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On the Strong-Mixing Property of Skew Product of Binary Transformation on 2-Dimensional Torus by Irrational Rotation

Satoshi TAKANOBU

Kanazawa University

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1. Presentation of Theorem.

Let $(\mathbb{T}, \mathcal{F}, \mathbf{P})$ be 1-dimensional Lebesgue probability space, i.e., $\mathbb{T}$ = 1-dimensional torus $\cong [0, 1)$, $\mathcal{F} = \mathcal{B}(\mathbb{T})$, $\mathbf{P}(dx)$ = 1-dimensional Lebesgue measure on $\mathbb{T}$.

Let T be binary transformation on 2-dimensional torus $\mathbb{T}^2$, i.e., $T(x_1, x_2) = (2x_1, 2x_2)$ (mod 1). T preserves $\mathbf{P}^2 = \mathbf{P} \times \mathbf{P}$, and moreover is strong-mixing (cf. (1)).

DEFINITION 1. For a measurable function $f : \mathbb{T}^2 \to \{-1, 1\}$, we define a transformation T_f on $\mathbb{T}^2 \times \{-1, 1\}$ by

$$T_f(x, \varepsilon) = (Tx, f(x)\varepsilon), \quad (x, \varepsilon) \in \mathbb{T}^2 \times \{-1, 1\}.$$

This T_f preserves $\mu(dxd\varepsilon) := \mathbf{P}^2(dx) \times \frac{1}{2}(\delta_{-1} + \delta_1)(d\varepsilon)$, and is called a *skew product* of T by f.

We are concerned with the strong-mixing property of T_f where f is defined by means of irrational rotation. More precisely, we define $r : \mathbb{T} \to \{-1, 1\}$ by $r(x) = 1_{[0,\frac{1}{2})}(x) - 1_{[\frac{1}{2},1)}(x)$, and for $k \in \mathbb{N}$ and $n_1, \cdots, n_k \in \mathbb{N}$; $n_1 < \cdots < n_k$, we consider $f : \mathbb{T}^2 \to \{-1, 1\}$ as

$$f(x_1, x_2) = r(x_1)r(x_1 + n_1x_2) \cdots r(x_1 + n_kx_2).$$

(We identify the function of $\mathbb{T}$ with the 1-periodic function of $\mathbb{R}$ in an obvious way. Thus r is regarded as a 1-periodic function, so that the f above is well-defined !) Then we have the following:

THEOREM. *Let f be as above. Then T_f is strong-mixing, i.e., for $^\forall A, B \in \mathcal{B}(\mathbb{T}^2 \times \{-1, 1\})$*

$$\lim_{m\to\infty} \mu(A \cap T_f^{-m}B) = \mu(A)\mu(B). \tag{1}$$

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2. Reduction of the problem and its motivation.

By the general theory (e.g. [6, Theorem 1.23]), the strong-mixing property of T_f (where $f : \mathbb{T}^2 \to \{-1, 1\}$ is general) is equivalent to the following convergence: Let $\{\varphi_n\}_{n=1}^{\infty}$ be a CONS of $L^2(\mathbb{T}^2 \times \{-1, 1\} \to \mathbb{C}, \mu)$, then

$$\lim_{m\to\infty} \iint_{\mathbb{T}^2\times\{-1,1\}} \varphi_n(T_f^m(x, \varepsilon))\overline{\varphi_n(x, \varepsilon)}\mu(dxd\varepsilon) = \left| \iint_{\mathbb{T}^2\times\{-1,1\}} \varphi_n(x, \varepsilon)\mu(dxd\varepsilon)\right|^2, \quad {}^{\forall} n \in \mathbb{N}.$$

In the present case, we can take, as a CONS of $L^2(\mathbb{T}^2 \times \{-1, 1\} \to \mathbb{C}, \mu)$

$$\{e^{\sqrt{-1}2\pi(p_1x_1+p_2x_2)},\ e^{\sqrt{-1}2\pi(q_1x_1+q_2x_2)}\varepsilon\,;\,(p_1, p_2), (q_1, q_2) \in \mathbb{Z}^2\}.$$

For $e^{\sqrt{-1}2\pi(p_1x_1+p_2x_2)}$, the convergence above holds obviously. Therefore we obtain the following criterion for T_f to be strong-mixing:

PROPOSITION 1. *T_f is strong-mixing if and only if, for ${}^{\forall}(p_1, p_2) \in \mathbb{Z}^2$*

$$\lim_{m\to\infty} \int_{\mathbb{T}^2} e^{\sqrt{-1}2\pi(2^m-1)(p_1x_1+p_2x_2)} f(x_1, x_2) f(2x_1, 2x_2)\cdots f(2^{m-1}x_1, 2^{m-1}x_2)dx_1dx_2 = 0.$$

Now let us go to our problem. Our $f : \mathbb{T}^2 \to \{-1, 1\}$ was

$$f(x_1, x_2) = r(x_1)r(x_1 + n_1x_2)\cdots r(x_1 + n_kx_2).$$

For this we introduce $X^{(m)} : \mathbb{T} \to \{-1, 1\}$ by $X^{(m)}(x) := \prod_{j=1}^{m} r(2^{j-1}x)$. Then

$$f(x_1, x_2) f(2x_1, 2x_2)\cdots f(2^{m-1}x_1, 2^{m-1}x_2) = X^{(m)}(x_1)X^{(m)}(x_1 + n_1x_2)\cdots X^{(m)}(x_1 + n_kx_2).$$

Thus, by Proposition 1 we reduce Theorem to the following:

THEOREM 1. *There exists $0 \le \rho = \rho(n_1, \cdots, n_k) < 1$ such that for ${}^{\forall} p \in \mathbb{Z}$*

$$\int_0^1 d\alpha \left| \int_{\mathbb{T}} e^{\sqrt{-1}2\pi(2^m-1)px} X^{(m)}(x)X^{(m)}(x + n_1\alpha)\cdots X^{(m)}(x + n_k\alpha)dx \right| = O(\rho^m) \quad \text{as } m \to \infty.$$

In the next section Theorem 1 will be proved. Before proceeding, we here mention the motivation for the study carried out in this paper. In [1], Sugita presented a pseudo-random number generator by means of irrational rotation, and as its theoretical justification showed the following:

FACT 1. *For almost every $\alpha \in \mathbb{T}$, the stationary process $\{X^{(m)}(\cdot + n\alpha)\}_{n=0}^{\infty}$ on $(\mathbb{T}, \mathbf{P})$ converges in law to the fair coin tossing process (i.e., mean zero $\{-1, 1\}$-valued i.i.d. process), as $m \to \infty$.*

After that, in [3] he tried to simplify the proof and stated the following (, though its statement is weaker than that of Fact 1):

FACT 2. *The stationary process* $\{X^{(m)}(x_1+nx_2)\}_{n=0}^{\infty}$ *on* $(\mathbb{T}^2, \mathbf{P}^2)$ *converges in law to the fair coin tossing process, as* $m \to \infty$.

