TOKYO J. MATH.
VOL. 24, NO. 2, 2001

On the Hyers-Ulam Stability of Real Continuous Function Valued Differentiable Map

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(Communicated by K. Kobayasi)

Abstract. We consider a differentiable map f from an open interval to a real Banach space of all bounded continuous real-valued functions on a topological space. We show that f can be approximated by the solution to the differential equation $x'(t) = \lambda x(t)$, if $\|f'(t) - \lambda f(t)\|_\infty \leq \varepsilon$ holds

1. Introduction.

In this paper, I denotes an open interval of $\mathbf{R}$, the real number field. We consider not only bounded intervals but also unbounded one. That is,

$$I = (a, b), \quad -\infty \leq a < b \leq \infty .$$

The letters ε and λ stand non-negative real number and non-zero real number, respectively. We define $J = \{e^{-\lambda t} : t \in I\}$.

DEFINITION 1.1. *Let A be a Banach space, f a map from I to A. We say that f is differentiable, if for every $t \in I$ there exists an $f'(t) \in A$ such that*

$$\lim_{s \to 0} \left\| \frac{f(t+s) - f(t)}{s} - f'(t) \right\|_A = 0 ,$$

where $\|\cdot\|_A$ denotes the norm on A. We call the map $f' : I \to A$ the derivative of f.

By definition, f is differentiable if and only if f is Fréchet differentiable at each point of I. While Fréchet derivative and our one differ from each other at first glance, we can identify them since $L_t(1) = f'(t)$ holds, where L_t denotes the Fréchet derivative for f at t.

Alsina and Ger [1] proved the following results in case where $\lambda = 1$. In a way similar to the proofs in [1], we obtain the following Propositions and the proofs are omitted.

PROPOSITION 1.1. *Let f be a real-valued differentiable function on I. Then the following conditions are equivalent.*

(i) $|f'(t) - \lambda f(t)| \leq \varepsilon$ *holds for every $t \in I$.*

(ii) *There exists a real-valued differentiable function θ on J such that*

$$0 \leq -\lambda\theta'(u) \leq 2\varepsilon \quad (u \in J) ,$$

Received May 16, 2000

$$f(t) = \frac{\varepsilon}{\lambda} + \theta(e^{-\lambda t})e^{\lambda t} \quad (t \in I).$$

NOTE 1.1. By the mean value theorem, the function θ in the condition (ii) is a $2\varepsilon/|\lambda|$-lipschitz function. That is,

$$|\theta(u) - \theta(v)| \le \frac{2\varepsilon}{|\lambda|}|u - v|$$

holds for every $u, v \in J$.

Here and after $\lim_{u \searrow s}$ denotes the right limit.

PROPOSITION 1.2. *Let f be a real-valued differentiable function on I. If the inequality*

$$|f'(t) - \lambda f(t)| \le \varepsilon$$

holds for every $t \in I$, then $\lim_{u \searrow \inf J} \theta(u)$ *exists and*

$$\left| f(t) - \left\{ \lim_{u \searrow \inf J} \theta(u) \right\} e^{\lambda t} \right| \le \frac{3\varepsilon}{|\lambda|}$$

holds for every $t \in I$, where θ is the function given in Proposition 1.1.

Proposition 1.2 states that f can be approximated by the solution $ce^{\lambda t}$ to the differential equation $x'(t) = \lambda x(t)$, if $|f'(t) - \lambda f(t)| \le \varepsilon$ holds for every $t \in I$. According to [1], we call the stability in the sense of Proposition 1.2 the Hyers-Ulam stability.

In this paper, X denotes a topological space. Let $\mathbf{C}$ be the complex number field and $\mathbf{F} \in \{\mathbf{R}, \mathbf{C}\}$. We write $C(X, \mathbf{F})$ for the Banach space of all bounded continuous $\mathbf{F}$-valued functions on X and $C_0(X, \mathbf{F})$ for the Banach space of all functions of $C(X, \mathbf{F})$ which vanish at infinity, in the following sense:

> f vanishes at infinity if and only if for every $\delta > 0$ there exists a compact subset K in X such that $|f(x)| < \delta$ holds for every $x \in X \setminus K$.

