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Stability of Cubic 3-folds

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Introduction.

Hilbert's idea of *null forms* appeared again as the *stability* and plays an important role in constructing the moduli space and its compactification in Geometric Invariant Theory of Mumford [7]. By virtue of the numerical criterion, one can determine the stable objects explicitly. For example, Hilbert proved the following. (See [3, §19] and [8, p15].)

THEOREM. *Let S be a cubic surface in the projective space $\mathbf{P}^3$.*

(1) *S is stable if and only if it has only rational double points of type A_1.*

(2) *S is semi-stable if and only if it has only rational double points of type A_1 or A_2.*

(3) *The moduli of stable ones is compactified by adding one point corresponding to the semi-stable cubic $xyz + w^3 = 0$ with 3 A_2 singularities.*

The stability of quartic surfaces is studied by Shah [11]. In this paper applying the same criterion to cubic 3-folds, i.e. hypersurfaces of degree 3 in $\mathbf{P}^4$, we prove the following.

MAIN THEOREM. *Let X be a cubic 3-fold.*

(1) *X is stable if and only if it has only double points of type $A_n : v^2+w^2+x^2+y^{n+1} = 0$ with $n \leq 4$.*

(2) *X is semi-stable if and only if it has only double points of type A_n with $n \leq 5$, $D_4 : v^2 + w^2 + x^3 + y^3 = 0$ or $A_\infty : v^2 + w^2 + x^2 = 0$. And if a semi-stable cubic 3-fold has A_∞ singularity, then it is isomorphic to the secant 3-fold, that is, the secant variety of rational normal curve in $\mathbf{P}^4$.*

(3) *The moduli of stable ones is compactified by adding two components. One is isomorphic to $\mathbf{P}^1$ and the other is an isolated point corresponding to the semi-stable cubic 3-fold $xyz + v^3 + w^3 = 0$ with 3 D_4 singularities.*

We remark that Collino [1] studies the degeneration of intermediate Jacobians for a family of cubic 3-folds approaching to the secant 3-fold.

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According to the numerical criterion for hypersurfaces in $\mathbf{P}^n$, in order to classify stable ones, it is enough to determine certain finite number of hyperplane sections in an n-dimensional simplex. In Hilbert's and Shah's cases, since the dimension n is 3, we can determine these hyperplanes by intuition. In our case $n = 4$, it becomes more difficult for us. The author determined such ones by aid of computer. In this paper, we use a combinatorial way to prove that without an assistant of computer.

As for the moduli of Fano 3-folds, we want to know whether it can be compactified by adding 3-folds with canonical singularities. (See [5] and [6, §8].) By (2) of Main Theorem, the answer is affirmative for cubic 3-folds. Moreover, by virtue of (3), except for one point corresponding to the secant 3-fold, the moduli is compactified by adding 3-folds with terminal singularities. Similar results are obtained for cubic 4-folds in [12].

This paper consists of 5 sections. In Section 1 we determine six 1-PS's characterizing the (semi-)stability by the numerical criterion. In Section 2 we show that, modulo SL(5)-action, two 1-PS's are maximal among them. In Section 3 we determine cubic 3-folds with a closed orbit in the space of cubic polynimials and prove (3) in Main Theorem. We note that Luna's criterion [4] is useful to prove it. In Section 4 we consider cubic 3-folds with A_∞ singularity and prove the latter half of (2) in Main Theorem. In Section 5 we translate two 1-PS's above into the analytic local condition, that is, we prove (1) and the first half of (2) in Main Theorem.

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1. Numerical criterion for cubic 3-folds.

(1.1) Numerical criterion of hypersurfaces. A one-parameter subgroup, 1-PS for short, of $\mathrm{SL}(n+1)$ is a homomorphism $\lambda : \mathbf{G_m} \to \mathrm{SL}(n+1)$ of algebraic groups. Such λ can always be diagonalized in a suitable basis:

$$\lambda(t) = \mathrm{diag}\,(t^{r_0}, t^{r_1}, \cdots, t^{r_n}) \quad \text{and} \quad r_0 \geq r_1 \geq \cdots \geq r_n\,.$$

It is simply expressed by $\lambda = [r_0, r_1, \cdots, r_n]$. Since $[r_0, r_1, \cdots, r_n] \neq [0, 0, \cdots, 0]$, r_0 is positive and r_n is negative.

A hypersurface of degree d in $\mathbf{P}^n$ defined by a homogeneous polynomial $f(x_0, x_1, \cdots, x_n)$ of degree d is not stable (resp. semi-stable) if and only if there exists an element σ of $\mathrm{SL}(n+1)$ and a 1-PS $\lambda(t) = \mathrm{diag}\,(t^{r_0}, t^{r_1}, \cdots, t^{r_n}) \in \mathrm{SL}(n+1)$ such that $\lim_{t\to 0} \lambda(t)(\sigma f)$ exists (resp. exists and is equal to 0). Expressing $\sigma f = \sum a_{ij\cdots k} x_0^i x_1^j \cdots x_n^k$, this is equivalent to the condition

$$\exists \text{ 1-PS } [r_0, r_1, \cdots, r_n] \text{ s.t. } r_0 i + r_1 j + \cdots + r_n k \geq 0 \quad (\text{resp. } > 0) \quad \text{if } a_{ij\cdots k} \neq 0\,.$$

Let $\mathbf{I}$ be the set of exponents of monomilas $x_0^i x_1^j \cdots x_n^k$, that is,

$$\mathbf{I} = \{(i, j, \cdots, k) \in \mathbf{Z}^{n+1} \mid i, j, \cdots, k \geq 0 \text{ and } i + j + \cdots + k = d\}\,.$$

Then the determination of all non-stable (resp. unstable) hypersurfaces is reduced to that of the subsets in $\mathbf{I}$

$$\mathbf{M}^{\oplus}(\mathbf{r}) = \{\mathbf{i} \in \mathbf{I} \mid \mathbf{i} \cdot \mathbf{r} \geq 0\} \quad (\text{resp. } \mathbf{M}^{+}(\mathbf{r}) = \{\mathbf{i} \in \mathbf{I} \mid \mathbf{i} \cdot \mathbf{r} > 0\})$$

for all 1-PS $\mathbf{r} = [r_0, r_1, \cdots, r_n]$. We seek for only maximal ones instead of all such subsets.

From now on, we suppose $n = 4$ and $d = 3$.

(1.2) THEOREM.

(1) *The maximal subsets of* $\{\mathbf{M}^{\oplus}(\mathbf{r}) \mid \mathbf{r}$ *is a 1-PS* $\}$ *are* $\mathbf{M}^{\oplus}(\gamma^i)$ *for* $i = 1, 2, \cdots, 6$, *where*

$$\gamma^1 = [1, 1, 1, -1, -2], \quad \gamma^2 = [2, 1, -1, -1, -1], \quad \gamma^3 = [2, 1, 0, -1, -2],$$
$$\gamma^4 = [1, 0, 0, 0, -1], \quad \gamma^5 = [1, 1, 0, 0, -2], \quad \gamma^6 = [2, 0, 0, -1, -1].$$

(2) *For a sufficiently small number* $1 \gg \varepsilon > 0$ (*e.g.* $\varepsilon = 0.1$ *is sufficient*), *the maximal subsets of* $\{\mathbf{M}^{+}(\mathbf{r}) \mid \mathbf{r}$ *is a 1-PS* $\}$ *are* $\mathbf{M}^{+}(\lambda^i)$ *for* $i = 1, 2, \cdots, 6$, *where*

$$\lambda^1 = \gamma^1 + [1, 1, 1, -2, -1]\varepsilon, \quad \lambda^2 = \gamma^2 + [3, 0, -1, -1, -1]\varepsilon,$$
$$\lambda^3 = \gamma^3 + [7, 2, -3, -3, -3]\varepsilon, \quad \lambda^4 = \gamma^4 + [1, 1, 1, 1, -4]\varepsilon,$$
$$\lambda^5 = \gamma^5 + [0, 0, 2, -3, 1]\varepsilon, \quad \lambda^6 = \gamma^4 + [2, 2, 2, -3, -3]\varepsilon.$$

To prove (1.2), we prepare Table. The symbols $+$, $-$, $\oplus$ and $\ominus$ mean $\mathbf{i} \cdot \mathbf{r} > 0$, $\mathbf{i} \cdot \mathbf{r} < 0$, $\mathbf{i} \cdot \mathbf{r} \geq 0$ and $\mathbf{i} \cdot \mathbf{r} \leq 0$ for $\mathbf{i} \in \mathbf{I}$, respectively.

We define the partial order of $\mathbf{I}$ as follows.

$$\begin{aligned}
& (a, b, c, d, e) \geq (a', b', c', d', e') \\
\overset{def}{\Longleftrightarrow}\ & (a, b, c, d, e)\mathbf{r} \geq (a', b', c', d', e')\mathbf{r} \quad \text{for any 1-PS } \mathbf{r}\ (r_0 \geq r_1 \geq \cdots \geq r_4) \\
\Longleftrightarrow\ & a \geq a',\ a + b \geq a' + b',\ a + b + c \geq a' + b' + c',\ a + b + c + d \geq a' + b' + c' + d', \\
& a + b + c + d + e \geq a' + b' + c' + d' + e'.
\end{aligned}$$

Put $\mathbf{M}^{-}(\mathbf{r}) = \mathbf{I} \setminus \mathbf{M}^{\oplus}(\mathbf{r})$ and $\mathbf{M}^{\ominus}(\mathbf{r}) = \mathbf{I} \setminus \mathbf{M}^{+}(\mathbf{r})$. Then we have

$$\begin{aligned}
\mathbf{M}^{\oplus}(\mathbf{r}) \subset \mathbf{M}^{\oplus}(\gamma^i) &\Leftrightarrow \mathbf{M}^{-}(\gamma^i) \subset \mathbf{M}^{-}(\mathbf{r}) \\
&\Leftrightarrow \mathbf{i} \cdot \mathbf{r} < 0 \text{ for any } \mathbf{i} \in \mathbf{M}^{-}(\gamma^i) \\
&\Leftrightarrow \mathbf{i} \cdot \mathbf{r} < 0 \text{ for any } \textit{maximal} \text{ element } \mathbf{i} \in \mathbf{M}^{-}(\gamma^i).
\end{aligned}$$

Hence we obtain the following:

(1.3) CRITERION. Let $\mathbf{r}$ be an arbitrary 1-PS.