In the process of showing this fact, he found the strong-mixing property of T_f, and utilizing this property, he clarified the reasons behind the validity of Fact 2. Actually he confirmed its ergodicity and weak-mixing property, and showed an equivalent statement to this fact:

FACT 2′. *For* $\forall k \in \mathbb{N}$ *and* $\forall n_1, \cdots, n_k \in \mathbb{N}$; $n_1 < \cdots < n_k$

$$\lim_{m\to\infty} \int_{\mathbb{T}^2} X^{(m)}(x_1) X^{(m)}(x_1+n_1x_2) \cdots X^{(m)}(x_1+n_kx_2) dx_1 dx_2 = 0\,.$$

Fact 2′ follows obviously from Theorem 1 with $p = 0$; however, his proof is purely of ergodic theory in comparison with ours, and it itself is of independent interest. Our method for proving the theorem is based on a method of cancellation, which is the same as in [1]. But, compared with that in [1], our cancellation is more transparent, so that our proof might be more readily acceptable. In this paper we solve his conjecture by proving the theorem, and at the same time succeed in his scheme of simplifying the proof of Fact 1 !

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3. Proof of Theorem 1.

We begin with the following proposition:

PROPOSITION 2 (cf. [5]). *For each* $m \in \mathbb{N}$

$$X^{(m)}(x) = (\sqrt{-1})^{m-2} \sum_{k=1}^{2^{m-1}} \left(\prod_{i=1}^{m} \sin \frac{2k-1}{2^i}\pi \right) e^{\sqrt{-1}\frac{2k-1}{2^m}\pi} e^{\sqrt{-1}2\pi(2k-1)\frac{[2^m x]}{2^m}}\,.$$

LEMMA 1 (cf. [2]). *Let* $\widehat{X^{(m)}}(n)$ *be the nth Fourier coefficient of* $X^{(m)}$, *i.e.,* $\widehat{X^{(m)}}(n) = \int_{\mathbb{T}} X^{(m)}(x) e^{-\sqrt{-1}2n\pi x} dx$. *Then*

$$\widehat{X^{(m)}}(n) = \begin{cases} 0 & \text{if} \quad n \in 2\mathbb{Z} \\ (\sqrt{-1})^{m-2} \left(\displaystyle\prod_{j=1}^{m} \sin \frac{n\pi}{2^j} \right) \displaystyle\int_0^1 e^{-\sqrt{-1}2\pi \frac{n}{2^m}(x-\frac{1}{2})} dx & \text{if} \quad n \in 2\mathbb{Z}-1\,. \end{cases}$$

PROOF. We define $d : \mathbb{T} \to \{0, 1\}$ by $d(x) = 1_{[\frac{1}{2},1)}(x)$, and set $r_j : \mathbb{T} \to \{-1, 1\}$, $d_j : \mathbb{T} \to \{0, 1\}$ by

$$r_j(x) = r(2^{j-1}x)\,, \; d_j(x) = d(2^{j-1}x)\,.$$

Then $r_j = (-1)^{d_j}$ ($^\forall j \in \mathbb{N}$), $\{d_j\}_{j=1}^\infty$ is an i.i.d. random sequence on $(\mathbb{T}, \mathbf{P})$, and

$$x = \sum_{j=1}^\infty \frac{d_j(x)}{2^j}, \quad {}^\forall x \in \mathbb{T}.$$

Using these facts, we actually compute $\widehat{X^{(m)}}(n)$:

$$\begin{aligned}
\widehat{X^{(m)}}(n) &= \int_{\mathbb{T}} \left(\prod_{j=1}^m r_j(x) \right) e^{-\sqrt{-1}2n\pi \sum_{j=1}^\infty \frac{d_j(x)}{2^j}} dx \\
&= \int_{\mathbb{T}} \prod_{j=1}^m (-1)^{d_j(x)} \prod_{j=1}^m e^{-\sqrt{-1}2n\pi \frac{d_j(x)}{2^j}} e^{-\sqrt{-1}\frac{2n\pi}{2^m}\sum_{j=1}^\infty \frac{d_{j+m}(x)}{2^j}} dx \\
&= \left(\prod_{j=1}^m \int_{\mathbb{T}} (-1)^{d_j(x)} e^{-\sqrt{-1}2n\pi \frac{d_j(x)}{2^j}} dx \right) \int_{\mathbb{T}} e^{-\sqrt{-1}\frac{2n\pi}{2^m}\sum_{j=1}^\infty \frac{d_{j+m}(x)}{2^j}} dx \\
&= \left(\prod_{j=1}^m \frac{1}{2}(1 - e^{-\sqrt{-1}2n\pi \frac{1}{2^j}}) \right) \int_{\mathbb{T}} e^{-\sqrt{-1}\frac{2n\pi}{2^m}x} dx \\
&= e^{-\sqrt{-1}n\pi \sum_{j=1}^m \frac{1}{2^j}} \left(\prod_{j=1}^m \sqrt{-1} \sin \frac{n\pi}{2^j} \right) \int_{\mathbb{T}} e^{-\sqrt{-1}\frac{2n\pi}{2^m}x} dx \\
&= (\sqrt{-1})^m \left(\prod_{j=1}^m \sin \frac{n\pi}{2^j} \right) (-1)^n \int_{\mathbb{T}} e^{-\sqrt{-1}\frac{2n\pi}{2^m}(x-\frac{1}{2})} dx\,.
\end{aligned}$$

From this the conclusion follows at once. □

Proof of Proposition 2. For simplicity we denote the RHS of the identity in Proposition 2 by $Y^{(m)}(x)$. Since $X^{(m)}(x)$ and $Y^{(m)}(x)$ are right continuous in x, it is enough to show that $X^{(m)} = Y^{(m)}$ in $L^2(\mathbb{T}, \mathbf{P})$.

By Lemma 1

$$\begin{aligned}
X^{(m)}(x) &= \sum_{n \in 2\mathbb{Z}-1} (\sqrt{-1})^{m-2} \left(\prod_{j=1}^m \sin \frac{n\pi}{2^j} \right) \int_0^1 e^{-\sqrt{-1}2\pi \frac{n}{2^m}(x-\frac{1}{2})} dx \; e^{\sqrt{-1}2n\pi x} \\
&= \sum_{\substack{q \in \mathbb{Z}, \\ r \in \{1, \cdots, 2^{m-1}\}}} (\sqrt{-1})^{m-2} \left(\prod_{j=1}^m \sin \frac{q2^m + 2r - 1}{2^j}\pi \right) \\
&\quad \times \int_0^1 e^{-\sqrt{-1}2\pi \frac{q2^m+2r-1}{2^m}(x-\frac{1}{2})} dx \; e^{\sqrt{-1}2(q2^m+2r-1)\pi x} \\
&= \sum_{r=1}^{2^{m-1}} \sum_{q \in \mathbb{Z}} (\sqrt{-1})^{m-2} \left(\prod_{j=1}^m \sin \frac{2r-1}{2^j}\pi \right) e^{\sqrt{-1}\frac{2r-1}{2^m}\pi}
\end{aligned}$$

$$\times \int_0^1 e^{-\sqrt{-1}2\pi qx} e^{-\sqrt{-1}2\pi \frac{2r-1}{2^m}\{x\}} dx \ e^{\sqrt{-1}2q\pi 2^m x} e^{\sqrt{-1}2\pi(2r-1)x}$$

