition. □

COROLLARY 3. *Let G be a non-discrete locally compact abelian group. Then there exists a measure $\mu \in NS(G)$ such that the corresponding multiplier on $L^1(G)$ is not decomposable. Furthermore, if G is not compact, then we can choose such a measure μ in $M_0(G)$.*

PROOF. $DM(G)$ is a subset of $NS(G)$ and is closed under addtion by a theorem of Albrecht [1, Thorem 3.1]. It follows by Corollary 2 that $DM(G)$ is a proper subset of $NS(G)$. Unless G is compact, then a set $NS(G) \cap M_0(G)$ is not closed under addition [4, Corollary 3], henceforce $DM(G) \cap M_0$ is a proper subset of $NS(G) \cap M_0(G)$. □

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