We consider a differentiable map f from I to $C(X, \mathbf{R})$ (resp. $C_0(X, \mathbf{R})$) with the inequality $\|f'(t) - \lambda f(t)\|_\infty \le \varepsilon$, where $\|\cdot\|_\infty$ denotes the supremum norm on X. Then we show that the Hyers-Ulam stability holds for f. That is, f can be approximated by the solution $e^{\lambda t} g$ to the differential equation $x'(t) = \lambda x(t)$ for some $g \in C(X, \mathbf{R})$ (resp. $C_0(X, \mathbf{R})$). As a corollary, we obtain the Hyers-Ulam stability of a certain differentiable map from I to $C(X, \mathbf{C})$.

To prove the Hyers-Ulam stability of the map f from I to $C(X, \mathbf{R})$ with the inequality $\|f'(t) - \lambda f(t)\|_\infty \le \varepsilon$, let us consider for every $x \in X$ the function f_x from I to $\mathbf{R}$ defined by

$$f_x(t) = f(t)(x) \quad (t \in I).$$

Then f_x is a real-valued differentiable function on I with the equality

$$f_x'(t) = f'(t)(x)$$

for each $x \in X$ and each $t \in I$. Therefore, we have the inequality

$$|f_x'(t) - \lambda f_x(t)| \le \varepsilon \quad (x \in X, t \in I).$$

Thus by Proposition 1.1, for every $x \in X$ there exists a real-valued differentiable function θ_x on J such that

$$0 \le -\lambda\theta_x'(u) \le 2\varepsilon \quad (u \in J), \quad f_x(t) = \frac{\varepsilon}{\lambda} + \theta_x(e^{-\lambda t})e^{\lambda t} \quad (t \in I).$$

By Proposition 1.2 the function

$$g(x) = \lim_{u \searrow \inf J} \theta_x(u)$$

is well-defined and the inequality

$$|f_x(t) - e^{\lambda t} g(x)| \le \frac{3\varepsilon}{|\lambda|} \quad (t \in I)$$

holds for every $x \in X$. The function g obtained above plays an important role in this paper.

From now on, θ_x denotes the real-valued differentiable function on J with

$$0 \le -\lambda\theta_x'(u) \le 2\varepsilon \quad (u \in J), \quad f(t)(x) = \frac{\varepsilon}{\lambda} + \theta_x(e^{-\lambda t})e^{\lambda t} \quad (t \in I)$$

for every $x \in X$, if f is a differentiable map from I to $C(X, \mathbf{R})$ with the inequality

$$\|f'(t) - \lambda f(t)\|_\infty \le \varepsilon$$

for every $t \in I$. Moreover, g stands for the function defined by

$$g(x) = \lim_{u \searrow \inf J} \theta_x(u) \quad (x \in X)$$

which satisfies the inequality

$$\|f(t) - e^{\lambda t} g\|_\infty \le \frac{3\varepsilon}{|\lambda|} \quad (t \in I).$$

2. Main results.

Before we turn to our main theorem, we consider a differential equation $x'(t) = \lambda x(t)$ for a differentiable map x from I to a Banach space. While the following proposition is well-known, we give a proof.

PROPOSITION 2.1. *Let A be a real (resp. complex) Banach space, f a differentiable map from I to A. If $\mu \in \mathbf{R} \setminus \{0\}$ (resp. $\mu \in \mathbf{C} \setminus \{0\}$), the following conditions are equivalent.*

(i) *$f'(t) = \mu f(t)$ holds for every $t \in I$.*

(ii) *There exists an $h \in A$ such that $f(t) = e^{\mu t} h$ $(t \in I)$.*

PROOF. (ii) $\Rightarrow$ (i) By definition, it is clear and a proof is omitted.

(i) $\Rightarrow$ (ii) We define $h(t) = e^{-\mu t} f(t)$ for every $t \in I$. Then h is differentiable and the equality

$$h'(t) = \{-\mu f(t) + f'(t)\}e^{-\mu t} = 0$$

holds for every $t \in I$, by hypothesis. We show that h is a constant map. In fact, fix any $t_0 \in I$ and put

$$h_0(t) = h(t) - h(t_0) \quad (t \in I).$$

Let A^* be the dual space of A. For every $\Lambda \in A^*$ the composed function $\Lambda \circ h_0$ from I to $\mathbf{C}$ is differentiable and the equality

$$(\Lambda \circ h_0)'(t) = \Lambda(h_0'(t)) = 0$$

holds for every $t \in I$, since Λ is bounded linear and since $h_0'(t) = 0$. Therefore, for every $\Lambda \in A^*$ there exists a $c_\Lambda \in \mathbf{C}$ such that $\Lambda(h_0(t)) = c_\Lambda$ holds for every $t \in I$. We have $c_\Lambda = \Lambda(h_0(t_0)) = 0$, since $h_0(t_0) = 0$. By the Hahn-Banach theorem, $h_0(t) = 0$ holds for every $t \in I$. Hence h is a constant map. If we write $h(t) = h$, we have $f(t) = e^{\mu t}h$. This completes the proof. □