[where $\{x\}$ = the fractional part of x]

$$= (\sqrt{-1})^{m-2} \sum_{r=1}^{2^{m-1}} \left(\prod_{j=1}^{m} \sin \frac{2r-1}{2^j}\pi \right) e^{\sqrt{-1}\frac{2r-1}{2^m}\pi} e^{\sqrt{-1}2(2r-1)\pi x}$$

$$\times \sum_{q\in\mathbb{Z}} \int_0^1 e^{-\sqrt{-1}2\pi qx} e^{-\sqrt{-1}2\pi\frac{2r-1}{2^m}\{x\}} dx \ e^{\sqrt{-1}2q\pi 2^m x}$$

$$= (\sqrt{-1})^{m-2} \sum_{r=1}^{2^{m-1}} \left(\prod_{j=1}^{m} \sin \frac{2r-1}{2^j}\pi \right) e^{\sqrt{-1}\frac{2r-1}{2^m}\pi} e^{\sqrt{-1}2(2r-1)\pi x} e^{-\sqrt{-1}2\pi\frac{2r-1}{2^m}\{2^m x\}}$$

$$= (\sqrt{-1})^{m-2} \sum_{r=1}^{2^{m-1}} \left(\prod_{j=1}^{m} \sin \frac{2r-1}{2^j}\pi \right) e^{\sqrt{-1}\frac{2r-1}{2^m}\pi} e^{\sqrt{-1}2\pi\frac{2r-1}{2^m}(2^m x - \{2^m x\})}$$

$$= (\sqrt{-1})^{m-2} \sum_{r=1}^{2^{m-1}} \left(\prod_{j=1}^{m} \sin \frac{2r-1}{2^j}\pi \right) e^{\sqrt{-1}\frac{2r-1}{2^m}\pi} e^{\sqrt{-1}2\pi(2r-1)\frac{[2^m x]}{2^m}}$$

$$= Y^{(m)}(x). \qquad \square$$

DEFINITION 2. Let $k, m \in \mathbb{N}$ and $p \in \mathbb{Z}$. We define $S_p^{(m)} : \mathbb{Z}^k \to \mathbb{R}$ by

$$S_p^{(m)}(l_1, \cdots, l_k) := \left(\prod_{i=1}^{m} \sin \frac{2l_1 - 1 + \cdots + 2l_k - 1 - p}{2^i}\pi \right)$$
$$\times \left(\prod_{i=1}^{m} \sin \frac{2l_1 - 1}{2^i}\pi \right) \times \cdots \times \left(\prod_{i=1}^{m} \sin \frac{2l_k - 1}{2^i}\pi \right), \quad (l_1, \cdots, l_k) \in \mathbb{Z}^k .$$

Clearly $S_p^{(m)}$ is symmetric and $S_p^{(m)}(\cdots, l_j + 2^{m-1}, \cdots) = S_p^{(m)}(\cdots, l_j, \cdots)$. Also $S_p^{(m)} = 0$ if $p \equiv k \pmod 2$.

LEMMA 2. *Let $k, m \in \mathbb{N}$ and $p \in \mathbb{Z}$. Then for $^\forall(\alpha_1, \cdots, \alpha_k) \in \mathbb{R}^k$*

$$\int_{\mathbb{T}} e^{\sqrt{-1}2\pi(2^m-1)px} X^{(m)}(x) X^{(m)}(x+\alpha_1) \cdots X^{(m)}(x+\alpha_k) dx$$

$$= (\sqrt{-1})^{(m-2)(k+1)} (-1)^m \int_0^1 \sum_{1 \le l_1, \cdots, l_k \le 2^{m-1}} S_p^{(m)}(l_1, \cdots, l_k) e^{\sqrt{-1}\frac{p}{2^m}\pi} e^{\sqrt{-1}2\pi p(1-\frac{1}{2^m})y}$$
$$\times \prod_{i=1}^{k} e^{\sqrt{-1}2\pi(2l_i - 1)\frac{[y+2^m\alpha_i]}{2^m}} dy .$$

PROOF. First let $p \equiv k \pmod 2$. In this case, since $X^{(m)}(x+1/2) = -X^{(m)}(x)$,

$$\begin{aligned}
\text{LHS} &= \int_{\mathbb{T}} e^{\sqrt{-1}2\pi(2^m-1)p(x+\frac{1}{2})} X^{(m)}\left(x+\frac{1}{2}\right) X^{(m)}\left(x+\frac{1}{2}+\alpha_1\right)\cdots X^{(m)}\left(x+\frac{1}{2}+\alpha_k\right)dx \\
&\quad \text{[by the shift invariance of } dx\text{]} \\
&= (-1)^{p+1+k}\text{LHS} \\
&= -\text{LHS},
\end{aligned}$$

so that LHS $=0$. On the other hand, since $S_p^{(m)} = 0$, RHS $=0$. Hence we have the conclusion in the case of $p \equiv k \pmod 2$.

Next let $p \not\equiv k \pmod 2$. Since, by Proposition 2

$$\begin{aligned}
&X^{(m)}(x)X^{(m)}(x+\alpha_1)\cdots X^{(m)}(x+\alpha_k) \\
&= (\sqrt{-1})^{(m-2)(k+1)} \sum_{1\le l_0,l_1,\cdots,l_k\le 2^{m-1}} \left(\prod_{i=1}^m \sin\frac{2l_0-1}{2^i}\pi\right)\times\cdots\times\left(\prod_{i=1}^m \sin\frac{2l_k-1}{2^i}\pi\right) \\
&\quad \times e^{\sqrt{-1}\left(\frac{2l_0-1}{2^m}+\cdots+\frac{2l_k-1}{2^m}\right)\pi} e^{\sqrt{-1}2\pi(2l_0-1)\frac{[2^m x]}{2^m}} \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[2^m x+2^m\alpha_i]}{2^m}},
\end{aligned}$$

it turns out that

$$\begin{aligned}
\text{LHS} &= (\sqrt{-1})^{(m-2)(k+1)} \sum_{1\le l_0,l_1,\cdots,l_k\le 2^{m-1}} \left(\prod_{i=1}^m \sin\frac{2l_0-1}{2^i}\pi\right) \\
&\quad \times\cdots\times\left(\prod_{i=1}^m \sin\frac{2l_k-1}{2^i}\pi\right) e^{\sqrt{-1}\left(\frac{2l_0-1}{2^m}+\cdots+\frac{2l_k-1}{2^m}\right)\pi} \\
&\quad \times \int_0^1 e^{\sqrt{-1}2\pi(2^m-1)px} e^{\sqrt{-1}2\pi(2l_0-1)\frac{[2^m x]}{2^m}} \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[2^m x+2^m\alpha_i]}{2^m}} dx\,.
\end{aligned}$$

