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Theorems of Gauss-Bonnet and Chern-Lashof Types in a Simply Connected Symmetric Space of Non-Positive Curvature

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Abstract. In this paper, we shall generalize the Gauss-Bonnet and Chern-Lashof theorems to compact submanifolds in a simply connected symmetric space of non-positive curvature. Those proofs are performed by applying the Morse theory to squared distance functions because height functions are not defined.

1. Introduction

For an n-dimensional compact immersed submanifold M in the m-dimensional Euclidean space $\mathbf{R}^m$, it is well-known that the following Gauss-Bonnet and Chern-Lashof theorems hold:

$$\frac{1}{\mathrm{Vol}(S^{m-1}(1))}\int_{\xi\in U^\perp M}\det A_\xi\,\omega_{U^\perp M}=\chi(M)\,, \tag{1.1}$$

$$\frac{1}{\mathrm{Vol}(S^{m-1}(1))}\int_{\xi\in U^\perp M}|\det A_\xi|\,\omega_{U^\perp M}\geq\sum_{k=0}^{n}b_k(M,\mathbf{F})\,, \tag{1.2}$$

where $\mathrm{Vol}(S^{m-1}(1))$ is the volume of the $(m-1)$-dimensional unit sphere, A is the shape tensor of M, $\omega_{U^\perp M}$ is the standard volume element on the unit normal bundle $U^\perp M$ of M, $\chi(M)$ is the Euler characteristic of M and $b_k(M,\mathbf{F})$ is the k-th Betti number of M with respect to an arbitrary coefficient field $\mathbf{F}$. These relations are proved by applying the Morse theory to height functions h_v ($v\in\mathbf{R}^m$). For an n-dimensional compact immersed submanifold M in the m-dimensional hyperbolic space $H^m(-1)$ of constant curvature -1 or the m-dimensional unit sphere $S^m(1)$, E. Teufel ([Teu1,2]) tried to obtain the inequality of Chern-Lashof type by applying the Morse theory to the restrictions (to M) of functions whose level sets are totally geodesic hypersurfaces in $H^m(-1)$ or $S^m(1)$. Concretely, in the case where the ambient space is $H^m(-1)$, he proved the following fact:

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If M is contained in some geodesic ball of radius r_0 (in $H^m(-1)$), then the following inequality holds:

$$\frac{1}{\mathrm{Vol}(S^{m-1}(1))}\int_{\xi\in U^\perp M}|\det A_\xi|\omega_{U^\perp M} > \frac{1}{\cosh r_0}\sum_{k=0}^{n} b_k(M,\mathbf{F}).$$

However, in the case where the ambient space is $S^m(1)$, the similar fact has not been obtained. We consider that the topology of a submanifold in a general complete and simply connected Riemannian manifold should be determined by both the extrinsic curvature (i.e., the shape tensor A) of the submanifold and the curvature R of the ambient Riemannian manifold. So we propose the following problem:

PROBLEM. *Find a function $F_{A,R}$ on $U^\perp M$ determined by both A and R such that*

$$\int_{\xi\in U^\perp M} F_{A,R}(\xi)\omega_{U^\perp M} = \chi(M)$$

and

$$\int_{\xi\in U^\perp M} |F_{A,R}(\xi)|\omega_{U^\perp M} \geq \sum_{k=0}^{n} b_k(M,\mathbf{F})$$

hold for each n-dimensional immersed compact submanifold M in an arbitrary complete and simply connected Riemannian manifold N.

For a submanifold in a Euclidean space, its tightness is defined in terms of height functions. However, for a submanifold in a general Riemannian manifold, its tightness is not defined because of the absence of height functions but its tautness is defined in terms of the energy functional on the space of H^1-paths. In particular, for a submanifold in a Hadamard manifold, its tautness can be defined in terms of squared distance functions. Thus it is natural to consider to apply the Morse theory to squared distance functions in order to obtain the theorems of Gauss-Bonnet and Chern-Lashof types for a compact immersed submanifold in a Hadamard manifold. In this paper, by applying the Morse theory to squared distance functions, we shall prove the following theorem of Gauss-Bonnet and Chern-Lashof types for a compact immersed submanifold in a simply connected symmetric space of non-positive curvature.