NOTE 2.1. Let f be a differentiable map from I to $C(X, \mathbf{R})$ (resp. $C_0(X, \mathbf{R})$) with

$$\|f'(t) - \lambda f(t)\|_\infty \leq \varepsilon \quad (t \in I).$$

If we consider the case where $\varepsilon = 0$, then g coincides with the function h in Proposition 2.1 in case where $A = C(X, \mathbf{R})$ (resp. $C_0(X, \mathbf{R})$). In fact, suppose that the inequality above holds for $\varepsilon = 0$. On one hand, there exists an $h \in C(X, \mathbf{R})$ (resp. $C_0(X, \mathbf{R})$) such that $f(t) = e^{\lambda t}h$ for every $t \in I$, by Proposition 2.1. On the other hand, we can write

$$f(t)(x) = \theta_x(e^{-\lambda t})e^{\lambda t}$$

for every $t \in I$ and every $x \in X$. Therefore, we have

$$h(x) = \theta_x(e^{-\lambda t}) \quad (x \in X, t \in I).$$

By the definition of the function g,

$$g(x) = \lim_{u \searrow \inf J} \theta_x(u) = h(x)$$

holds for every $x \in X$. Hence, $g = h$ holds if $\varepsilon = 0$. In particular, g is an element of $C(X, \mathbf{R})$ (resp. $C_0(X, \mathbf{R})$), if $\varepsilon = 0$.

LEMMA 2.2. *Let f be a differentiable map from I to $C(X, \mathbf{R})$ with the inequality*

$$\|f'(t) - \lambda f(t)\|_\infty \leq \varepsilon \quad (t \in I).$$

Then g is continuous on X.

PROOF. By Note 2.1, it is enough to consider the case where $\varepsilon > 0$. Suppose that g is not continuous on X. Then there exist an $x_0 \in X$ and an $\eta_0 > 0$ such that for every open neighbourhood V of x_0 there corresponds a $z \in V$ with

$$|g(x_0) - g(z)| \geq \eta_0.$$

Since $g(x_0) = \lim_{u \searrow \inf J} \theta_{x_0}(u)$, there exists a $u_0 \in J$ such that

$$|g(x_0) - \theta_{x_0}(u)| < \frac{\eta_0}{4} \quad (u \in J : u < u_0).$$

Put $\alpha = \inf J$, and choose $u_1 \in J$ with $u_1 < \min\{u_0, \alpha + |\lambda|\eta_0/8\varepsilon\}$. Then we have

$$(1) \qquad |g(x_0) - \theta_{x_0}(u_1)| < \frac{\eta_0}{4},$$

$$u_1 < \alpha + \frac{|\lambda|\eta_0}{8\varepsilon}. \tag{2}$$

Since $x \mapsto \theta_x(u_1)$ is continuous function on X, there exists an open neighbourhood W_0 of x_0 such that

$$|\theta_{x_0}(u_1) - \theta_y(u_1)| < \frac{\eta_0}{4} \quad (y \in W_0). \tag{3}$$

By hypothesis, there corresponds a $z \in W_0$ with

$$|g(x_0) - g(z)| \geq \eta_0. \tag{4}$$

In a way similar to the inequality (1), we obtain

$$|g(z) - \theta_z(u_2)| < \frac{\eta_0}{4} \tag{5}$$

for some $u_2 \in J$ with $u_2 < u_1$. By (1), (3), (4) and (5), we have

$$\begin{aligned}
\eta_0 &\leq |g(z) - g(x_0)| \\
&\leq |g(z) - \theta_z(u_2)| + |\theta_z(u_2) - \theta_z(u_1)| \\
&\quad + |\theta_z(u_1) - \theta_{x_0}(u_1)| + |\theta_{x_0}(u_1) - g(x_0)| \\
&\leq |\theta_z(u_2) - \theta_z(u_1)| + \frac{3}{4}\eta_0 .
\end{aligned}$$