Here we further compute the integral $\int_0^1\cdots dx$ in the last line as

$$\begin{aligned}
&= \sum_{j=1}^{2^m} \int_{\frac{j-1}{2^m}}^{\frac{j}{2^m}} e^{\sqrt{-1}2\pi p(2^m x-x)} e^{\sqrt{-1}2\pi(2l_0-1)\frac{[2^m x]}{2^m}} \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[2^m x+2^m\alpha_i]}{2^m}} dx \\
&= \sum_{j=1}^{2^m} \int_0^1 e^{\sqrt{-1}2\pi p(1-\frac{1}{2^m})(y+j-1)} e^{\sqrt{-1}2\pi(2l_0-1)\frac{j-1}{2^m}} \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[y+2^m\alpha_i]+j-1}{2^m}} \frac{dy}{2^m} \\
&\quad \text{[by the change of variables } 2^m x-(j-1)=y\text{]} \\
&= \frac{1}{2^m}\sum_{j=1}^{2^m} e^{\sqrt{-1}2\pi(-p+2l_0-1+\cdots+2l_k-1)\frac{j-1}{2^m}} \int_0^1 e^{\sqrt{-1}2\pi p(1-\frac{1}{2^m})y} \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[y+2^m\alpha_i]}{2^m}} dy
\end{aligned}$$

$$= 1_{-p+2l_0-1+\cdots+2l_k-1\in 2^m\mathbb{Z}} \int_0^1 e^{\sqrt{-1}2\pi p(1-\frac{1}{2^m})y} \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[y+2^m\alpha_i]}{2^m}} dy$$

$$\left[\text{because, for } \nu\in\mathbb{Z}\ \frac{1}{2^m}\sum_{j=1}^{2^m} e^{\sqrt{-1}2\pi\nu\frac{j-1}{2^m}} = 1_{\nu\in 2^m\mathbb{Z}} = \begin{cases} 0 & \text{if } \nu\notin 2^m\mathbb{Z}, \\ 1 & \text{if } \nu\in 2^m\mathbb{Z} \end{cases}\right].$$

Substituting the last expression into the above, we have

$$\begin{aligned} \text{LHS} =& (\sqrt{-1})^{(m-2)(k+1)} \sum_{\substack{1\le l_0,l_1,\cdots,l_k\le 2^{m-1};\\ -p+2l_0-1+\cdots+2l_k-1\in 2^m\mathbb{Z}}} \left(\prod_{i=1}^m \sin\frac{2l_0-1}{2^i}\pi\right) \\ &\times\cdots\times\left(\prod_{i=1}^m \sin\frac{2l_k-1}{2^i}\pi\right) e^{\sqrt{-1}\left(\frac{2l_0-1}{2^m}+\cdots+\frac{2l_k-1}{2^m}\right)\pi} \\ &\times\int_0^1 e^{\sqrt{-1}2\pi p(1-\frac{1}{2^m})y} \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[y+2^m\alpha_i]}{2^m}} dy\,. \end{aligned}$$

Here, since $p+1\equiv k \pmod 2$, we see that for ${}^\forall(l_1,\cdots,l_k)\in\{1,\cdots,2^{m-1}\}^k$

$$1\le {}^{\exists 1}l_0\le 2^{m-1} \ \text{ such that } \ -p+2l_0-1+2l_1-1+\cdots+2l_k-1\in 2^m\mathbb{Z}\,.$$

For this l_0

$$\begin{aligned} &\left(\prod_{i=1}^m \sin\frac{2l_0-1}{2^i}\pi\right) e^{\sqrt{-1}\left(\frac{2l_0-1}{2^m}+\cdots+\frac{2l_k-1}{2^m}\right)\pi} \\ &\quad = (-1)^m\left(\prod_{i=1}^m \sin\frac{2l_1-1+\cdots+2l_k-1-p}{2^i}\pi\right) e^{\sqrt{-1}\frac{p}{2^m}\pi}\,. \end{aligned}$$

Therefore substituting this into the above, we have the conclusion in the case of $p\not\equiv k \pmod 2$. □

In the following, let $k\in\mathbb{N}$ and $n_1,\cdots,n_k\in\mathbb{N}$; $n_1<\cdots<n_k$ be fixed.

DEFINITION 3. For $p\in\mathbb{Z}$ and $m\in\mathbb{N}$, we define $A_p^{(m)}:\mathbb{Z}^k=\underbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}_{k}\to\mathbb{C}$ and $\rho_p(m)\in[0,\infty)$ by

$$A_p^{(m)}(K_1,\cdots,K_k) := \sum_{1\le l_1,\cdots,l_k\le 2^{m-1}} S_p^{(m)}(l_1,\cdots,l_k)\prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{K_i}{2^m}}, \qquad (K_1,\cdots,K_k)\in\mathbb{Z}^k\,,$$

$$\rho_p(m) := \max_{\substack{t_1\in\{0,\cdots,n_1\},\\ \cdots\cdots\\ t_k\in\{0,\cdots,n_k\}}} \frac{1}{2^m}\sum_{j=1}^{2^m} |A_p^{(m)}(n_1(j-1)+t_1,\cdots,n_k(j-1)+t_k)|\,.$$

LEMMA 3. *For* $^\forall p \in \mathbb{Z}$ *and* $^\forall m \in \mathbb{N}$

$$\int_0^1 d\alpha \left| \int_{\mathbb{T}} e^{\sqrt{-1}2\pi(2^m-1)px} X^{(m)}(x) X^{(m)}(x+n_1\alpha)\cdots X^{(m)}(x+n_k\alpha)dx \right| \le \rho_p(m)\,.$$

PROOF. By Lemma 2

$$\begin{aligned}
&\int_0^1 d\alpha \left| \int_{\mathbb{T}} e^{\sqrt{-1}2\pi(2^m-1)px} X^{(m)}(x) X^{(m)}(x+n_1\alpha)\cdots X^{(m)}(x+n_k\alpha)dx \right| \\
&\le \int_0^1 d\alpha \int_0^1 dy \left| \sum_{1\le l_1,\cdots,l_k\le 2^{m-1}} S_p^{(m)}(l_1,\cdots,l_k) \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[y+2^m n_i\alpha]}{2^m}} \right| \\
&= \int_0^1 dy \int_0^1 d\alpha |A_p^{(m)}([y+n_1 2^m\alpha],\cdots,[y+n_k 2^m\alpha])| \\
&= \int_0^1 dy \sum_{j=1}^{2^m} \int_{\frac{j-1}{2^m}}^{\frac{j}{2^m}} d\alpha |A_p^{(m)}([y+n_1 2^m\alpha],\cdots,[y+n_k 2^m\alpha])| \\
&= \int_0^1 dy \sum_{j=1}^{2^m} \int_0^1 \frac{d\beta}{2^m} |A_p^{(m)}([y+n_1\beta]+n_1(j-1),\cdots,[y+n_k\beta]+n_k(j-1))| \\
&\qquad \text{[by the change of variables } 2^m\alpha-(j-1)=\beta\text{]} \\
&= \int_0^1\int_0^1 dyd\beta \frac{1}{2^m}\sum_{j=1}^{2^m} |A_p^{(m)}(n_1(j-1)+[y+n_1\beta],\cdots,n_k(j-1)+[y+n_k\beta])| \\
&= \sum_{\substack{t_1\in\{0,\cdots,n_1\},\\ \cdots\cdots \\ t_k\in\{0,\cdots,n_k\}}} \iint_{\substack{(y,\beta)\in[0,1)\times[0,1);\\ [y+n_1\beta]=t_1,\cdots,[y+n_k\beta]=t_k}} dyd\beta \\
&\qquad\qquad \times \frac{1}{2^m}\sum_{j=1}^{2^m} |A_p^{(m)}(n_1(j-1)+t_1,\cdots,n_k(j-1)+t_k)| \\
&\qquad \text{[because } [y+n_i\beta]\in\{0,\cdots,n_i\}\ (i=1,\cdots,k)\text{]} \\
&\le \max_{\substack{t_1\in\{0,\cdots,n_1\},\\ \cdots\cdots \\ t_k\in\{0,\cdots,n_k\}}} \frac{1}{2^m}\sum_{j=1}^{2^m} |A_p^{(m)}(n_1(j-1)+t_1,\cdots,n_k(j-1)+t_k)| \\
&= \rho_p(m).
\end{aligned}$$