(1) $\mathbf{M}^{\oplus}(\mathbf{r}) \subset \mathbf{M}^{\oplus}(\gamma^i)$ if and only if $\mathbf{i} \cdot \mathbf{r} < 0$ for all the maximal elements $\mathbf{i} \in \mathbf{M}^{-}(\gamma^i)$.

(2) $\mathbf{M}^{+}(\mathbf{r}) \subset \mathbf{M}^{+}(\lambda^i)$ if and only if $\mathbf{i} \cdot \mathbf{r} \leq 0$ for all the maximal elements $\mathbf{i} \in \mathbf{M}^{\ominus}(\lambda^i)$.

We mark † on the maximal elements of $\mathbf{M}^{-}(\gamma^i)$ and $\mathbf{M}^{\ominus}(\lambda^i)$ in Table. For example, the maximal element of $\mathbf{M}^{-}(\gamma^3)$ are $(0, 0, 2, 1, 0)$, $(0, 1, 0, 2, 0)$, $(0, 1, 1, 0, 1)$ and $(1, 0, 0, 1, 1)$, which are marked.

TABLE.

$\mathbf{i} \setminus \mathbf{r}$	γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	λ^1	λ^2	λ^3	λ^4	λ^5	λ^6
(0, 0, 0, 0, 3)	$-$	$-$	$-$	$-$	$-$	$-$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$
(0, 0, 0, 1, 2)	$-$	$-$	$-$	$-$	$-$	$-$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$
(0, 0, 0, 2, 1)	$-$	$-$	$-$	$-$	$-$	$-$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$
(0, 0, 0, 3, 0)	$-$	$-$	$-$	$\oplus$	$\oplus$	$-$	$\ominus$	$\ominus$	$\ominus$	$+$	$\ominus$	$\ominus$
(0, 0, 1, 0, 2)	$-$	$-$	$-$	$-$	$-$	$-$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$
(0, 0, 1, 1, 1)	$-$	$-$	$-$	$-$	$-$	$-$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$
(0, 0, 1, 2, 0)	$-$	$-$	$-$	$\oplus$	$\oplus$	$-$	$\ominus$	$\ominus$	$\ominus$	$+$	$\ominus\dagger$	$\ominus$
(0, 0, 2, 0, 1)	$\oplus$	$-$	$-$	$-$	$-$	$-$	$+$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$
(0, 0, 2, 1, 0)	$\oplus$	$-$	$-\dagger$	$\oplus$	$\oplus$	$-$	$+$	$\ominus$	$\ominus$	$+$	$+$	$+$
(0, 0, 3, 0, 0)	$\oplus$	$-$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$\ominus$	$\ominus\dagger$	$+$	$+$	$+$
(0, 1, 0, 0, 2)	$-$	$-$	$-$	$-$	$-$	$-$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$
(0, 1, 0, 1, 1)	$-$	$-$	$-$	$-$	$-$	$-$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$
(0, 1, 0, 2, 0)	$-$	$-$	$-\dagger$	$\oplus$	$\oplus$	$-$	$\ominus$	$\ominus$	$\ominus$	$+$	$+$	$\ominus\dagger$
(0, 1, 1, 0, 1)	$\oplus$	$-$	$-\dagger$	$-$	$-$	$-$	$+$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$
(0, 1, 1, 1, 0)	$\oplus$	$-$	$\oplus$	$\oplus$	$\oplus$	$-$	$+$	$\ominus$	$\ominus\dagger$	$+$	$+$	$+$
(0, 1, 2, 0, 0)	$\oplus$	$-\dagger$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$\ominus\dagger$	$+$	$+$	$+$	$+$
(0, 2, 0, 0, 1)	$\oplus$	$\oplus$	$\oplus$	$-\dagger$	$\oplus$	$-$	$+$	$+$	$+$	$\ominus$	$+$	$\ominus\dagger$
(0, 2, 0, 1, 0)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$-\dagger$	$+$	$+$	$+$	$+$	$+$	$+$
(0, 2, 1, 0, 0)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$
(0, 3, 0, 0, 0)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$
(1, 0, 0, 0, 2)	$-$	$\oplus$	$-$	$-\dagger$	$-$	$\oplus$	$\ominus$	$+$	$\ominus$	$\ominus$	$\ominus$	$\ominus$
(1, 0, 0, 1, 1)	$-$	$\oplus$	$-\dagger$	$\oplus$	$-$	$\oplus$	$\ominus$	$+$	$\ominus\dagger$	$\ominus$	$\ominus$	$\ominus\dagger$
(1, 0, 0, 2, 0)	$-\dagger$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\ominus\dagger$	$+$	$+$	$+$	$+$	$+$
(1, 0, 1, 0, 1)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$-\dagger$	$\oplus$	$+$	$+$	$+$	$\ominus$	$\ominus\dagger$	$+$
(1, 0, 1, 1, 0)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$
(1, 0, 2, 0, 0)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$
(1, 1, 0, 0, 1)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$+$	$+$	$\ominus\dagger$	$+$	$+$
(1, 1, 0, 1, 0)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$
(1, 1, 1, 0, 0)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$
(1, 2, 0, 0, 0)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$
(2, 0, 0, 0, 1)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$
(2, 0, 0, 1, 0)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$
(2, 0, 1, 0, 0)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$
(2, 1, 0, 0, 0)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$
(3, 0, 0, 0, 0)	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$+$	$+$	$+$	$+$	$+$	$+$

(1.4) PROOF OF (1) OF (1.2). Assume that $\mathbf{M}^{\oplus}(\mathbf{r}) \not\subset \mathbf{M}^{\oplus}(\gamma^i)$ for any $i \neq 3$. It is enough to show that $\mathbf{M}^{\oplus}(\mathbf{r}) \subset \mathbf{M}^{\oplus}(\gamma^3)$, that is, $(0,0,2,1,0)\mathbf{r} < 0$, $(0,1,0,2,0)\mathbf{r} < 0$, $(0,1,1,0,1)\mathbf{r} < 0$ and $(1,0,0,1,1)\mathbf{r} < 0$ by (1.3). Since $\mathbf{M}^{\oplus}(\mathbf{r}) \not\subset \mathbf{M}^{\oplus}(\gamma^i)$ for $i = 1, 2, 5, 6$, we have $(1,0,0,2,0)\mathbf{r}$, $(0,1,2,0,0)\mathbf{r}$, $(1,0,1,0,1)\mathbf{r}$, $(0,2,0,1,0)\mathbf{r} \geq 0$ respectively by (1.3). We note that $r_i - r_j \leq 0$ for $i > j$, and $r_4 < 0$ by the assumption in (1.1). From

$$4(1,0,0,0,2)\mathbf{r} + 2(0,1,2,0,0)\mathbf{r} + (0,2,0,1,0)\mathbf{r} = (4,4,4,1,8)\mathbf{r}$$
$$= (0,0,0,-3,4)\mathbf{r} = 3(r_4 - r_3) + r_4 < 0,$$

we obtain $(1,0,0,0,2)\mathbf{r} < 0$. Since $\mathbf{M}^{\oplus}(\mathbf{r}) \not\subset \mathbf{M}^{\oplus}(\gamma^4)$, we have $(0,2,0,0,1)\mathbf{r} \geq 0$. From

$$(0,0,0,3,0)\mathbf{r} + (0,2,0,0,1)\mathbf{r} + (1,0,1,0,1)\mathbf{r} = (1,2,1,3,2)\mathbf{r}$$

$$= (-1,0,-1,1,0)\mathbf{r} = -r_0 + (r_3 - r_2) < 0\,,$$

we deduce $(0,0,0,3,0)\mathbf{r} < 0$. From the equations

$$3(0,1,0,2,0)\mathbf{r} + 3(1,0,1,0,1)\mathbf{r} - (0,0,0,3,0)\mathbf{r} = (3,3,3,3,3)\mathbf{r} = 0\,,$$
$$3(0,1,1,0,1)\mathbf{r} + 3(1,0,0,2,0)\mathbf{r} - (0,0,0,3,0)\mathbf{r} = (3,3,3,3,3)\mathbf{r} = 0\,,$$
$$3(0,0,2,1,0)\mathbf{r} + 6(1,0,1,0,1)\mathbf{r} + 6(1,0,0,2,0)\mathbf{r} + 6(0,2,0,0,1)\mathbf{r}$$
$$-(0,0,0,3,0)\mathbf{r} = 0$$

we have $(0,1,0,2,0)\mathbf{r}$, $(0,1,1,0,1)\mathbf{r}$, $(0,0,2,1,0)\mathbf{r} < 0$, respectively. Hence from

$$15(1,0,0,1,1)\mathbf{r} + 5(0,2,0,0,1)\mathbf{r} + 8(0,1,2,0,0)\mathbf{r} = (15,18,16,15,20)\mathbf{r}$$
$$= (-2,1,-1,-2,3)\mathbf{r} = -r_0 + (r_1 - r_0) + (r_4 - r_2) + 2(r_4 - r_3) < 0.$$