THEOREM A. *Let M be an n-dimensional compact immersed submanifold in an m-dimensional simply connected symmetric space N of non-positive curvature. Then, for each $o\in N$, we have*

$$\int_{\xi\in U^\perp M}\frac{1}{v(\xi)}e^{b_\xi(o)\mathrm{Tr}\sqrt{-R_\xi}}\det(\mathrm{pr}_T\circ\sqrt{-R_\xi}|_{T_{\pi(\xi)}M}-A_\xi)\omega_{U^\perp M} = \chi(M)\,, \tag{1.5}$$

$$\int_{\xi\in U^\perp M}\frac{1}{v(\xi)}e^{b_\xi(o)\mathrm{Tr}\sqrt{-R_\xi}}|\det(\mathrm{pr}_T\circ\sqrt{-R_\xi}|_{T_{\pi(\xi)}M}-A_\xi)|\omega_{U^\perp M} \geq \sum_{k=0}^{n} b_k(M,\mathbf{F}) \tag{1.6}$$

and the equality sign holds in the inequality (1.6) *if M is taut, where $\sqrt{-R_\xi}$ is the positive operator with $\sqrt{-R_\xi}^2 = -R(\cdot,\xi)\xi$ (R : the curvature tensor of N), $v(\xi) = \lim_{r\to\infty} r\mathrm{vol}_{N,r}$ $\det\frac{\sqrt{-R_\xi}}{\sinh(r\sqrt{-R_\xi})}$ ($\mathrm{vol}_{N,r}$: the volume of the geodesic sphere of radius r in N), b_ξ is the Busemann function for the geodesic ray γ_ξ with $\dot\gamma_\xi(0)=\xi$, pr_T is the orthogonal projection of $TN|_M$ (the bundle induced from TN by the immersion) onto TM and π is the bundle projection of $U^\perp M$. In particular, if M is contained in a geodesic ball of radius r_0, then we have*

$$\int_{\xi\in U^\perp M}\frac{1}{v(\xi)}e^{r_0\mathrm{Tr}\sqrt{-R_\xi}}|\det(\mathrm{pr}_T\circ\sqrt{-R_\xi}|_{T_{\pi(\xi)}M}-A_\xi)|\omega_{U^\perp M}\geq\sum_{k=0}^n b_k(M,\mathbf{F})\,. \tag{1.7}$$

REMARK 1.1. (i) Since $v(\xi)$ and $b_\xi(o)$ are determined by the curvature tensor R, this theorem answers the above problem in the case where the ambient space is a simply connected symmetric space of non-positive curvature.

(ii) In case of $N = \mathbf{R}^m$ (the m-dimensional Euclidean space), we have $\sqrt{-R_\xi} = 0$ and $v(\xi) = \mathrm{Vol}(S^{m-1}(1))$. Therefore, the relation (1.5) (resp. (1.6)) is rewritten as (1.1) (resp. (1.2)). Thus this theorem is a generalization of the Gauss-Bonnet and Chern-Lashof theorems. Hence the proof of this theorem in this paper gives a new proof of the Gauss-Bonnet and Chern-Lashof theorems in the case where the ambient space is a Euclidean space.

(iii) If N is of rank one, then we have $v(\xi) = \mathrm{Vol}(S^{m-1}(1))$.

(iv) For a submanifold in an arbitrary Riemannian manifold, the squared distance functions are defined. Hence we expect that the proof of this theorem is applied to a compact immersed submanifold in various Riemannian manifolds.

As the hypersurface version of Theorem A, we obtain the following result.