□

DEFINITION 4. We define $s_0, s_1 : \mathbb{Z}\to\mathbb{Z}$ by

$$s_1(r) = \frac{r+1_{r\in 2\mathbb{Z}-1}}{2} = \begin{cases} \dfrac{r}{2} & \text{if } \ r \text{ is even} \\ \dfrac{r+1}{2} & \text{if } \ r \text{ is odd}\,, \end{cases}$$

$$s_0(r) = \frac{r - 1_{r\in 2\mathbb{Z}-1}}{2} = \begin{cases} \dfrac{r}{2} & \text{if } \ r \text{ is even} \\ \dfrac{r-1}{2} & \text{if } \ r \text{ is odd}. \end{cases}$$

LEMMA 4. *Let $m \geq 2$ and $(K_1, \cdots, K_k) \in \mathbb{Z}^k$. Then*

$$A_p^{(m)}(K_1, \cdots, K_k) = \sum_{\delta\in\{0,1\}} \frac{(\sqrt{-1})^{k-1}}{2}$$
$$\times (\prod_{i=1}^{k}(-1)^{K_i})(-1)^{(k-1)(1-\delta)} e^{-\sqrt{-1}\pi \frac{p}{2^m}(2\delta-1)} A_p^{(m-1)}(s_\delta(K_1), \cdots, s_\delta(K_k)).$$

PROOF. Let $m \geq 2$ and $(K_1, \cdots, K_k) \in \mathbb{Z}^k$. First, noting that

$$\{1, \cdots, 2^{m-1}\}^k$$
$$= \{(l_1 + \varepsilon_1 2^{m-2}, \cdots, l_k + \varepsilon_k 2^{m-2}); (l_1, \cdots, l_k) \in \{1, \cdots, 2^{m-2}\}^k, (\varepsilon_1, \cdots, \varepsilon_k) \in \{0, 1\}^k\}$$

and

$$S_p^{(m)}(l_1 + \varepsilon_1 2^{m-2}, \cdots, l_k + \varepsilon_k 2^{m-2})$$
$$= S_p^{(m-1)}(l_1, \cdots, l_k) \sin\left(\frac{2l_1 - 1 + \cdots + 2l_k - 1 - p}{2^m}\pi + \frac{\varepsilon_1 + \cdots + \varepsilon_k}{2}\pi\right)$$
$$\times \sin\left(\frac{2l_1 - 1}{2^m}\pi + \frac{\varepsilon_1}{2}\pi\right) \times \cdots \times \sin\left(\frac{2l_k - 1}{2^m}\pi + \frac{\varepsilon_k}{2}\pi\right),$$

we see that

$$A_p^{(m)}(K_1, \cdots, K_k) = \sum_{1\leq l_1,\cdots,l_k\leq 2^{m-2}} \sum_{\varepsilon_1,\cdots,\varepsilon_k\in\{0,1\}} S_p^{(m)}(l_1 + \varepsilon_1 2^{m-2}, \cdots, l_k + \varepsilon_k 2^{m-2})$$
$$\times \prod_{i=1}^{k} e^{\sqrt{-1}2\pi\left(2(l_i+\varepsilon_i 2^{m-2})-1\right)\frac{K_i}{2^m}}$$

$$= \sum_{1\leq l_1,\cdots,l_k\leq 2^{m-2}} S_p^{(m-1)}(l_1, \cdots, l_k) \prod_{i=1}^{k} e^{\sqrt{-1}2\pi(2l_i-1)\frac{K_i}{2^m}}$$
$$\times \sum_{\varepsilon_1,\cdots,\varepsilon_k\in\{0,1\}} \prod_{i=1}^{k} \sin\left(\frac{2l_i - 1}{2^m}\pi + \frac{\varepsilon_i}{2}\pi\right)$$
$$\times \sin\left(\frac{2l_1 - 1 + \cdots + 2l_k - 1 - p}{2^m}\pi + \frac{\varepsilon_1 + \cdots + \varepsilon_k}{2}\pi\right) \prod_{i=1}^{k}(-1)^{\varepsilon_i K_i}.$$

Next, using the identity

$$\sum_{\varepsilon\in\{0,1\}} \sin\left(X+\frac{\varepsilon}{2}\pi\right)e^{\sqrt{-1}\,\frac{\varepsilon}{2}\pi}(-1)^{\varepsilon K} = \sqrt{-1}(-1)^K e^{-\sqrt{-1}(-1)^K X}, \quad {}^\forall X\in\mathbb{R}, {}^\forall K\in\mathbb{Z},$$

we compute the summation $\sum_{\varepsilon_1,\cdots,\varepsilon_k\in\{0,1\}}\cdots$ in the last line as