Therefore, we have $(1,0,0,1,1)\mathbf{r} < 0$. □

(1.5) PROOF OF (2) OF (1.2). Assuming $\mathbf{M}^{+}(\mathbf{r}) \not\subset \mathbf{M}^{+}(\lambda^i)$ for $1 \leq i \leq 5$, we show that $\mathbf{M}^{+}(\mathbf{r}) \subset \mathbf{M}^{+}(\lambda^6)$, that is,

$$(0,1,0,2,0)\mathbf{r} \leq 0,\ (0,2,0,0,1)\mathbf{r} \leq 0 \quad \text{and} \quad (1,0,0,1,1)\mathbf{r} \leq 0\,.$$

For $\mathbf{M}^{+}(\mathbf{r}) \not\subset \mathbf{M}^{+}(\lambda^1)$, $\mathbf{M}^{+}(\lambda^2)$, $\mathbf{M}^{+}(\lambda^4)$, we have $(1,0,0,2,0)\mathbf{r}$, $(0,1,2,0,0)\mathbf{r}$, $(1,1,0,0,1)\mathbf{r} > 0$ respectively by (1.3). From

$$2(0,0,1,2,0)\mathbf{r} + 3(1,1,0,0,1)\mathbf{r} = (3,3,2,4,3)\mathbf{r} = (0,0,-1,1,0)\mathbf{r} = r_3 - r_2 \leq 0\,,$$

we have $(0,0,1,2,0)\mathbf{r} < 0$, and since $\mathbf{M}^{+}(\mathbf{r}) \not\subset \mathbf{M}^{+}(\lambda^5)$, we obtain $(1,0,1,0,1)\mathbf{r} > 0$ by (1.3). From

$$3(0,1,0,2,0)\mathbf{r} + 2(1,1,0,0,1)\mathbf{r} + (0,1,2,0,0)\mathbf{r} + 4(1,0,1,0,1)\mathbf{r}$$
$$= (6,6,6,6,6)\mathbf{r} = 0\,, \quad 3(1,0,0,1,1)\mathbf{r} + 2(0,1,2,0,0)\mathbf{r} = (3,2,4,3,3)\mathbf{r}$$
$$= (0,-1,1,0,0)\mathbf{r} = r_2 - r_1 \leq 0\,,$$

we deduce $(0,1,0,2,0)\mathbf{r}$ and $(1,0,0,1,1)\mathbf{r} < 0$ respectively. Because $\mathbf{M}^{+}(\mathbf{r}) \not\subset \mathbf{M}^{+}(\lambda^3)$ and $(1,0,0,1,1)\mathbf{r} \leq 0$, either $(0,0,3,0,0)\mathbf{r} > 0$ or $(0,1,1,1,0)\mathbf{r} > 0$ by (1.3). In any case, from

$$3(0,2,0,0,1)\mathbf{r} + (0,0,3,0,0)\mathbf{r} + 3(1,0,0,2,0)\mathbf{r} + 3(1,0,1,0,1)\mathbf{r}$$
$$= (6,6,6,6,6)\mathbf{r} = 0\,, \quad (0,2,0,0,1)\mathbf{r} + (0,1,1,1,0)\mathbf{r}$$
$$+ (1,0,0,2,0)\mathbf{r} + 2(1,0,1,0,1)\mathbf{r} = (3,3,3,3,3)\mathbf{r} = 0\,,$$

Therefore we have $(0,2,0,0,1)\mathbf{r} < 0$. $\square$

2. Inclusion relations modulo SL(5)-action.

In this section we prove the following proposition by writing the linear transformation of polynomials explicitly.

(2.1) PROPOSITION. *Modulo* SL(5)*-action, there exist following inclusions.*

$$\begin{array}{ccccccccccc}
\mathbf{M}^{\oplus}(\gamma^2) & & \mathbf{M}^{\oplus}(\gamma^3) & & \mathbf{M}^{\oplus}(\gamma^4) & = & \mathbf{M}^{\oplus}(\gamma^5) & \supset & \mathbf{M}^{\oplus}(\gamma^6) & & \mathbf{M}^{\oplus}(\gamma^1) \\
\| & & \cup & & \cup & & \cup & & & & \| \\
\mathbf{M}^{+}(\lambda^2) & \subset & \mathbf{M}^{+}(\lambda^3) & & \mathbf{M}^{+}(\lambda^4) & \subset & \mathbf{M}^{+}(\lambda^5) & \supset & \mathbf{M}^{+}(\lambda^6) & \supset & \mathbf{M}^{+}(\lambda^1)\,.
\end{array}$$

Here $M \subset N$ mod SL(5) means that $\mathrm{SL}(5)\cdot M \subset \mathrm{SL}(5)\cdot N$. Let $[1.k]$ and $[2.k]$ be linear combinations of monomials of $\mathbf{M}^{\oplus}(\gamma^k)$ and $\mathbf{M}^{+}(\lambda^k)$ respectively where $k = 1, 2, \cdots, 6$. We take $(v{:}w{:}x{:}y{:}z)$ as a homogeneous coordinate system of $\mathbf{P}^4$. Then we have the following list.

(2.2) List.

[1.1] $yq_1(v,w,x) + zq_2(v,w,x) + c(v,w,x);$

[1.2] $xq_1(v,w) + yq_2(v,w) + zq_3(v,w) + vq_4(x,y,z) + c(v,w);$

[1.3] $a_1vy^2 + xyl_1(v,w) + yq_1(v,w) + zq_2(v,w) + a_2vxz + c(v,w,x);$

[1.4] $vzl(v,w,x,y) + c(v,w,x,y);$

[1.5] $zq(v,w) + c(v,w,x,y);$

[1.6] $vyl_1(v,w,x) + vzl_2(v,w,x) + vq(y,z) + c(v,w,x);$

[2.1] = [1.1];

[2.2] = [1.2];

[2.3] $avxz + vyl_1(x,y) + xq_1(v,w) + zq_2(v,w) + yq_3(v,w) + c(v,w) + x^2l_2(v,w);$

[2.4] $av^2z + c(v,w,x,y);$

[2.5] $y^2l(v,w) + yq_1(v,w,x) + zq_2(v,w) + c(v,w,x);$

[2.6] $avy^2 + vzl(v,w,x) + yq(v,w,x) + c(v,w,x).$

The symbols l, q and c denote a linear, quadratic and cubic homogeneous polynomials respectively. By (2.1), modulo SL(5)-action, all the maximal subsets among $\mathbf{M}^{\oplus}(\mathbf{r})$ are

$\mathbf{M}^{\oplus}(\gamma^3)$ and $\mathbf{M}^{\oplus}(\gamma^5)$, and all the maximal subsets among $\mathbf{M}^{+}(\mathbf{r})$ are $\mathbf{M}^{+}(\lambda^3)$ and $\mathbf{M}^{+}(\lambda^5)$. In other words:

(2.3) COROLLARY. *Let X be a cubic 3-fold defined by a homogeneous polynomial F.*

(1) *X is not stable if and only if F is either* [1.3] *or* [1.5] *for a suitable coordinate.*

(2) *X is unstable, i.e. not semi-stable, if and only if F is either* [2.3] *or* [2.5] *for a suitable coordinate.*

In particular, if a cubic 3*-fold has a double point of rank* 2 (*resp.* 1), *then it is not stable* (*resp. not semi-stable*).

We begin to prove (2.1). The following inclusions are obvious by List.

$$\mathbf{M}^{+}(\lambda^3) \subset \mathbf{M}^{\oplus}(\gamma^3)\,, \quad \mathbf{M}^{+}(\lambda^4) \subset \mathbf{M}^{\oplus}(\gamma^4) \quad \text{and} \quad \mathbf{M}^{+}(\lambda^5) \subset \mathbf{M}^{\oplus}(\gamma^5)\,.$$

Both [1.4] and [1.5] define all cubic 3-folds with a double point of rank < 3, so we have:

(2.4) PROPOSITION. $\mathbf{M}^{\oplus}(\gamma^4) = \mathbf{M}^{\oplus}(\gamma^5)$ mod SL(5).

Here $f \equiv g$ means that $f = \sigma g$ for some linear transformation σ.

(2.5) PROPOSITION. $\mathbf{M}^{\oplus}(\gamma^6) \subset \mathbf{M}^{\oplus}(\gamma^5)$ mod SL(5).

PROOF. Since $q(y,z) \equiv y(c_1y + c_2z)$ by a suitable linear transformation $(y,z) \mapsto (a_1y + b_1z, a_2y + b_2z)$ by (2.6) below, we have

$$\begin{aligned}[1.6] =&vyl_1(v,w,x) + vzl_2(v,w,x) + vq(y,z) + c(v,w,x)\\ \equiv&vl_3(y,z)l_1(v,w,x) + vl_4(y,z)l_2(v,w,x) + vy(c_1y + c_2z) + c(v,w,x)\\ =&zv\{l_1'(v,w,x) + l_2'(v,w,x) + c_2y\} + vyl_3'(v,w,x) + vyl_4'(v,w,x)\\ &+ avy^2 + c(v,w,x)\end{aligned}$$

which has a double point (0:0:0:0:1) of rank < 3. Therefore [1.6] $\Rightarrow$ [1.5]. □

(2.6) LEMMA. *For a suitable linear transformation σ of $(x_1, x_2, \cdots, x_n)$, $q(x_1, x_2, \cdots, x_n) = \sigma q'(x_1, x_2, \cdots, x_n)$ if and only if*

$$\operatorname{rank} q(x_1, x_2, \cdots, x_n) = \operatorname{rank} q'(x_1, x_2, \cdots, x_n)\,.$$

(2.7) PROPOSITION. $\mathbf{M}^{+}(\lambda^2) \subset \mathbf{M}^{+}(\lambda^3)$ mod SL(5).