That is, we obtain the inequality

$$|\theta_z(u_2) - \theta_z(u_1)| \geq \frac{\eta_0}{4}. \tag{6}$$

By the mean value theorem, there exists a $v \in (u_2, u_1)$ such that

$$\theta_z'(v) = \frac{\theta_z(u_2) - \theta_z(u_1)}{u_2 - u_1}.$$

On one hand, we have

$$-\lambda\theta_z'(v) \geq -\frac{|\lambda|\eta_0}{4(u_2 - u_1)} > \frac{|\lambda|\eta_0}{4(u_1 - \alpha)},$$

by the inequality (6), whether λ is positive or negative. On the other hand, the inequality

$$\frac{|\lambda|\eta_0}{u_1 - \alpha} > 8\varepsilon$$

holds by (2). Therefore, we have the inequality

$$-\lambda\theta_z'(v) > 2\varepsilon.$$

This contradicts with $0 \leq -\lambda\theta_z'(v) \leq 2\varepsilon$. Thus we proved that g is continuous on X. □

We obtain the Hyers-Ulam stability of a differentiable map from I to $C(X, \mathbf{R})$.

THEOREM 2.3. *Let f be a differentiable map from I to $C(X, \mathbf{R})$ with the inequality*

$$\|f'(t) - \lambda f(t)\|_\infty \leq \varepsilon \quad (t \in I).$$

Then g is an element of $C(X, \mathbf{R})$ with

$$\|f(t) - e^{\lambda t} g\|_\infty \leq \frac{3\varepsilon}{|\lambda|} \quad (t \in I).$$

PROOF. By Lemma 2.2, g is continuous. Therefore, it is enough to show that g is bounded on X. In fact, fix any element $u_0 \in J$. Since θ_x is $2\varepsilon/|\lambda|$-lipschitz,

$$|\theta_x(u) - \theta_x(u_0)| \leq \frac{2\varepsilon}{|\lambda|}|u - u_0|$$

holds for every $x \in X$ and every $u \in J$. Therefore, we have the inequality

$$\begin{aligned} |g(x) - \theta_x(u_0)| &= \lim_{u \searrow \inf J} |\theta_x(u) - \theta_x(u_0)| \\ &\leq \frac{2\varepsilon}{|\lambda|} u_0 \quad (x \in X). \end{aligned}$$

Put $t_0 = -\lambda^{-1} \log u_0 \in I$. Since $f(t_0)$ is bounded on X, there exists an $M > 0$ such that $|f(t_0)(x)| \leq M$ holds for every $x \in X$. By the definition of the function θ_x,

$$\begin{aligned} |\theta_x(e^{-\lambda t_0})| &= \left| \left\{ f(t_0)(x) - \frac{\varepsilon}{\lambda} \right\} e^{-\lambda t_0} \right| \\ &\leq \left\{ M + \frac{\varepsilon}{|\lambda|} \right\} e^{-\lambda t_0} \end{aligned}$$

holds for every $x \in X$. Therefore, we have

$$\begin{aligned} |g(x)| &\leq \frac{2\varepsilon}{|\lambda|} u_0 + |\theta_x(u_0)| \\ &\leq \left\{ \frac{3\varepsilon}{|\lambda|} + M \right\} u_0 \end{aligned}$$

for every $x \in X$. That is, g is bounded on X and this completes the proof. □

Next we consider a differentiable map from I to $C_0(X, \mathbf{R})$. The function g need not vanish at infinity, but for a suitable constant c we have $g + c \in C_0(X, \mathbf{R})$.

LEMMA 2.4. *Let f be a differentiable map from I to $C_0(X, \mathbf{R})$ with the inequality*

$$\|f'(t) - \lambda f(t)\|_\infty \leq \varepsilon \quad (t \in I).$$

Then $g_0 = g + \alpha\varepsilon/\lambda$ vanishes at infinity, where $\alpha = \inf J$.

PROOF. By Note 2.1, it is enough to consider the case where $\varepsilon > 0$. In this case, assume to the contrary that g_0 does not vanish at infinity. That is, there exists a $\delta_0 > 0$ with the following property:

For every compact subset K in X, there exists a $y \in X \setminus K$ such that $|g_0(y)| \geq \delta_0$.