COROLLARY B. *Let M be a compact immersed hypersurface in an $(n+1)$-dimensional simply connected symmetric space N of non-positive curvature. Then, for each $o\in N$, we have*

$$\int_{x\in M}\frac{1}{v(\xi_x)}\sum_{i=1}^2 e^{b_{(-1)^i\xi_x}(o)\mathrm{Tr}\sqrt{-R_{\xi_x}}}\det(\sqrt{-R_{\xi_x}}|_{T_xM}-(-1)^iA_{\xi_x})\omega_M=\chi(M)\,,$$
$$\int_{x\in M}\frac{1}{v(\xi_x)}\sum_{i=1}^2 e^{b_{(-1)^i\xi_x}(o)\mathrm{Tr}\sqrt{-R_{\xi_x}}}|\det(\sqrt{-R_{\xi_x}}|_{T_xM}-(-1)^iA_{\xi_x})|\omega_M\geq\sum_{k=0}^n b_k(M,\mathbf{F})$$

and the equality sign holds in this inequality if M is taut, where ξ is the unit normal vector field on M determined by the orientation of M and ω_M is the volume element of M. In particular, if M is contained in a geodesic ball of radius r_0, then we have

$$\int_{x\in M}\frac{1}{v(\xi_x)}e^{r_0\mathrm{Tr}\sqrt{-R_{\xi_x}}}\sum_{i=1}^2|\det(\sqrt{-R_{\xi_x}}|_{T_xM}-(-1)^iA_{\xi_x})|\omega_M\geq\sum_{k=0}^n b_k(M,\mathbf{F})\,.$$

In the case where the ambient space is the m-dimensional hyperbolic space $H^m(c)$ of constant curvature c, we can obtain the following result from Theorem A.

COROLLARY C. *Let M be an n-dimensional compact immersed submanifold in $H^m(c)$. Then, for each $o \in N$, we have*

$$\frac{1}{\mathrm{Vol}(S^{m-1}(1))}\int_{\xi\in U^\perp M} e^{(m-1)\sqrt{-c}b_\xi(o)}\det(\sqrt{-c}\mathrm{id}-A_\xi)\omega_{U^\perp M}=\chi(M)\,,$$
$$\frac{1}{\mathrm{Vol}(S^{m-1}(1))}\int_{\xi\in U^\perp M} e^{(m-1)\sqrt{-c}b_\xi(o)}|\det(\sqrt{-c}\mathrm{id}-A_\xi)|\omega_{U^\perp M}\geq\sum_{k=0}^{n}b_k(M,\mathbf{F})$$

and the equality sign holds in this inequality if M is taut, where id *is the identity transformation of TM. In particular, if M is contained in a geodesic ball of radius r_0, then we have*

$$\frac{e^{(m-1)\sqrt{-c}r_0}}{\mathrm{Vol}(S^{m-1}(1))}\int_{\xi\in U^\perp M}|\det(\sqrt{-c}\mathrm{id}-A_\xi)|\omega_{U^\perp M}\geq\sum_{k=0}^{n}b_k(M,\mathbf{F})\,.$$

REMARK 1.2. The quantity $e^{(m-1)\sqrt{-c}b_\xi(o)}$ is explicitly described as

$$e^{(m-1)\sqrt{-c}b_\xi(o)}=\{\cosh(\sqrt{-c}d(o,\pi(\xi)))-\sinh(\sqrt{-c}d(o,\pi(\xi)))\cos\theta(\xi)\}^{1-m}$$

(see §2), where d is the Riemannian distance function of $H^m(c)$ and $\theta(\xi)$ is the angle of ξ and $\overrightarrow{\pi(\xi)o}$ ($\overrightarrow{\pi(\xi)o}$: the initial vector of the geodesic γ in N with $\gamma(0)=\pi(\xi)$ and $\gamma(1)=0$).

Denote by $\mathbf{C}H^m(c)$, $\mathbf{Q}H^m(c)$ and $\mathbf{Cay}H^2(c)$ the m-dimensional complex hyperbolic space of constant holomorphic sectional curvature c, the m-dimensional quaternionic hyperbolic space of constant quaternionic sectional curvature c and the Cayley hyperbolic plane of constant Cayley sectional curvature c, respectively. Also, let J_1, $\{J_1, J_2, J_3\}$ and $\{J_1,\cdots,J_7\}$ be the complex structure of $\mathbf{C}H^m(c)$, the quaternionic structure of $\mathbf{Q}H^m(c)$ and the Cayley structure of $\mathbf{Cay}H^2(c)$, respectively.