PROOF. Since $q_4(x,y,z) \equiv axz + yl_1(x,y)$ by a suitable linear transformation $(x,y,z) \mapsto (l_1'(x,y,z), l_2'(x,y,z), l_3'(x,y,z))$, we have

$$\begin{aligned}[2.2] &= xq_1(v,w) + yq_2(v,w) + zq_3(v,w) + c(v,w) + vq_4(x,y,z)\\ &\equiv xq_1(v,w) + zq_2(v,w) + yq_3(v,w) + c(v,w) + v\{axz + yl_1(x,y)\}\\ &\Rightarrow xq_1(v,w) + zq_2(v,w) + yq_3(v,w) + c(v,w) + v\{axz + yl_1(x,y)\}\\ &\quad + x^2l_2(v,w)\\ &= avxz + vyl_1(x,y) + xq_1(v,w) + zq_2(v,w) + yq_3(v,w) + c(v,w)\\ &\quad + x^2l_2(v,w) = [2.3]. \quad \square\end{aligned}$$

(2.8) PROPOSITION. $\mathbf{M}^+(\lambda^4) \subset \mathbf{M}^+(\lambda^5)$ mod SL(5).

PROOF. For any cubic form $c(w, x, y)$ there exist a, $q(w, x)$ and $c'(w, x)$ such that

$$\{c(w, x, y) = 0\} \equiv \{awy^2 + q(w, x)y + c'(w, x) = 0\} \quad \text{in } \mathbf{P}^2(w{:}x{:}y)\,.$$

Hence $c(v, w, x, y) \equiv y^2 l(v, w) + yq(v, w, x) + c'(v, w, x)$ by a suitable linear transformation $(w{:}x{:}y) \mapsto (l_1(w, x, y){:}l_2(w, x, y){:}l_3(w, x, y))$. So we have

$$\begin{aligned}
[2.4] &= av^2 z + c(v, w, x, y) \\
&\equiv av^2 z + y^2 l(v, w) + yq(v, w, x) + c'(v, w, x) \\
&= y^2 l(v, w) + yq(v, w, x) + av^2 z + c'(v, w, x) \\
&\Rightarrow y^2 l(v, w) + yq_1(v, w, x) + zq_2(v, w) + c'(v, w, x) = [2.5]\,. \quad \square
\end{aligned}$$

(2.9) PROPOSITION. $\mathbf{M}^+(\lambda^6) \subset \mathbf{M}^+(\lambda^5)$ mod SL(5).

PROOF. We put

$$[2.6] = avy^2 + vz(a_1 v + a_2 w + a_3 x) + yq(v, w, x) + c(v, w, x)\,.$$

If $a_2 = a_3 = 0$, then our proposition is obvious. In case $(a_2, a_3) \neq (0, 0)$, exchanging w and x if necessary, we may assume $a_2 \neq 0$. So we have

$$\begin{aligned}
[2.6] &\equiv avy^2 + vz(a_1 v + a_4 w) + yq'(v, w, x) + c'(v, w, x) \quad \text{by } a_2 w + a_3 x \mapsto a_4 w \\
&\Rightarrow y^2 l(v, w) + zq_2(v, w) + yq_1(v, w, x) + c'(v, w, x) \\
&= y^2 l(v, w) + yq_1(v, w, x) + zq_2(v, w) + c'(v, w, x) = [2.5]\,. \quad \square
\end{aligned}$$

(2.10) PROPOSITION. $\mathbf{M}^+(\lambda^1) \subset \mathbf{M}^+(\lambda^6)$ mod SL(5).

PROOF. Let F be a polynomial of type [2.1]. Then we have

$$\begin{aligned}
F &= yq_1(v, w, x) + zq_2(v, w, x) + c(v, w, x) \\
&\equiv y\{aq_1(v, w, x) + bq_2(v, w, x)\} + z\{a'q_1(v, w, x) + b'q_2(v, w, x)\} + c(v, w, x) \\
&\qquad \text{by } (y, z) \mapsto (ay + a'z, by + b'z) \\
&= yq_1'(v, w, x) + zl_1(v, w, x)l_2(v, w, x) + c(v, w, x) \quad \text{by (2.11) below} \\
&\equiv yq'(v, w, x) + zvl_2'(v, w, x) + c'(v, w, x) \quad \text{by } l_1(v, w, x) \mapsto v \\
&\Rightarrow yq'(v, w, x) + vzl_2'(v, w, x) + c'(v, w, x) + avy^2 \\
&= avy^2 + vzl_2'(v, w, x) + yq(v, w, x) + c'(v, w, x) = [2.6]\,. \quad \square
\end{aligned}$$

(2.11) LEMMA. *For any pair of non-zero quadratic forms $q_1(v, w, x)$ and $q_2(v, w, x)$ there exists a pair of complex number a and b such that $aq_1(v, w, x) + bq_2(v, w, x) = l_1(v, w, x)l_2(v, w, x)$.*

Now the proof of (2.1) has completed. A cubic 3-fold defined by any polynomial in List (2.2) is singular. We describe the geometric situation of (2.1) associating it with a space sextic curve.

(2.12) DEFINITION. Let p be a singular point of a cubic 3-fold X. We choose a homogeneous coordinate of $\mathbf{P}^4$ such that $p = (0{:}0{:}0{:}0{:}1)$. Then the defining equation of X is

$$q(v, w, x, y)z + c(v, w, x, y)\,.$$

The projection $X \cdots \to \mathbf{P}^3$ from p is a birational map. The indeterminacy set of its inverse map is defined by

$$q(v, w, x, y) = c(v, w, x, y) = 0 \quad \text{in } \mathbf{P}^3(v{:}w{:}x{:}y)\,,$$

which we denote by $\mathcal{I}(X, p)$. We note that $\dim \mathcal{I}(X, p) = 1$ if and only if X is irreducible and p is not a triple point.

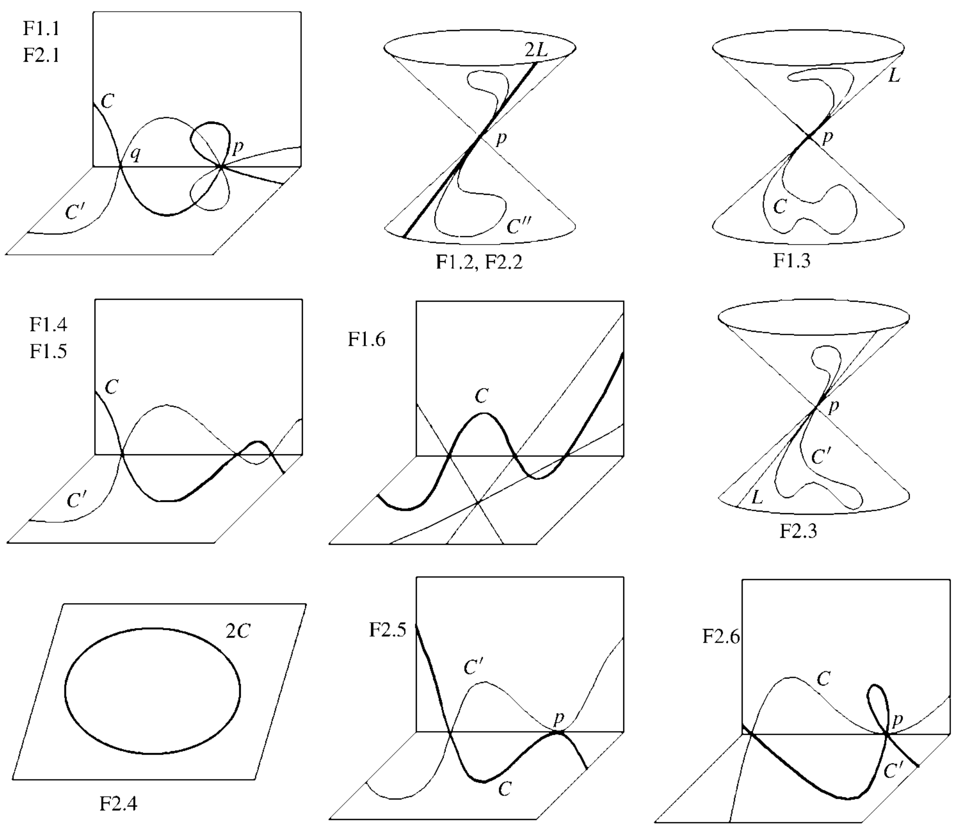

FIGURE.

(2.13) In Figure, the illustrations of the sextic curve F$i.j$ are corresponding to general cubic 3-folds defined by $[i.j]$.

F1.4 and F1.5 are two plane cubics C and C' with common three points. F1.4 and F1.5 become F1.6 if C' degenerates the union of 3 lines. F1.4 and F1.5 become F2.4 if $C = C'$. F1.4 and F1.5 become F2.5 if C and C' touch at a point p. F2.5 becomes F2.6 if C is singular at p. F2.6 becomes F1.1 and F2.1 if C' is also singular at p.