Since $\alpha = \inf J$, we can choose a $u_0 \in J$ with

$$u_0 < \alpha + \frac{|\lambda|\delta_0}{8\varepsilon}. \tag{7}$$

Let $t_0 = -\lambda^{-1} \log u_0 \in I$. Since $f(t_0) \in C_0(X, \mathbf{R})$, there corresponds a compact subset K_0 in X such that

$$|f(t_0)(x)| < \frac{\delta_0}{4} e^{\lambda t_0}$$

holds for every $x \in X \setminus K_0$. Hence,

$$|\theta_x(u_0) + \frac{\varepsilon}{\lambda} u_0| < \frac{\delta_0}{4} \quad (x \in X \setminus K_0). \tag{8}$$

By hypothesis, there exists a $y \in X \setminus K_0$ such that

$$|g_0(y)| \geq \delta_0 .$$

That is,

$$\left| g(y) + \frac{\alpha\varepsilon}{\lambda} \right| \geq \delta_0 . \tag{9}$$

By the definition of the function g, we have

$$|g(y) - \theta_y(v_0)| < \frac{\delta_0}{4} \tag{10}$$

for some $v_0 \in J$ with $v_0 < u_0$. By the inequalities (7), (8), (9) and (10), we have

$$\begin{aligned}
\delta_0 &\leq \left| g(y) + \frac{\alpha\varepsilon}{\lambda} \right| \\
&\leq |g(y) - \theta_y(v_0)| + |\theta_y(v_0) - \theta_y(u_0)| \\
&\quad + \left| \theta_y(u_0) + \frac{\varepsilon}{\lambda} u_0 \right| + \frac{\varepsilon}{|\lambda|} |\alpha - u_0| \\
&< |\theta_y(v_0) - \theta_y(u_0)| + \frac{3}{4} \delta_0 .
\end{aligned}$$

Therefore, we obtain the following inequality.

$$|\theta_y(v_0) - \theta_y(u_0)| > \frac{\delta_0}{4} . \tag{11}$$

By the mean value theorem, there exists a $w \in (v_0, u_0)$ such that

$$\theta_y'(w) = \frac{\theta_y(v_0) - \theta_y(u_0)}{v_0 - u_0} .$$

Then we have the following inequality

$$-\lambda \theta_y'(w) > -\frac{|\lambda|\delta_0}{4(v_0 - u_0)} > \frac{|\lambda|\delta_0}{4(u_0 - \alpha)} ,$$

by (11), whether λ is positive or negative. On the other hand, we have

$$\frac{|\lambda|\delta_0}{u_0 - \alpha} > 8\varepsilon ,$$

by the inequality (7). Therefore, we obtain the inequality $-\lambda \theta_y'(w) > 2\varepsilon$. We arrived at a contradiction, since $0 \leq -\lambda \theta_y'(w) \leq 2\varepsilon$. We have proved that g_0 vanishes at infinity. □

THEOREM 2.5. *Let f be a differentiable map from I to $C_0(X, \mathbf{R})$ with the inequality*

$$\|f'(t) - \lambda f(t)\|_\infty \leq \varepsilon \quad (t \in I) .$$

Then $g_0 = g + \alpha\varepsilon/\lambda$ *is an element of* $C_0(X, \mathbf{R})$ *with*

$$\|f(t) - e^{\lambda t} g_0\|_\infty \le \frac{4\varepsilon}{|\lambda|} \quad (t \in I),$$

where $\alpha = \inf J$.

PROOF. By Lemma 2.2 and Lemma 2.4, g_0 is an element of $C_0(X, \mathbf{R})$. Since $\alpha = \inf J \le e^{-\lambda t}$ holds for every $t \in I$, we have

$$\begin{aligned}\|f(t) - e^{\lambda t} g_0\|_\infty &\le \|f(t) - e^{\lambda t} g\|_\infty + \frac{\varepsilon}{|\lambda|}\alpha e^{\lambda t} \\ &\le \frac{\varepsilon}{|\lambda|}(3 + \alpha e^{\lambda t}) \\ &\le \frac{4\varepsilon}{|\lambda|} \quad (t \in I).\end{aligned}$$

This completes the proof. □

COROLLARY 2.6. *Let* f *be a differentiable map from* $\mathbf{R}$ *to* $C(X, \mathbf{R})$ *with the inequality*

$$\|f'(t) - \lambda f(t)\|_\infty \le \varepsilon \quad (t \in \mathbf{R}).$$

Suppose that the inequality

$$\|f(t) - e^{\lambda t} h\|_\infty \le k\varepsilon \quad (t \in \mathbf{R})$$

holds for some $h \in C(X, \mathbf{R})$ *and some* $k \ge 0$, *then* $g = h$ *holds. In particular, if* f *is a map from* $\mathbf{R}$ *to* $C_0(X, \mathbf{R})$ *then* g *itself is an element of* $C_0(X, \mathbf{R})$ *and* $g = h$ *holds, if* h *belongs to* $C(X, \mathbf{R})$ *which satisfies the inequality above.*