In the case where the ambient space is one of these spaces, we obtain the following result from Theorem A.

COROLLARY D. *Let M be an n-dimensional compact immersed submanifold in $\mathbf{F}H^m(c)$, where $\mathbf{F}=\mathbf{C}$, $\mathbf{Q}$ or $\mathbf{Cay}$ and $m=2$ when $\mathbf{F}=\mathbf{Cay}$. Then, for each $o\in N$, we have*

$$\frac{1}{\mathrm{Vol}(S^{qm-1}(1))}\int_{\xi\in U^\perp M} e^{\frac{(qm+q-2)\sqrt{-c}b_\xi(o)}{2}}\det\left(\sum_{i=1}^{2}\frac{\sqrt{-c}}{i}\mathrm{pr}_T\circ\mathrm{pr}_i^\xi|_{T_{\pi(\xi)}M}-A_\xi\right)\omega_{U^\perp M}$$
$$=\chi(M)\,,$$

$$\frac{1}{\mathrm{Vol}(S^{qm-1}(1))}\int_{\xi\in U^\perp M} e^{\frac{(qm+q-2)\sqrt{-c}b_\xi(o)}{2}}\left|\det\left(\sum_{i=1}^{2}\frac{\sqrt{-c}}{i}\mathrm{pr}_T\circ\mathrm{pr}_i^\xi|_{T_{\pi(\xi)}M}-A_\xi\right)\right|\omega_{U^\perp M} \geq \sum_{k=0}^{n} b_k(M,\mathbf{F})$$

and the equality sign holds in this inequality if M is taut, where $q=\dim_{\mathbf{R}}\mathbf{F}$ and pr_1^ξ (resp. pr_2^ξ) is the orthogonal projection of $T_{\pi(\xi)}\mathbf{F}H^m(c)$ onto $\mathrm{Span}\{J_1\xi,\cdots,J_{q-1}\xi\}$ *(resp.* $\mathrm{Span}\{\xi,J_1\xi,\cdots,J_{q-1}\xi\}^\perp$*). In particular, if M is contained in a geodesic ball of radius r_0, then we have*

$$\frac{e^{\frac{(qm+q-2)\sqrt{-c}r_0}{2}}}{\mathrm{Vol}(S^{qm-1}(1))}\int_{\xi\in U^\perp M}\left|\det\left(\sum_{i=1}^{2}\frac{\sqrt{-c}}{i}\mathrm{pr}_T\circ\mathrm{pr}_i^\xi|_{T_{\pi(\xi)}M}-A_\xi\right)\right|\omega_{U^\perp M}\geq\sum_{k=0}^{n}b_k(M,\mathbf{F}).$$

REMARK 1.3. The quantity $e^{\frac{(qm+q-2)\sqrt{-c}b_\xi(o)}{2}}$ is evaluated from above and below as follows:

$$e^{\frac{(qm+q-2)\sqrt{-c}b_\xi(o)}{2}}\leq\{\cosh(\tfrac{\sqrt{-c}}{2}d(o,\pi(\xi)))-\sinh(\tfrac{\sqrt{-c}}{2}d(o,\pi(\xi)))\cos\theta(\xi)\}^{-(qm+q-2)},$$
$$e^{\frac{(qm+q-2)\sqrt{-c}b_\xi(o)}{2}}\geq\{\cosh(\sqrt{-c}d(o,\pi(\xi)))-\sinh(\sqrt{-c}d(o,\pi(\xi)))\cos\theta(\xi)\}^{-\frac{qm+q-2}{2}}$$

(see §2), where d is the Riemannian distance function of $\mathbf{F}H^m(c)$ and $\theta(\xi)$ is the angle of ξ and $\overrightarrow{\pi(\xi)o}$.