F1.3 is a complete intersection C of a quadratic cone Q with a vertex q and a cubic surface S passing through q such the tangent cone of $Q \cap S$ at q is a double line L, and $L \cap S$ is a (triple) point. F1.3 becomes F2.3 if C degenerate the union of a line L and a curve C' with $L \cap C' = \{p\}$. F2.3 becomes F1.2 and F2.2 if $C' = L \cup C''$.

3. Closed orbits of cubic 3-folds.

In this section we consider cubic 3-fold with a closed orbit in $\mathrm{Sym}^3\mathbf{C}^5$ and prove the following theorem, where $\mathrm{Sym}^d\mathbf{C}^n$ is the family of homogeneous polynomials in n variables of degree d. As a corollary we obtain (3) in Main Theorem.

(3.1) THEOREM. (1) *A semi-stable cubic* 3*-fold is contained in a closed orbit in* $\mathrm{Sym}^3\mathbf{C}^5$ *if and only if either it is stable or its defining equation is projectively equivalent to*:

$$[3.1] \qquad \phi_{\alpha,\beta} = vy^2 + w^2z - vxz - \alpha wxy + \beta x^3 \quad \text{with} \quad (\alpha, \beta) \neq (0,0) \quad \text{or}$$

$$[3.2] \qquad \varphi = vwz + x^3 + y^3\,.$$

(2) *Equations $\phi_{\alpha,\beta}$ and $\phi_{\alpha',\beta'}$ are equivalent under the* SL(5)*-action if and only if* $\alpha^2 : \beta = \alpha'^2 : \beta'$.

First we show the 'only if' part of (1) in (3.1).

(3.2) PROPOSITION. *If a semi-stable cubic* 3*-fold X is contained in a closed orbit, then either it is stable or its defining equation is projectively equivalent to $\phi_{\alpha,\beta}$ or φ.*

PROOF. Since X is not stable, we may assume that the defining equation f is either [1.3] or [1.5] by (2.3). If $f = [1.5]$, then

$$\lim_{t\to 0} \gamma^5(t)(f) = \lim_{t\to 0} \mathrm{diag}\,(t, t, 1, 1, t^{-2})(f) = zq(v, w) + c(0, 0, x, y) \equiv vwz + x^3 + y^3 = \varphi$$

because X is semi-stable. If $f = [1.3]$, then

$$\lim_{t\to 0} \gamma^3(t)(f) = \lim_{t\to 0} \mathrm{diag}\,(t^2, t, 1, t^{-1}, t^{-2})(f) = a_1vy^2 + a_2w^2z + a_3vxz + a_4wxy + a_5x^3$$

which is projectively equivalent to $\phi_{\alpha,\beta}$ for some α and β by (3.3). □

We put $\mathcal{S}(\sum a_{ij\cdots k}v^iw^j\cdots z^k) = \{(i, j, \cdots, k) \in \mathbf{I} \mid a_{ij\cdots k} \neq 0\}$.

(3.3) LEMMA. *If a homogeneous cubic polynomial $f = a_1vy^2 + a_2w^2z + a_3vxz + a_4wxy + a_5x^3$ is semi-stable, then $f \equiv \phi_{\alpha,\beta}$ for some $(\alpha, \beta) \neq (0,0)$.*

PROOF. We show that $a_1, a_2, a_3 \neq 0$ and $(a_4, a_5) \neq (0, 0)$. If $a_4 = a_5 = 0$, then

$$\mathcal{S}(f) = \{(0,2,0,0,1),\ (1,0,0,2,0),\ (1,0,1,0,1)\} \subset \mathbf{M}^+(\lambda^2)\,,$$

so f is not semi-stable. Hence we must have that $(a_4, a_5) \neq (0, 0)$. Similarly $a_1, a_2, a_3 \neq 0$ follows from

$$\{(0,0,3,0,0),\ (0,1,1,1,0),\ (0,2,0,0,1),\ (1,0,1,0,1)\} \subset \mathbf{M}^+(\lambda^1),$$
$$\{(0,0,3,0,0),\ (0,1,1,1,0),\ (1,0,0,2,0),\ (1,0,1,0,1)\} \subset \mathbf{M}^+(\lambda^6),$$
$$\{(0,0,3,0,0),\ (0,2,0,0,1),\ (1,0,0,2,0),\ (1,0,1,0,1)\} \subset \mathbf{M}^+(\lambda^5).$$

Our lemma follows from (3.4) below immediately. □

(3.4) LEMMA. *For any non-zero constant k, the diagonal transformation*

$$\operatorname{diag}\,(k, (ka_2)^{-1/2}, (k^2a_3)^{-1}, (ka_1)^{-1/2}, k)\,,$$

sends the polynomial $f = a_1vy^2 + a_2w^2z + a_3vxz + a_4wxy + a_5x^3$ to

$$vy^2 + w^2z + vxz + (k^{-3}a_1^{-1/2}a_2^{-1/2}a_3^{-1}a_4)wxy + (k^{-6}a_3^{-3}a_5)x^3\,.$$

The following Lemmas are useful to show (3.7).

(3.5) LEMMA (Luna's criterion [4] or [9, 6.17]). *Suppose that a reductive group G acts on an affine variety X, H is a reductive subgroup of G, and x belongs to the set X^H of fixed points of H. Then the following are equivalent*:

(1) *the orbit Gx is closed*;
(2) *the orbit $N_G(H)x$ over the normalizer is closed*;
(3) *the orbit $Z_G(H)x$ over the centralizer is closed.*

(3.6) LEMMA ([9, 6.15]). *Suppose that T is an algebraic torus acting linearly on a finite-dimensional vector space V and $v \in V$ be a vector. Then the following conditions are equivalent*:

(1) *the orbit Tv is closed in V*;
(2) 0 *is an interior point of the set* $\operatorname{supp} v$ *in $X(T) \otimes_{\mathbf{Z}} \mathbf{Q}$, where $X(T)$ is the group of character of T.*

In our case, $X(T)$ is $\mathbf{Z}^{\oplus 5}/\mathbf{Z}(1,1,1,1,1)$ and $\operatorname{supp}\varphi$ is the convex hull of $\mathcal{S}(\varphi)$ in $X(T) \otimes_{\mathbf{Z}} \mathbf{Q}$. Obviously the composite $\mathbf{I} \hookrightarrow \mathbf{Z}^{\oplus 5} \to X(T)$ is injective. We begin to prove the 'if part' of (1) in (3.1).

(3.7) PROPOSITION. *The orbits of $\phi_{\alpha,\beta}$ and φ are closed. Hence they are semi-stable.*

PROOF. First we show that the orbit of φ is closed. Put

$$H = \{\operatorname{diag}\,(t^2, t, \omega, \omega^2, t^{-3}) \mid t \in \mathbf{C}^*,\ \omega^3 = 1\}\,.$$

Then $\phi_{\alpha,\beta}$ is invariant under H and the center $Z_G(H)$ is the maximal torus

$$T = \{\operatorname{diag}\,(t_0, t_1, t_2, t_3, t_4) \mid t_0t_1\cdots t_4 = 1\}\,.$$

By Luna's criterion (3.5), it is enough to check that $T\varphi$ is closed, which is equivalent to that 0 is an interior point of the set supp φ by (3.6). This is immediately from

$$\mathcal{S}(\varphi) = \{(1,1,0,0,1),\ (0,0,3,0,0),\ (0,0,0,3,0)\} \quad \text{and}$$

$$3(1,1,0,0,1) + (0,0,3,0,0) + (0,0,0,3,0) = (3,3,3,3,3) = 0 \quad \text{in } X(T)\,.$$

The closedness of the orbit of $\phi_{\alpha,\beta}$ similarly follows if we put

$$H = \{\gamma^3 = \operatorname{diag}(t^2, t, 1, t^{-1}, t^{-2}) \mid t \in \mathbf{C}^*\}. \qquad \square$$

We complete the proof of (3.1). For the symbol $\mathcal{I}(X, p)$ refer to (2.12).

(3.8) PROOF OF (2) IN (3.1). If $\alpha_1^2 : \beta_1 = \alpha_2^2 : \beta_2$, then ϕ_{α_1,β_1} and ϕ_{α_2,β_2} are isomorphic by (3.4). To prove the converse, in view of (3.4), it is enough to show that if $\phi_{2,1-t^2}$ and $\phi_{2,1-s^2}$ are isomorphic then $t = \pm s$.

We note that singular points of cubic 3-fold X defined by $\phi_{2,1-t^2}$ are only $p = (1{:}0{:}0{:}0{:}0)$ and $q = (0{:}0{:}0{:}0{:}1)$ for $t \neq 0$. Since $\mathcal{I}(X, p)$ is defined by $y^2 - xz = w^2 z - 2wxy + (1 - t^2)x^3 = 0$, the projection from $\overline{q} = (0{:}0{:}0{:}1)$ is defined by the following polynomial in $\mathbf{P}^2(w{:}x{:}y)$

$$w^2y^2 - 2wyx^2 + (1-t^2)x^4 = \{wy - (1+t)x^2\}\{wy - (1-t)x^2\}\,.$$

Hence its image consists of two conics. If $\phi_{2,1-t^2} \equiv \phi_{2,1-s^2}$, then for a suitable linear transformation σ we have either:

$$\text{(i)} \quad \begin{cases} \sigma\{wy-(1+t)x^2\} = wy-(1-s)x^2 \\ \sigma\{wy-(1-t)x^2\} = wy-(1+s)x^2 \end{cases} \quad \text{or} \quad \text{(ii)} \quad \begin{cases} \sigma\{wy-(1+t)x^2\} = wy-(1+s)x^2 \, (1) \\ \sigma\{wy-(1-t)x^2\} = wy-(1-s)x^2 \, (2) \end{cases}$$

In case (ii) we have from $(1) \times (1-t) - (2) \times (1+t)$

$$-2t\sigma(wy) = -2twy + 2(t-s)x^2\,.$$

Since σ preserves the rank of a quadric form, we obtain $t = s$.