$$\begin{aligned}
&= \Im\Bigg(e^{\sqrt{-1}\pi\frac{2l_1-1+\cdots+2l_k-1-p}{2^m}} \sum_{\varepsilon_1,\cdots,\varepsilon_k\in\{0,1\}} \prod_{i=1}^k (\sin\left(\frac{2l_i-1}{2^m}\pi+\frac{\varepsilon_i}{2}\pi\right)e^{\sqrt{-1}\,\frac{\varepsilon_i}{2}\pi}(-1)^{\varepsilon_i K_i})\Bigg) \\
&= \Im\Bigg(e^{\sqrt{-1}\pi\frac{2l_1-1+\cdots+2l_k-1-p}{2^m}} \prod_{i=1}^k (\sum_{\varepsilon\in\{0,1\}} \sin\left(\frac{2l_i-1}{2^m}\pi+\frac{\varepsilon}{2}\pi\right)e^{\sqrt{-1}\,\frac{\varepsilon}{2}\pi}(-1)^{\varepsilon K_i})\Bigg) \\
&= \Im\Bigg(e^{\sqrt{-1}\pi\frac{2l_1-1+\cdots+2l_k-1-p}{2^m}} (\sqrt{-1})^k(-1)^{K_1+\cdots+K_k} e^{-\sqrt{-1}\pi\sum_{i=1}^k(-1)^{K_i}\frac{2l_i-1}{2^m}}\Bigg) \\
&= (-1)^{K_1+\cdots+K_k}\Im\Bigg(e^{\sqrt{-1}\pi\left(\frac{k}{2}+\sum_{i=1}^k(1-(-1)^{K_i})\frac{2l_i-1}{2^m}-\frac{p}{2^m}\right)}\Bigg) \\
&= (-1)^{K_1+\cdots+K_k}\frac{1}{2\sqrt{-1}}\Bigg\{e^{\sqrt{-1}\pi\left(\frac{k}{2}+\sum_{i=1}^k(1-(-1)^{K_i})\frac{2l_i-1}{2^m}-\frac{p}{2^m}\right)} \\
&\qquad - e^{-\sqrt{-1}\pi\left(\frac{k}{2}+\sum_{i=1}^k(1-(-1)^{K_i})\frac{2l_i-1}{2^m}-\frac{p}{2^m}\right)}\Bigg\}.
\end{aligned}$$

Substituting the last expression into the above, we have

$$\begin{aligned}
&A_p^{(m)}(K_1,\cdots,K_k) \\
&\quad = \sum_{1\le l_1,\cdots,l_k\le 2^{m-2}} S_p^{(m-1)}(l_1,\cdots,l_k)e^{\sqrt{-1}2\pi\sum_{i=1}^k K_i\frac{2l_i-1}{2^m}} \\
&\qquad \times (-1)^{K_1+\cdots+K_k}\frac{1}{2\sqrt{-1}}\Bigg\{(\sqrt{-1})^k e^{-\sqrt{-1}\pi\frac{p}{2^m}} e^{\sqrt{-1}2\pi\sum_{i=1}^k\frac{1-(-1)^{K_i}}{2}\frac{2l_i-1}{2^m}} \\
&\qquad\quad - (-\sqrt{-1})^k e^{\sqrt{-1}\pi\frac{p}{2^m}} e^{-\sqrt{-1}2\pi\sum_{i=1}^k\frac{1-(-1)^{K_i}}{2}\frac{2l_i-1}{2^m}}\Bigg\} \\
&\quad = \frac{(\sqrt{-1})^{k-1}}{2}(-1)^{K_1+\cdots+K_k}\sum_{1\le l_1,\cdots,l_k\le 2^{m-2}} S_p^{(m-1)}(l_1,\cdots,l_k) \\
&\qquad \times \Bigg\{e^{-\sqrt{-1}\pi\frac{p}{2^m}} e^{\sqrt{-1}2\pi\sum_{i=1}^k\left(K_i+\frac{1-(-1)^{K_i}}{2}\right)\frac{2l_i-1}{2^m}} \\
&\qquad\quad + (-1)^{k-1}e^{\sqrt{-1}\pi\frac{p}{2^m}} e^{\sqrt{-1}2\pi\sum_{i=1}^k\left(K_i-\frac{1-(-1)^{K_i}}{2}\right)\frac{2l_i-1}{2^m}}\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{(\sqrt{-1})^{k-1}}{2}(-1)^{K_1+\cdots+K_k} \\
&\quad\times \Bigg\{ e^{-\sqrt{-1}\pi\frac{p}{2^m}} \sum_{1\le l_1,\cdots,l_k\le 2^{m-2}} S_p^{(m-1)}(l_1,\cdots,l_k)\prod_{i=1}^{k} e^{\sqrt{-1}2\pi(2l_i-1)\frac{s_1(K_i)}{2^{m-1}}} \\
&\qquad + (-1)^{k-1}e^{\sqrt{-1}\pi\frac{p}{2^m}} \sum_{1\le l_1,\cdots,l_k\le 2^{m-2}} S_p^{(m-1)}(l_1,\cdots,l_k)\prod_{i=1}^{k} e^{\sqrt{-1}2\pi(2l_i-1)\frac{s_0(K_i)}{2^{m-1}}} \Bigg\} \\
&= \tfrac{(\sqrt{-1})^{k-1}}{2}(-1)^{K_1+\cdots+K_k} \Big\{ e^{-\sqrt{-1}\pi\frac{p}{2^m}} A_p^{(m-1)}(s_1(K_1),\cdots,s_1(K_k)) \\
&\qquad + (-1)^{k-1}e^{\sqrt{-1}\pi\frac{p}{2^m}} A_p^{(m-1)}(s_0(K_1),\cdots,s_0(K_k)) \Big\} \\
&= \sum_{\delta\in\{0,1\}} \frac{(\sqrt{-1})^{k-1}}{2}\left(\prod_{i=1}^{k}(-1)^{K_i}\right)(-1)^{(k-1)(1-\delta)}e^{-\sqrt{-1}\pi\frac{p}{2^m}(2\delta-1)} \\
&\quad\times A_p^{(m-1)}(s_\delta(K_1),\cdots,s_\delta(K_k)),
\end{aligned}
$$

which is just the desired expression. □

LEMMA 5. *There exists an $m_0\in\mathbb{N}$ such that for $^\forall p\in\mathbb{Z}$ and $^\forall m>m_0$*

$$
\rho_p(m) \le \left(1-\frac{1}{2^{2m_0-1}}\left(1-|\sin\frac{\pi p}{2^{m-m_0+2}}|\right)\right)\rho_p(m-m_0).
$$

PROOF. Choose $\eta_0,\mu_0\in\mathbb{N}$ such that $2^{\mu_0}\le n_k\eta_0<2^{\mu_0+1}$ and $n_{k-1}\eta_0<2^{\mu_0}$, and set $h_0, m_0\in\mathbb{N}$ by

$$
\begin{aligned}
h_0 &:= 2^{\left[\frac{\log n_k}{\log 2}\right]+1}\eta_0+1, \\
m_0 &:= \left[\frac{\log n_k}{\log 2}\right]+\mu_0+3.
\end{aligned}
$$

Then, for $^\forall t_i\in\{0,\cdots,n_i\}$ $(1\le i\le k)$

$$
\begin{aligned}
& n_i(h_0-1)+t_i < 2^{m_0-2} \quad (1\le {}^\forall i\le k-1), \\
& 2^{m_0-2}\le n_k(h_0-1)+t_k<2^{m_0-1},
\end{aligned}
$$

i.e., for $^\exists\varepsilon_{i\,0},\cdots,\varepsilon_{i\,m_0-3}\in\{0,1\}$ $(1\le i\le k)$

$$
n_i(h_0-1)+t_i = \varepsilon_{i\,0}+\varepsilon_{i1}2+\cdots+\varepsilon_{i\,m_0-3}2^{m_0-3}, \quad 1\le i\le k-1, \tag{2}
$$

$$
n_k(h_0-1)+t_k = \varepsilon_{k\,0}+\varepsilon_{k1}2+\cdots+\varepsilon_{k,m_0-3}2^{m_0-3}+2^{m_0-2}. \tag{3}
$$

In the following, let $t_i\in\{0,\cdots,n_i\}$ $(1\le i\le k)$ be fixed arbitrarily.