In Section 2, we shall recall the notion of the Busemann function and define the tautness of a submanifold in a Hadamard manifold. In Section 3, we shall obtain key equality and inequality used to prove Theorem A. In Section 4, we shall prove Theorem A and Corollaries C and D. Finally, in Section 5, we shall confirm by calculations that the equality and the inequality in Corollary C (resp. D) hold for a geodesic sphere in a hyperbolic space (resp. a simply connected rank one symmetric space of non-compact type (other than a hyperbolic space)).

Throughout this paper, we assume that all objects are of class C^∞ and that all manifolds are oriented and connected ones without boundary.

I would like to thank Professor Eberhard Teufel for his valuable advice about the results in [Teu1,2].

2. Busemann functions and the tautness

In this section, we first recall the notion of the Busemann function. Let N be a Hadamard manifold and $N(\infty)$ be its ideal boundary, that is, the set of all asymptotic classes of geodesic rays in N. Denote by $\gamma(\infty)$ the asymptotic class of a geodesic ray $\gamma:[0,\infty)\to N$. Take $p\in N$ and $z\in N(\infty)$. Let γ be the geodesic ray in N with $\gamma(0)=p$ and $\gamma(\infty)=z$. Denote by $S_\infty(z,p)$ the horosphere through p and z, and by $B_\infty(z,p)$ the closed domain in N surrounded by $S_\infty(z, p)$. In the case where N is a Euclidean space, $S_\infty(z, p)$ is the hyperplane through p which is orthogonal to $\dot\gamma(0)$. Let ξ be a unit tangent vector of N at p_0 and $\gamma_\xi : [0, \infty) \to N$ be the geodesic ray with $\dot\gamma_\xi(0) = \xi$. The Busemann function $b_\xi : N \to \mathbf{R}$ is defined by

$$b_\xi(p) := \begin{cases} d(p,\ S_\infty(\gamma_\xi(\infty), p_0)) & (\text{when } p \in B_\infty(\gamma_\xi(\infty), p_0)) \\ -d(p,\ S_\infty(\gamma_\xi(\infty), p_0)) & (\text{when } p \notin B_\infty(\gamma_\xi(\infty), p_0)) \end{cases}$$

for $p \in N$, where $d(p,\ S_\infty(\gamma_\xi(\infty), p_0))$ is the distance of p and $S_\infty(\gamma_\xi(\infty), p_0)$.

In case of $N = H^m(c)$, it follows from the cosine theorem that

$$b_\xi(p) = \frac{-1}{\sqrt{-c}} \log(\cosh(\sqrt{-c}d(p, p_0)) - \sinh(\sqrt{-c}d(p, p_0)) \cos\theta(\xi)),$$

where $\theta(\xi)$ is the angle of ξ and $\overrightarrow{p_0 p}$. Here we note that $\overrightarrow{p_0 p}$ implies the initial velocity vector of the geodesic γ with $\gamma(0) = p_0$ and $\gamma(1) = p$. Further, in case of $N = \mathbf{F}H^m(c)$ ($\mathbf{F} = \mathbf{C},\ \mathbf{Q}$ or $\mathbf{Cay}$), we have

$$\begin{aligned} b_\xi(p) &\le \tfrac{-2}{\sqrt{-c}} \log(\cosh(\tfrac{\sqrt{-c}}{2}d(p, p_0)) - \sinh(\tfrac{\sqrt{-c}}{2}d(p, p_0)) \cos\theta(\xi)), \\ b_\xi(p) &\ge \tfrac{-1}{\sqrt{-c}} \log(\cosh(\sqrt{-c}d(p, p_0)) - \sinh(\sqrt{-c}d(p, p_0)) \cos\theta(\xi)) \end{aligned}$$

because the minimal (resp. maximal) sectional curvature is c (resp. $c/4$), where we use the comparison theorem.