In case (i) we similarly have $t = -s$. $\square$

(3.9) If $\alpha^2 = 4\beta$, then the space sextic $\mathcal{I}(X, p)$ of $\phi_{\alpha,\beta}$ is a twisted cubic with a double structure and X is the secant variety of rational normal curve R_4 in $\mathbf{P}^4$, it is called *the secant 3-fold* in [1]. Since the stabilizer group of X coincides with that of the rational curve R_4, it is PGL (2). If $\phi_{\alpha,\beta}$ defines the secant 3-fold, then its singular locus is the curve R_4. Otherwise, its singular locus consists of 2 points (1:0:0:0:0), (0:0:0:0:1).

4. Non-isolated singular loci of cubic 3-folds.

In this section we consider cubic 3-fold with non-isolated singular locus and prove the following theorem.

(4.1) THEOREM. *Let X be a cubic 3-fold with non-isolated singular locus. Then the following conditions are equivalent*:

(i) *X is semi-stable*;

(ii) *X is the secant* 3-*fold*;

(iii) *X has only double points of type* $A_\infty : v^2 + w^2 + x^2 = 0$.

We begin to list up the defining equation of cubic 3-folds with non-isolated singular locus.

(4.2) PROPOSITION. *Let X be an irreducible cubic* 3-*fold.*

(i) *If* Sing$X$ *contains a surfaces S*, *then S is a plane and X is defined by*

$$[4.0] \quad vq_1(y, z) + wq_2(y, z) + xq_3(y, z) + c(y, z)\,.$$

(ii) *Assume that* Sing$X$ *does not contains a surface. If* Sing$X$ *contains a curve C, then it is either a line*, *a conic or a rational normal curve of degree* 4 *and X is defined by one of the following*:

$$[4.1] \quad vq_1(x, y, z) + wq_2(x, y, z) + c(x, y, z)\,;$$

$$[4.2] \quad c(y, z) + vq_1(y, z) + wq_2(y, z) + xq_3(y, z) + l(y, z)q(v, w, x)\,;$$

$$[4.3] \quad vy^2 + w^2z - vxz - 2wxy + x^3 \text{ (the secant 3-fold)}\,.$$

PROOF. (i) Put $d = \deg S \geq 1$. Cutting X by 2 general hyperplanes, we obtain an irreducible cubic curve with d singular points and $p_a = 1$. Since $d \leq p_a$, $d = 1$ and we have [4.0].

(ii) If C is a line $x = y = z = 0$, then we have [4.1] immediately. If C lies in a plane $P : y = z = 0$, then it is defined by $y = z = f(v, w, x) = 0$. If $\deg f > 2$, then we have $P \subset \mathrm{Sing}X$, because $F_v, F_w, F_x, F_y, F_z \in (y, z, f(v, w, x))$. Hence we have $\deg f(v, w, x) = 2$. If we write

$$F = c_0(y, z) + vq_1(y, z) + wq_2(y, z) + xq_3(y, z) + yq_3(v, w, x) + zq_4(v, w, x) + c(v, w, x)\,,$$

then we have $F_y = F_z = 0$ because $F_y = q_3(v, w, x),\ F_z = q_4(v, w, x) \in (y, zf(v, w, x))$. Since the cubic curve $\{c(v, w, x) = 0\}$ in $\mathbf{P}^3(v{:}w{:}x)$ is singular along conic $\{f(v, w, x) = 0\}$, we have $c(v, w, x) = 0$. Hence we obtain $F = [4.2]$.

Assume C spans a 3-space, say, $z = 0$. We may suppose that C contains 4 points (1:0:0:0:0), (0:1:0:0:0), (0:0:1:0:0), (0:0:0:1:0). So we can write

$$F = a_1vwx + a_2vwy + a_3vxy + a_4wxy + zq(v, w, x, y, z)\,.$$

Since $a_4F_v - a_3F_w \mid_{z=0} = (a_1x + a_2y)(a_4w - a_3v)$, C is a plane curve. This is a contradiction.

Assume that C is a non-degenerate curve of degree d. Then we have $d \geq 4$. Since the general hyperplane section of X is a cubic surface with d singular points, we have $d = 4$ by (4.3) below and C is a rational normal curve of degree 4. Since a cubic 3-fold X contains the secant variety of Sing$X$, X is the secant 3-fold (of C). □

(4.3) LEMMA (See [2, p. 644] for example). *A cubic surfaces cannot contain more than 4 isolated singularities.*

In [13] we list up the defining equations of cubic n-folds X's with codimSing$X \leq 3$.

(4.4) PROPOSITION. *Let X be a cubic 3-fold with non-isolated singular locus. If it is not the secant 3-fold, then it has a non-isolated double point of rank ≤ 2. In particular, if* $\mathrm{Sing}X$ *contains either a line or a conic, then X has a non-isolated double point of rank ≤ 2.*

PROOF. If X is reducible, then we may assume $F = z\{vl(w, x, y, z) + q(w, x, y, z)\}$. Hence $p = (1{:}0{:}0{:}0{:}0)$ is a double point of rank < 3 and $p \in \mathrm{Sing}X$.

Assume that X is irreducible. Then F is either [4.0], [4.1] or [4.2] according to (4.2). [4.0] has a non-isolated double point of rank 2 at (1:0:0:0:0). [4.1] has a double point of rank 2 on the line $x = y = z = 0$ by (2.11).

We show [4.2] has a non-isolated double point of rank ≤ 2. Since $vq_1(y, z)+wq_2(y, z)+xq_3(y, z) \equiv vq_1'(y, z)+wyl_1(y, z)+xyl_2(y, z)$ by a suitable linear transformation $(v, w, x) \mapsto (l(v, w, x), l'(v, w, x), l''(v, w, x))$, we have

$$\begin{aligned} F &\equiv c(y, z) + vq_1(y, z) + wq_2(y, z) + xq_3(y, z) + yq(v, w, x) \\ &\equiv c(y, z) + vq_1'(y, z) + wyl_1(y, z) + xyl_2(y, z) + yq(v, w, x) \\ &= c(y, z) + vq_1'(y, z) + y\{wl_1(y, z) + xl_2(y, z) + q(v, w, x)\} \\ &\equiv c(y, z) + vq_1'(y, z) + y\{wl_1'(y, z) + xl_2'(y, z) + awx + vl_3(v, w, x)\} \\ &\qquad \text{by } (w, x) \mapsto (l_4(w, x), l_5(w, x)) . \end{aligned}$$

Hence both $p = (0{:}1{:}0{:}0{:}0)$ and $q = (0{:}0{:}1{:}0{:}0)$ are double points of rank < 3 and

$$p,\ q \in \{y = z = awx + vl_3(v, w, x) = 0\} \subset \mathrm{Sing}X . \quad \square$$

(4.5) PROOF OF (4.1). Since a non-isolated double point of rank 2 is of type $v^2 + w^2 + x^n$ $(n > 2)$, (4.4) means that (iii) implies (ii) in (4.1). According to (3.9), the converse is obvious.

We show (i) and (ii) are equivalent. The secant 3-fold is semi-stable by (3.7). It is enough to check that any cubic 3-folds defined by [4.0], [4.1] and [4.2] is unstable from (4.2). Since

$$[4.0] \subset \mathbf{M}^+([-2, -2, -2, 3, 3]) \quad \text{and} \quad [4.1] \subset \mathbf{M}^+([-3, -3, 2, 2, 2]) ,$$

[4.0] and [4.1] are unstable. Since

$$[4.2] \equiv c(y, z)+vq_1(y, z)+wq_2(y, z)+xq_3(y, z)+zq(v, w, x) \subset \mathbf{M}^+([-3, -3, -3, 2, 7]) ,$$

[4.2] is also unstable. $\square$

5. Analytic local characterization of stability.

So far we characterized stability in terms of global equations. Now we translate them into local analytic conditions and prove Main Theorem. If F and G are analytically isomorphic, then we denote $F \sim G$.

Let X be a cubic 3-fold which has a double point at p =(0:0:0:0:1). Then the defining affine equation at p is $F = q(v, w, x, y) + c(v, w, x, y)$. If the quadratic part q is of rank 4, then we have $F \sim v^2 + w^2 + x^2 + y^2$, and, p is called of type A_1. If rank $q = 3$, then we have $F \sim v^2 + w^2 + x^2 + ay^{n+1}$. When $a \neq 0$, p is of type A_n with $n \geq 2$. Otherwise p is of type A_∞. As for simple singularity, we refer to [14, Ch. 3] for example.

First we describe [1.3] in terms of the curve $\mathcal{I}(X, p)$ defined in (2.12).

(5.1) PROPOSITION. *Let X be a singular cubic* 3*-fold which has no double points of rank* < 3. *Assume $\mathcal{I}(X, p_0)$ is an intersection of a quadric Q and cubic surface S such that*

(i) *Q is a cone with a vertex p,*

(ii) *S passes through p,*

(iii) *the intersection of Q and the tangent space T_pS of S at p is a* (*double*) *line L,*

(iv) *$L \cap S = \{p\}$.*

Then X is defined by [1.3].