PROOF. By Theorem 2.3, g belongs to $C(X, \mathbf{R})$ and the inequality

$$\|f(t) - e^{\lambda t} g\|_\infty \le \frac{3\varepsilon}{|\lambda|} \quad (t \in \mathbf{R})$$

holds. We show that $g = h$, if

$$\|f(t) - e^{\lambda t} h\|_\infty \le k\varepsilon \quad (t \in \mathbf{R}).$$

In fact,

$$\begin{aligned}\|g - h\|_\infty &\le \|g - e^{-\lambda t} f(t)\|_\infty + \|e^{-\lambda t} f(t) - h\|_\infty \\ &\le \left\{\frac{3}{|\lambda|} + k\right\} \varepsilon e^{-\lambda t} \quad (t \in \mathbf{R}).\end{aligned}$$

Note that $e^{-\lambda t} \to 0$ as $t \to \infty$ if $\lambda > 0$, and $e^{-\lambda t} \to 0$ as $t \to -\infty$ if $\lambda < 0$. In any case $g = h$ holds. In particular, if f is a map from $\mathbf{R}$ to $C_0(X, \mathbf{R})$, then g is an element of $C_0(X, \mathbf{R})$ since $g + \alpha\varepsilon/\lambda$ belongs to $C_0(X, \mathbf{R})$ and since $\alpha = 0$, where $\alpha = \inf J$. In a way similar to the above, we have $g = h$, if h is an element of $C(X, \mathbf{R})$ with $\|f(t) - e^{\lambda t} h\|_\infty \le k\varepsilon$ for some $k \ge 0$. This completes the proof. □

Finally we consider a differentiable map f from I to $C(X, \mathbf{C})$. Since $f(t)$ is an element of $C(X, \mathbf{C})$ for every $t \in I$, we can write

$$f(t) = \mathrm{Re}\{f(t)\} + i\mathrm{Im}\{f(t)\},$$

where $\mathrm{Re}\{f(t)\}$ and $\mathrm{Im}\{f(t)\}$ denote the real part of $f(t)$ and the imaginary part of $f(t)$, respectively. Let $\mathrm{Re} f$ and $\mathrm{Im} f$ be the maps from I to $C(X, \mathbf{R})$ defined by

$$(\mathrm{Re}\, f)(t) = \mathrm{Re}\{f(t)\}, \quad (\mathrm{Im}\, f)(t) = \mathrm{Im}\{f(t)\} \quad (t \in I).$$

If we apply Theorem 2.3, Theorem 2.5 and Corollary 2.6 to $\mathrm{Re} f$ and $\mathrm{Im} f$, then we obtain the following Corollaries.

COROLLARY 2.7. *Let f be a differentiable map from I to $C(X, \mathbf{C})$ with the inequality*

$$\|f'(t) - \lambda f(t)\|_\infty \le \varepsilon \quad (t \in I).$$

Then there exists a $\tilde{g} \in C(X, \mathbf{C})$ such that

$$\|f(t) - e^{\lambda t}\tilde{g}\|_\infty \le \frac{3\sqrt{2}\varepsilon}{|\lambda|} \quad (t \in I).$$

COROLLARY 2.8. *Let f be a differentiable map from I to $C_0(X, \mathbf{C})$ with the inequality*

$$\|f'(t) - \lambda f(t)\|_\infty \le \varepsilon \quad (t \in I).$$

Then there exists a $\tilde{g}_0 \in C_0(X, \mathbf{C})$ such that

$$\|f(t) - e^{\lambda t}\tilde{g}_0\|_\infty \le \frac{4\sqrt{2}\varepsilon}{|\lambda|} \quad (t \in I).$$

COROLLARY 2.9. *Let f be a differentiable map from $\mathbf{R}$ to $C(X, \mathbf{C})$ with the inequality*

$$\|f'(t) - \lambda f(t)\|_\infty \le \varepsilon \quad (t \in \mathbf{R}).$$

Then there exists a unique function $\tilde{g} \in C_0(X, \mathbf{C})$ such that

$$\|f(t) - e^{\lambda t}\tilde{g}\|_\infty \le \frac{4\sqrt{2}\varepsilon}{|\lambda|} \quad (t \in \mathbf{R}).$$

ACKNOWLEDGEMENT. This paper owes much to the thoughtful and helpful comments of Professors Osamu Hatori and Keiichi Watanabe. The second author is partly supported by the Grants-in-Aid for Scientific Research, The Ministry of Education, Science, Sports and Culture, Japan (10640150).

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