At the end of this section, we define the tautness of a submanifold in a Hadamard manifold N. Let M be a submanifold in N. If the squared distance function d_p^2 ($x \in M \to d(p, x)^2$) is a perfect Morse function for each $p \in N \setminus F$, then we shall say that M is *taut*, where F is the focal set of M.

3. Key equality and inequality

Let $N = G/K$ be an m-dimensional simply connected symmetric space of non-positive curvature. For arbitrary two points p and q of N, there exists a unique geodesic γ with $\gamma(0) = p$ and $\gamma(1) = q$. Denote by γ_{pq} this geodesic. Also, denote by $\overrightarrow{pq}$ the initial velocity vector $\dot\gamma_{pq}(0)$. Let M be an n-dimensional immersed submanifold in N. In this section, we obtain an equality and an inequality (see Proposition 3.4) used to prove Theorem A. Also, we calculate the volume of a geodesic sphere in N. For simplicity, we set

$$\begin{aligned} D_\xi &:= \sqrt{-R_\xi}, \quad & D_\xi^{co} &:= \cosh\sqrt{-R_\xi}, \\ D_\xi^{si} &= \frac{\sinh\sqrt{-R_\xi}}{\sqrt{-R_\xi}}, \quad & D_\xi^{ct} &= \frac{\sqrt{-R_\xi}}{\tanh\sqrt{-R_\xi}} \end{aligned}$$

for each $\xi \in TN$. Now we recall a very useful description of a Jacobi field in N. A Jacobi field J along a geodesic γ in N is described as

$$J(s) = P_{\gamma|_{[0,s]}}(D^{co}_{s\dot\gamma(0)} J(0) + s D^{si}_{s\dot\gamma(0)} J'(0)), \tag{3.1}$$

where $P_{\gamma|_{[0,s]}}$ is the parallel translation along $\gamma|_{[0,s]}$ with respect to the Levi-Civita connection $\tilde{\nabla}$ of N, $\dot{\gamma}(0)$ is the velocity vector of γ at 0 and $J'(0) = \tilde{\nabla}_{\dot{\gamma}(0)}J$. See [TT] (or [Ko3]) in detail. For the squared distance function d_p^2, we have the following fact.

LEMMA 3.1. *Let x be a critical point of d_p^2 (hence $\overrightarrow{xp}$ is normal to M). The Hessian $(\mathrm{Hess}\, d_p^2)_x$ of d_p^2 at x is given by*

$$(\mathrm{Hess}\, d_p^2)_x(X, Y) = 2 < X, (D^{ct}_{\overrightarrow{xp}} - A_{\overrightarrow{xp}})Y > \quad (X, Y \in T_xM)\,.$$

PROOF. Take tangent vectors X and Y of M at x. Take a two-parameter map $\bar{\delta}$ to M with $\bar{\delta}_*(\frac{\partial}{\partial u}|_{u=t=0}) = X$ and $\bar{\delta}_*(\frac{\partial}{\partial t}|_{u=t=0}) = Y$, where u (resp. t) is the first (resp. the second) parameter of $\bar{\delta}$. Define a three-parameter map δ to N by $\delta(u,t,s) = \gamma_{\overrightarrow{\bar{\delta}(u,t)p}}(s)$. For simplicity, we denote $\delta_*(\frac{\partial}{\partial u})$, $\delta_*(\frac{\partial}{\partial t})$ and $\delta_*(\frac{\partial}{\partial s})$ by $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$, respectively. Set $J_t(s) := \frac{\partial}{\partial u}|_{u=0}$, which is a Jacobi field along $\gamma_{\overrightarrow{\delta(0,t,0)p}}$. From (3.1), we have $J_t(s) = P_{\gamma_{\overrightarrow{\delta(0,t,0)p}}|_{[0,s]}}(D^{co}_{\overrightarrow{s\cdot\delta(0,t,0)p}}J_t(0) + sD^{si}_{\overrightarrow{s\cdot\delta(0,t,0)p}}J'_t(0))$. This together with $J_t(1) = 0$ deduces $J'_t(0) = -D^{ct}_{\overrightarrow{\delta(0,t,0)p}}J_t(0)$. Also, since x is a critical point of d_p^2, $\frac{\partial}{\partial s}|_{u=t=s=0}$ is normal to M. These facts deduce