PROOF. We take coordinate $(v{:}w{:}x{:}y{:}z)$ such that p_0 =(0:0:0:0:1) as in (2.12) and such that $L : v = w = 0$ and p =(0:0:0:1). By (i), we have

$$Q : q(v, w, x) = 0\,, \quad S : c(v, w, x, y) = 0\,.$$

By (ii), $c(v, w, x, y)$ does not contain y^3, that is,

$$Q : q(v, w, x) = 0\,, \quad S : y^2 l(v, w, x) + yq'(v, w, x) + c(v, w, x) = 0\,.$$

We note that $l(v, w, x) \neq 0$, otherwise Sing$X$ contains a line $v = w = x = 0$, which contradicts to our assumption by (4.4). Since T_pS is $\{l(v, w, x) = 0\}$ and $Q \cap T_pS = L$ by (iii), we have

$$\{l(v, w, x) = q(v, w, x) = 0\} = \{v = w = 0\}\,,$$

which implies that $l(v, w, x) = l(v, w)$ and $q(v, w, x) = q(v, w) + xl(v, w)$. By $l(v, w) \mapsto v$, we have

$$Q : q(v, w) + vx = 0\,, \quad S : vy^2 + yq'(v, w, x) + c(v, w, x) = 0\,. \qquad (*)$$

By (iv), $q'(v, w, x)$ does not contain x^2, that is, $q'(v, w, x) = xl'(v, w) + q'(v, w)$. Therefore, we have

$$Q : q(v, w) + xl(v, w) = 0\,, \quad S : y^2 l(v, w) + yxl'(v, w) + yq'(v, w) + c(v, w, x) = 0\,,$$

$$F = z\{q(v, w) + xl(v, w)\} + y^2 l(v, w) + yxl'(v, w) + yq'(v, w) + c(v, w, x) = [1.3]\,. \quad \square$$

Now we translate this into the analytic local condition.

(5.2) PROOF OF (1) IN MAIN THEOREM. Let X be a cubic 3-fold which has no double points of rank < 3. By (5.4) and (5.3) below, X is not defined by [1.3] if and only if X has only the double points of type either A_1, A_2, A_3 or A_4. Since [1.5] is equivalent to the existence of a double point of rank ≤ 2, we have proved (1) in Main Theorem.

(5.3) LEMMA. *If X is not defined by* [1.3], *then any double point p of rank* 3 *on X is of type either A_2, A_3 or A_4.*

PROOF. Let F be the affine defining equation of X at p =(0:0:0:0:1). According to (5.1), all of (i) to (iv) do not hold. If (i) does not hold, then $q(v, w, x, y)$ is of rank 4. If (i) holds but (ii) dose not, then we have

$$\begin{aligned} F &= v^2 + w^2 + x^2 + y^3 + l(v, w, x)y^2 + q(v, w, x)y + c(v, w, x) \\ &\sim v^2 + w^2 + x^2 + y^3 , \end{aligned}$$

which is of type A_2. If (i) and (ii) hold but (iii) does not, then from (i) and (ii) we have

$$F = v^2 + w^2 + x^2 + 2(av + bw + cx)y^2 + q(v, w, x)y + c(v, w, x) .$$

Since $\{v^2 + w^2 + x^2 = av + bw + cx = 0\}$ is not a (double) line by (iii), we obtain $a^2 + b^2 + c^2 \neq 0$. Hence we have

$$\begin{aligned} F &= (v + ay^2)^2 + (w + by^2)^2 + (x + cy^2)^2 - (a^2 + b^2 + c^2)y^4 + q(v, w, x)y + c(v, w, x) \\ &\sim v^2 + w^2 + x^2 - (a^2 + b^2 + c^2)y^4 \quad \text{by } (v + ay^2, w + by^2, x + cy^2) \mapsto (v, w, x) \\ &\sim v^2 + w^2 + x^2 + y^4 , \end{aligned}$$

which is of type A_3.

Finally assume that (i), (ii) and (iii) hold but (iv) does not. Then from (i), (ii) and (iii), we have

$$F = \{q(v, w) + vx\}z + vy^2 + yq'(v, w, x) + c(v, w, x)$$

by $(*)$ in the proof of (5.1). Since (iv) does not hold, $q'(v, w, x)$ contains x^2. Hence we have

$$\begin{aligned} F &= \{w^2 + vl_0(v, w, x)\} + y^2v + \{yx^2 + yxl'(v, w) + yq'(v, w)\} + c(v, w, x) \\ &\equiv \{vx + w^2\} + vy^2 + xyl_1(v, w) + yq_1(v, w) + c(v, w, x) + x^2y \quad \text{by } l_0(v, w, x) \mapsto x \\ &= w^2 + v(x + y^2) + xyl_1(v, w) + yq_1(v, w) + c(v, w, x) + x^2y \\ &\sim w^2 + vx + (x - y^2)yl_1(v, w) + yq_1(v, w) + c(v, w, x - y^2) + (x - y^2)^2y \quad \text{by } x + y^2 \mapsto x \\ &\sim w^2 + vx + y^5 \sim v^2 + w^2 + x^2 + y^5, \end{aligned}$$

which is of type A_4. □

(5.4) LEMMA. *If X is defined by* [1.3], *then there exists a double point p of type A_n with $5 \leq n \leq \infty$.*

PROOF. Let F be the affine equation of [1.3] at p =(0:0:0:0:1), that is

$$F \equiv w^2 + vx + a_0vy^2 + q(v, w)y + l(v, w)xy + c(v, w, x) .$$

If $a_0 = 0$, then X is singular along the line $v = w = x = 0$. Hence X has a double point of rank < 3 by (4.4). If $a_0 \neq 0$, then we have

$$\begin{aligned} F &\equiv w^2 + vx + vy^2 + q(v, w)y + l(v, w)xy + c(v, w, x) \\ &= w^2 + v(x + y^2) + q(v, w)y + l(v, w)xy + c(v, w, x) \\ &\sim w^2 + vx + q(v, w)y + l(v, w)(x - y^2)y + c(v, w, x - y^2) \quad \text{by } x + y^2 \mapsto x \\ &\sim w^2 + vx + ay^{n+1} \sim v^2 + w^2 + x^2 + ay^{n+1} \end{aligned}$$

with $n \geq 5$. □

Now we complete the proof of (1) in Main Theorem. Next we consider the analytic local condition of *semi-stability*.

(5.5) PROPOSITION. *Let X be a singular cubic 3-fold which has no double points of rank < 3. Assume that $\mathcal{I}(X, p_0)$ is an intersection of a quadric Q and cubic surface S such that*

(i) *Q is a cone with vertex p,*
(ii) *$Q \cap S$ consists of a line L and a quintic C,*
(iii) *$C \cap L = \{p\}$.*

Then X is defined by [2.3].

PROOF. We take coordinate $(v{:}w{:}x{:}y{:}z)$ such that p =(0:0:0:0:1) $L : v = w = 0$ and p =(0:0:0:1) as in the proof of (5.1). By (i), we have

$$Q : q(v, w, x) = 0\,, \quad S : c(v, w, x, y) = 0\,.$$

Since $L \subset S$ and $L \subset Q$ by (ii), we can write

$$Q : q(v, w) + vx = 0\,, \quad S : c(v, w) + xq_1(v, w) + yq_2(v, w) + vq_3(x, y) + wq_4(x, y) = 0\,.$$

Since $\mathrm{Sing}(Q \cap S) = \{p\}$ by (iii), we have

$$\mathrm{rank}\left.\begin{pmatrix} Q_v & S_v \\ Q_w & S_w \\ Q_x & S_x \\ Q_y & S_y \end{pmatrix}\right|_{v=w=0} = \mathrm{rank}\begin{pmatrix} x & q_3(x, y) \\ 0 & q_4(x, y) \\ 0 & 0 \\ 0 & 0 \end{pmatrix} < 2 \Leftrightarrow x = 0\,.$$

Hence $q_4(x, y) = ax^2$ and

$$Q : q(v, w) + vx = 0\,, \quad S : c(v, w) + xq_1(v, w) + yq_2(v, w) + vq_3(x, y) + awx^2 = 0\,.$$

Therefore, we obtain

$$F = z\{q(v, w) + vx\} + c(v, w) + xq_1(v, w) + yq_2(v, w) + vq_3(x, y) + awx^2 = [2.3]\,. \quad \square$$

We translate this to the analytic local condition.

(5.6) PROPOSITION. *Let X be a cubic 3-fold which has no double points of rank < 3. Then X is not defined by* [2.3] *if and only if X has only double points of type A_1, A_2, A_3, A_4, A_5 or A_∞.*

PROOF. If X has only isolated double points, then our theorem follows (5.7) and (5.8) below immediately. Assume that X has non-isolated singular locus. Then it is not defined by [2.3] if and only if it is semi-stable by (2.3), which is also equivalent to that it has only A_∞ singular points by (4.4).

(5.7) LEMMA. *If it is not defined by* [2.3], *then any double point p of rank 3 on X is of type A_n ($n = 2, 3, 4, 5$ or ∞).*

PROOF. Let F be the affine defining equation of X at p =(0:0:0:0:1). If (i) in (5.5) does not hold, then p is of rank 4.