1° Noting that for $m > m_0 \geq 1$

$$\{0, 1, \cdots, 2^m - 1\} = \{h - 1 + 2^{m_0}(g-1); h \in \{1, \cdots, 2^{m_0}\}, g \in \{1, \cdots, 2^{m-m_0}\}\},$$

we see

$$\begin{aligned}
&\frac{1}{2^m}\sum_{j=1}^{2^m} |A_p^{(m)}(n_1(j-1)+t_1, \cdots, n_k(j-1)+t_k)| \\
&= \frac{1}{2^m}\sum_{h=1}^{2^{m_0}}\sum_{g=1}^{2^{m-m_0}} |A_p^{(m)}(n_1(h-1+2^{m_0}(g-1))+t_1, \cdots, n_k(h-1+2^{m_0}(g-1))+t_k)| \\
&= \frac{1}{2^{m_0}}\sum_{h=1}^{2^{m_0}}\frac{1}{2^{m-m_0}}\sum_{g=1}^{2^{m-m_0}} |A_p^{(m)}(2^{m_0}n_1(g-1)+n_1(h-1)+t_1, \cdots, 2^{m_0}n_k(g-1) \\
&\qquad\qquad + n_k(h-1)+t_k)|\,.
\end{aligned}$$

2° By Lemma 4

$$\begin{aligned}
&A_p^{(m)}(2^{m_0}n_1(g-1)+n_1(h-1)+t_1, \cdots, 2^{m_0}n_k(g-1)+n_k(h-1)+t_k) \\
&= \sum_{\delta_1, \cdots, \delta_{m_0}\in\{0,1\}} \left(\frac{(\sqrt{-1})^{k-1}}{2}\right)^{m_0} \prod_{i=1}^{k}(-1)^{\sum_{a=1}^{m_0} s_{\delta_{a-1}}\cdots s_{\delta_1}(n_i(h-1)+t_i)} \\
&\quad \times (-1)^{(k-1)(1-\delta_{m_0}+\cdots+1-\delta_1)} e^{-\sqrt{-1}\pi p(\frac{2\delta_1-1}{2^m}+\frac{2\delta_2-1}{2^{m-1}}+\cdots+\frac{2\delta_{m_0}-1}{2^{m-m_0+1}})} \\
&\quad \times A_p^{(m-m_0)}(n_1(g-1)+s_{\delta_{m_0}}\cdots s_{\delta_1}(n_1(h-1)+t_1), \cdots, n_k(g-1) \\
&\qquad + s_{\delta_{m_0}}\cdots s_{\delta_1}(n_k(h-1)+t_k))\,,
\end{aligned}$$

where

$$s_{\delta_{a-1}}\cdots s_{\delta_1} = \begin{cases} \mathrm{id} & \text{if} \quad a = 1\,, \\ s_{\delta_{a-1}} \circ \cdots \circ s_{\delta_1} & \text{if} \quad a > 1\,. \end{cases}$$

3° Note that for $0 \leq r \leq n2^{m_0}$, $0 \leq s_{\delta_{m_0}}\cdots s_{\delta_1}(r) \leq n$ ($\delta_1, \cdots, \delta_{m_0} \in \{0,1\}$). By 2°, this implies that for each $h = 1, \cdots, 2^{m_0}$

$$\begin{aligned}
&\frac{1}{2^{m-m_0}}\sum_{g=1}^{2^{m-m_0}} |A_p^{(m)}(2^{m_0}n_i(g-1)+n_i(h-1)+t_i)| \\
&\leq \frac{1}{2^{m-m_0}}\sum_{g=1}^{2^{m-m_0}}\sum_{\delta_1,\cdots,\delta_{m_0}\in\{0,1\}}\frac{1}{2^{m_0}}|A_p^{(m-m_0)}(n_i(g-1)+s_{\delta_{m_0}}\cdots s_{\delta_1}(n_i(h-1)+t_i))| \\
&= \frac{1}{2^{m_0}}\sum_{\delta_1,\cdots,\delta_{m_0}\in\{0,1\}}\frac{1}{2^{m-m_0}}\sum_{g=1}^{2^{m-m_0}} |A_p^{(m-m_0)}(n_i(g-1)+s_{\delta_{m_0}}\cdots s_{\delta_1}(n_i(h-1)+t_i))|
\end{aligned}$$

$$\leq \max_{\substack{\tau_1\in\{0,\cdots,n_1\},\\ \cdots\cdots \\ \tau_k\in\{0,\cdots,n_k\}}} \frac{1}{2^{m-m_0}} \sum_{g=1}^{2^{m-m_0}} |A_p^{(m-m_0)}(n_i(g-1)+\tau_i)|$$
$$= \rho_p(m-m_0)\,.$$

Here and in the sequel, for simplicity we write $A_p^{(m)}(K_1,\cdots,K_k)$ as $A_p^{(m)}(K_i)$.

4° Let $\delta^{(1)}=(0,\cdots,0,0,1)$, $\delta^{(2)}=(0,\cdots,0,1,0)\in\{0,1\}^{m_0}$. Then, by (2) and (3)

$$s_{\delta^{(1)}_{m_0-1}}\cdots s_{\delta^{(1)}_1}(n_i(h_0-1)+t_i)=0\,,\quad 1\leq {}^\forall i\leq k\,,$$
$$s_{\delta^{(1)}_{m_0}}\cdots s_{\delta^{(1)}_1}(n_i(h_0-1)+t_i)=0\,,\quad 1\leq {}^\forall i\leq k\,,$$
$$s_{\delta^{(2)}_{m_0-1}}\cdots s_{\delta^{(2)}_1}(n_i(h_0-1)+t_i)=\begin{cases}0 & 1\leq {}^\forall i\leq k-1,\\ 1 & i=k\end{cases}$$
$$s_{\delta^{(2)}_{m_0}}\cdots s_{\delta^{(2)}_1}(n_i(h_0-1)+t_i)=0,\quad 1\leq {}^\forall i\leq k\,,$$

and hence

$$\prod_{i=1}^{k}(-1)^{\sum_{a=1}^{m_0} s_{\delta^{(1)}_{a-1}}\cdots s_{\delta^{(1)}_1}(n_i(h_0-1)+t_i)}(-1)^{(k-1)(1-\delta^{(1)}_{m_0}+\cdots+1-\delta^{(1)}_1)}$$
$$=\prod_{i=1}^{k}(-1)^{\sum_{a=1}^{m_0-1} s_0^{a-1}(n_i(h_0-1)+t_i)}(-1)^{(k-1)(m_0-1)}\,,$$
$$\prod_{i=1}^{k}(-1)^{\sum_{a=1}^{m_0} s_{\delta^{(2)}_{a-1}}\cdots s_{\delta^{(2)}_1}(n_i(h_0-1)+t_i)}(-1)^{(k-1)(1-\delta^{(2)}_{m_0}+\cdots+1-\delta^{(2)}_1)}$$
$$=(-1)\prod_{i=1}^{k}(-1)^{\sum_{a=1}^{m_0-1} s_0^{a-1}(n_i(h_0-1)+t_i)}(-1)^{(k-1)(m_0-1)}\,.$$