Assume (i) holds but (ii) does not hold. By (i), we can put

$$\begin{aligned}F &= v^2 + w^2 + x^2 + c(v, w, x, y)\\ &= v^2 + w^2 + x^2 + a_0 y^3 + (a_1 v + a_2 w + a_3 x)y^2 + q_0(v, w, x)y + c_0(v, w, x)\,.\end{aligned}$$

If $a_0 \neq 0$, then we have $F \sim w^2 + v^2 + x^2 + y^3$. If $a_0 = a_1 = a_2 = a_3 = 0$, then Sing$X$ contains a line $v = w = x = 0$, which is a contradiction by (4.4). If $a_0 = 0$ and $a_1^2 + a_2^2 + a_3^2 \neq 0$, then we have $F \sim v^2 + w^2 + x^2 + y^4$.

If $a_0 = 0$ and $a_1^2 + a_2^2 + a_3^2 = 0$, then we may assume $a_1 = 1$. By the linear transform $v + a_2 w + a_3 x \mapsto v$, we have

$$\begin{aligned}F &\equiv -(a_3 w - a_2 x)^2 + v(v - 2a_2 w - 2a_3 x) + v^2 y + q_1(v, w, x) + c_1(v, w, x)\\ &\equiv w^2 - vx + vy^2 + q(v, w, x)y + c(v, w, x)\\ &= w^2 - v(x - y^2) + q(v, w, x)y + c(v, w, x)\\ &\sim w^2 - vx + q(v, w, x + y^2)y + c(v, w, x + y^2) \quad \text{by } x - y^2 \mapsto x\\ &\sim w^2 - vx + ay^{n+1} \sim v^2 + w^2 + x^2 + ay^{n+1}\,,\end{aligned}$$

where n is either 4 or 5 because $q(0, 0, x)y + c(0, 0, x) \neq 0$ by the assumption. We note that a can be 0. For example, in the case $F = w^2 - vx + vy^2 - 2wxy + x^3$, we have

$$F = (w - xy)^2 - (x - y^2)(v - x^2) \;\sim\; w^2 - vx \;\sim\; w^2 + v^2 + x^2\,.$$

Finally assume (i) and (ii) hold but (iii) does not hold. By (i) and (ii), we can write

$$F = w^2 - vx + l(v, w)y^2 + q(v, w, x)y + c(v, w, x)$$

where $q(0, 0, x) = c(0, 0, x) = 0$. If $l(v, w) = 0$, then Sing$X$ contains the line $\{v = w = x = 0\}$, hence $l(v, w) \neq 0$ by (4.4). If $l(v, w)$ contains w, then

$$\begin{aligned}F &\equiv w^2 - vx + 2wy^2 + \cdots = (w - y^2)^2 - vx - y^4 + \cdots\\ &\sim w^2 - vx - y^4 \;\sim\; w^2 + v^2 + x^2 + y^4\,.\end{aligned}$$

If $l(v, w) = v$, then $q(v, w, x)$ contains wx since (iii) does not hold. So we have

$$\begin{aligned}F &= w^2 - vx + vy^2 + 2wxy + \cdots = (w + xy)^2 - v(x - y^2) - x^2y^2 + \cdots\\ &\sim w^2 - vx + (x + y^2)^2y^2 + \cdots \quad \text{by } (w + xy, x - y^2) \mapsto (w, x)\\ &\sim w^2 - vx + y^6 \;\sim\; w^2 + v^2 + x^2 + y^6\,. \quad \square\end{aligned}$$

(5.8) LEMMA. *If X is defined by* [2.3], *then there exists a double point is of type A_7.*

PROOF. Since X has no double point of rank < 3, we have

$$\begin{aligned}[2.3] &= z\{a_1 w^2 + v l_1(v, w, x)\} + a_2 v y^2 + \{q_1(v, w) + a_3 v x\}y + c(v, w) + x q_2(v, w) + x^2 l_2(v, w)\\ &\equiv z(w^2 - vx) + vy^2 + \{q_1'(v, w) + a_2' vx\}y + c'(v, w) + q_2'(v, w)x + (a_3' v + 2a_4' w)x^2\,.\end{aligned}$$

We claim $a_4 \neq 0$. Otherwise Sing$X$ contains a conic $v = w = -zx + y^2 + a_1xy + a_3x^2 = 0$, so X has a double point of rank 2 by (4.4). Put $z = 1$. Then we have

$$\begin{aligned}[2.3] &\equiv w^2 - vx + vy^2 + 2wx^2 + \cdots \\ &= (w + x^2) - v(x - y^2)^2 - x^4 + \cdots \\ &\sim w^2 - vx - (x + y^2)^4 + \cdots \quad \text{by } (w + x^2, x - y^2) \mapsto (w, x) \\ &\sim w^2 - vx - y^8 \ \sim \ v^2 + w^2 + x^2 + y^8\,,\end{aligned}$$

which is of type A_7. □

Now we have completed the proof of (5.6). In the proof of (5.8), we have in fact proved the following:

(5.9) PROPOSITION. *If the double point* $(0:0:0:0:1)$ *on* [2.3] *is of rank* 3, *then it is of type either* A_7 *or* A_∞.

Finally we prove (2) in Main Theorem.

(5.10) LEMMA. *Assume that a cubic* 3*-fold* X *has no double points of rank* < 2. *Then* X *is not defined by* [2.5] *if and only if any double point* p *of rank* 2 *on* X *is of type* D_4.

PROOF. Suppose X is not defined by [2.5]. Let p be an arbitrary double point of rank 2 on X. We may assume that p =(0:0:0:0:1) and the defining equation of X is $F = zq'(v, w) + c'(v, w, x, y)$. Since X is not defined by

$$[2.5] = zq_2(v, w) + y^2l(v, w) + yq_1(v, w, x) + c(v, w, x)\,,$$

then $c'(0, 0, x, y) = 0$ has no double roots. Since

$$\begin{aligned}F|_{z=1} &= v^2 + w^2 + c_0(v, w) + vq_3(x, y) + wq_4(x, y) + c'(0, 0, x, y) \\ &\sim v^2 + w^2 + c'(0, 0, x, y) + \cdots\,,\end{aligned}$$

p is of type $D_4 : v^2 + w^2 + x^3 + y^3$.

If X is defined by [2.5], then (0:0:0:0:1) is a double point of rank 2 and not of type D_4. This shows 'if' part. □

(5.11) PROOF OF (2) IN MAIN THEOREM. In view of (2.3), a cubic 3-fold X is semi-stable if and only if it is defined by neither [2.3] nor [2.5]. In the case that X has no double points of rank < 3, X is semi-stable if and only if it has only double points of type A_n (n =1, 2, 3, 4, 5 or ∞) by (5.6).

In the case that X has no double points of rank < 2, X is semi-stable if and only if X is not defined by [2.5] by (5.12) below. Hence the first half of (2) follows from (5.10). The latter half of (2) is already proved in (4.1).

(5.12) LEMMA. *If* [2.3] *has a double point of rank* 2, *then it is a special case of* [2.5].

PROOF. If p =(0:0:0:0:1) is a double point of rank 2, then our claim is obvious. Suppose that there exists a double point $q \neq p$ of rank 2 on [2.5]. According to (5.13), the defining equation F is either (i) or (ii). In the case (i), F is a special case of [2.5] by (5.10). In the case (ii), since (0:0:0:0:1) is of type neither A_2 nor A_3 by (5.9), y^3 or wy^2 is not contain in $c(v, w, y)$. Hence F is a special case of [2.5] by (5.10). □

(5.13) LEMMA. *If a cubic* 3*-fold has double points of rank* 2 *and* 3, *then its defining equation is projectively equivalent to either*:

$$\text{(i)} \quad wxz + vq(w, x, y) + c(w, x, y) \quad \text{or} \quad \text{(ii)} \quad (vx + w^2)z + c(v, w, y)\,.$$

PROOF. Suppose (1:0:0:0:0) and (0:0:0:0:1) are double points of rank 2 and 3, respectively. Then the defining equation can be written

$$F = vq_1(w, x, y) + zq_2(w, x, y) + vzl_1(w, x, y) + c(w, x, y)\,,$$

If $l_1(w, x, y) = 0$, then we have

$$vq_1(w, x, y) + zq_2(w, x, y) + c(w, x, y) \equiv wxz + vq'(w, x, y) + c'(w, x, y)\,,$$

which is of type (i).

If $l_1(w, x, y) \neq 0$, then we may assume that $l(w, x, y) = x$. Since rank $\{q_1(w, x, y) + xz\} = 2$, we have $q_1(w, x, y) = xl_2(w, x, y)$. Since rank $\{q_2(w, x, y) + vx\} = 3$, we have $q_2(w, x, y) = xl_3(w, x, y) + l_4(w, x, y)^2$. Hence we have

$$\begin{aligned}
F &= vxl_2(w, x, y) + z\{xl_3(w, x, y) + l_4(w, x, y)^2\} + vxz + c(w, x, y) \\
&= vx\{l_2(w, x, y) + z\} + z\{xl_3(w, x, y) + l_4(w, x, y)^2\} + c(w, x, y) \\
&\equiv vxz + z\{xl_3(w, x, y) + l_4(w, x, y)^2\} + c'(w, x, y) \quad \text{by} \ \ l_2(w, x, y) + z \mapsto z \\
&= xz\{v + l_3(w, x, y)\} + l_4(w, x, y)^2 z + c'(w, x, y) \\
&\equiv vxz + w^2 z + c''(w, x, y) \quad \text{by} \ \ (v + l_3(w, x, y), l_4(w, x, y)) \mapsto (v, w) \\
&= (vx + w^2)z + c''(w, x, y),
\end{aligned}$$

which is of type (ii). □

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