By 2°, this implies

$$A_p^{(m)}(2^{m_0}n_i(g-1)+n_i(h_0-1)+t_i)$$
$$=\left(\frac{(\sqrt{-1})^{k-1}}{2}\right)^{m_0}\prod_{i=1}^{k}(-1)^{\sum_{a=1}^{m_0-1} s_0^{a-1}(n_i(h_0-1)+t_i)}(-1)^{(k-1)(m_0-1)}$$
$$\times\Bigg(e^{-\sqrt{-1}\pi p(\frac{-1}{2^m}+\cdots+\frac{-1}{2^{m-m_0+2}}+\frac{1}{2^{m-m_0+1}})}$$
$$-e^{-\sqrt{-1}\pi p(\frac{-1}{2^m}+\cdots+\frac{-1}{2^{m-m_0+3}}+\frac{1}{2^{m-m_0+2}}+\frac{-1}{2^{m-m_0+1}})}\Bigg)A_p^{(m-m_0)}(n_i(g-1))$$
$$+\sum_{\delta\in\{0,1\}^{m_0}\setminus\{\delta^{(1)},\delta^{(2)}\}}\left(\frac{(\sqrt{-1})^{k-1}}{2}\right)^{m_0}\prod_{i=1}^{k}(-1)^{\sum_{a=1}^{m_0} s_{\delta_{a-1}}\cdots s_{\delta_1}(n_i(h_0-1)+t_i)}$$

$$\times (-1)^{(k-1)(1-\delta_{m_0}+\cdots+1-\delta_1)} e^{-\sqrt{-1}\pi p(\frac{2\delta_1-1}{2^m}+\cdots+\frac{2\delta_{m_0}-1}{2^{m-m_0+1}})}$$
$$\times A_p^{(m-m_0)}(n_i(g-1)+s_{\delta_{m_0}}\cdots s_{\delta_1}(n_i(h_0-1)+t_i)),$$

and by taking the absolute value

$$\begin{aligned}
&|A_p^{(m)}(2^{m_0}n_i(g-1)+n_i(h_0-1)+t_i)| \\
&\quad \le \frac{1}{2^{m_0}}2|\sin\frac{\pi p}{2^{m-m_0+2}}||A_p^{(m-m_0)}(n_i(g-1))| \\
&\qquad + \frac{1}{2^{m_0}}\sum_{\delta\in\{0,1\}^{m_0}\setminus\{\delta^{(1)},\delta^{(2)}\}}|A_p^{(m-m_0)}(n_i(g-1)+s_{\delta_{m_0}}\cdots s_{\delta_1}(n_i(h_0-1)+t_i))|.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{2^{m-m_0}}\sum_{g=1}^{2^{m-m_0}}|A_p^{(m)}(2^{m_0}n_i(g-1)+n_i(h_0-1)+t_i)| \\
&\quad \le \frac{1}{2^{m_0}}2\left|\sin\frac{\pi p}{2^{m-m_0+2}}\right|\frac{1}{2^{m-m_0}}\sum_{g=1}^{2^{m-m_0}}|A_p^{(m-m_0)}(n_i(g-1))| \\
&\qquad + \frac{1}{2^{m_0}}\sum_{\delta\in\{0,1\}^{m_0}\setminus\{\delta^{(1)},\delta^{(2)}\}}\frac{1}{2^{m-m_0}}\sum_{g=1}^{2^{m-m_0}}|A_p^{(m-m_0)}(n_i(g-1) \\
&\qquad\qquad + s_{\delta_{m_0}}\cdots s_{\delta_1}(n_i(h_0-1)+t_i))| \\
&\quad \le \left(1-\frac{1}{2^{m_0-1}}\left(1-\left|\sin\frac{\pi p}{2^{m-m_0+2}}\right|\right)\right)\rho_p(m-m_0).
\end{aligned}$$

5° Collecting 1°, 3° and 4°, we have

$$\begin{aligned}
&\frac{1}{2^m}\sum_{j=1}^{2^m}|A_p^{(m)}(n_i(j-1)+t_i)| \\
&\quad \le \frac{1}{2^{m_0}}\sum_{h\in\{1,\cdots,2^{m_0}\}\setminus\{h_0\}}\rho_p(m-m_0) \\
&\qquad + \frac{1}{2^{m_0}}\left(1-\frac{1}{2^{m_0-1}}\left(1-\left|\sin\frac{\pi p}{2^{m-m_0+2}}\right|\right)\right)\rho_p(m-m_0) \\
&\quad = \left(1-\frac{1}{2^{2m_0-1}}\left(1-\left|\sin\frac{\pi p}{2^{m-m_0+2}}\right|\right)\right)\rho_p(m-m_0),
\end{aligned}$$

from which the conclusion follows at once. □

PROOF OF THEOREM 1. Let $p \in \mathbb{Z}$ be fixed. Since $\left|\sin \frac{\pi p}{2^{m-m_0+2}}\right| < 1/2$ for $m \geq \left[\frac{\log|p| \vee 1}{\log 2}\right] + 2 + m_0$ (where m_0 is an integer in Lemma 5), Lemma 5 tells us that

$$\rho_p(m) \leq \left(1 - \frac{1}{4^{m_0}}\right)\rho_p(m - m_0)\,.$$

This yields that for $^\forall q \geq 0$, $0 \leq {}^\forall r < m_0$

$$\rho_p\left(\left[\frac{\log|p| \vee 1}{\log 2}\right] + 2 + qm_0 + r\right) \leq \left(1 - \frac{1}{4^{m_0}}\right)^q \rho_p\left(\left[\frac{\log|p| \vee 1}{\log 2}\right] + 2\right).$$

If we express $m \geq 0$ as $m = qm_0 + r$ ($q \geq 0$, $0 \leq r < m_0$), then $q > m/m_0 - 1$ and so

$$\left(1 - \frac{1}{4^{m_0}}\right)^q \leq \left(1 - \frac{1}{4^{m_0}}\right)^{\frac{m}{m_0} - 1}.$$

Combining this observation with the above, we have

$$\rho_p\left(\left[\frac{\log|p| \vee 1}{\log 2}\right] + 2 + m\right) \leq \left(1 - \frac{1}{4^{m_0}}\right)^{\frac{m}{m_0} - 1} \rho_p\left(\left[\frac{\log|p| \vee 1}{\log 2}\right] + 2\right), \quad {}^\forall m \geq 0\,.$$

Consequently, by Lemma 3, we conclude that the assertion of Theorem 1 is valid with $\rho = (1 - 1/4^{m_0})^{\frac{1}{m_0}}$. □

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Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE,
KANAZAWA UNIVERSITY, KANAZAWA, 920–1192 JAPAN.
e-mail: takanob@kenroku.kanazawa-u.ac.jp