$$\begin{aligned}
(\mathrm{Hess}\, d_p^2)_x(X,Y) &= \frac{d}{dt}\left(\frac{\partial}{\partial u}\left\langle\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right\rangle\Big|_{u=s=0}\right)\Big|_{t=0} \\
&= 2\frac{d}{dt}\left\langle J'_t(0),\ \frac{\partial}{\partial s}\Big|_{u=s=0}\right\rangle\Big|_{t=0} \\
&= -2\frac{d}{dt}\langle D^{ct}_{\overrightarrow{\delta(0,t,0)p}}J_t(0),\ \overrightarrow{\delta(0,t,0)p}\rangle|_{t=0} \qquad (3.2)\\
&= -2\frac{d}{dt}\langle J_t(0),\ (\overrightarrow{\delta(0,t,0)p})_T\rangle|_{t=0} \\
&= -2\left\langle X,\ \nabla_Y\left(\frac{\partial}{\partial s}\Big|_{u=s=0}\right)_T\right\rangle,
\end{aligned}$$

where $(\cdot)_T$ is the tangent component of $\cdot$. On the other hand, we can show $\tilde{\nabla}_{\overrightarrow{xp}}\frac{\partial}{\partial t} = \nabla_Y(\frac{\partial}{\partial s}\Big|_{u=s=0})_T - A_{\overrightarrow{xp}}Y + \nabla^{\perp}_Y(\frac{\partial}{\partial s}|_{u=s=0})_{\perp}$ and $\tilde{\nabla}_{\overrightarrow{xp}}\frac{\partial}{\partial t} = -D^{ct}_{\overrightarrow{xp}}Y$, where $(\cdot)_{\perp}$ is the normal component of $\cdot$. These relations deduce $\nabla_Y(\frac{\partial}{\partial s}|_{u=s=0})_T = -(\mathrm{pr}_T \circ D^{ct}_{\overrightarrow{xp}} - A_{\overrightarrow{xp}})Y$. From (3.2) and this relation, we obtain the desired relation. □

Denote by $\tilde{\omega}$ (resp. $\omega_{T^{\perp}M}$) the volume element of N (resp. $T^{\perp}M$). Let $\exp^{\perp}$ be the normal exponential map of M. Then we have the following relation.

LEMMA 3.2. *For each $\xi \in T^{\perp}M$, the following relation holds:*

$$((\exp^{\perp})^*\tilde{\omega})_{\xi} = \det(\mathrm{pr}_T \circ D^{ct}_{\xi}|_{T_{\pi(\xi)}M} - A_{\xi})\det D^{si}_{\xi}(\omega_{T^{\perp}M})_{\xi}\,,$$

where π is the bundle projection of $T^{\perp}M$.

PROOF. Set $x := \pi(\xi)$. Let $(e_1, \cdots, e_n)$ be an orthonormal tangent frame of M at x and $(\xi_1^0, \cdots, \xi_{m-n}^0)$ be an orthonormal frame of $T_x^{\perp}M$. Let ξ_i $(i = 1, \cdots, m-n)$ be the element of $T_\xi(T_x^{\perp}M)$ corresponding to ξ_i^0 under the natural identification of $T_x^{\perp}M$ and $T_\xi(T_x^{\perp}M)$. Denote by $\widetilde{(e_i)}_\xi$ the horizontal lift of e_i to ξ $(i = 1, \cdots, n)$. Fix $i \in \{1, \cdots, n\}$. Take a curve c in M with $\dot{c}(0) = e_i$. Let $\tilde{\xi}$ be the $\nabla^{\perp}$-parallel vector field along c with $\tilde{\xi}(0) = \xi$. Define a two-parameter map δ to N by $\delta(t, s) := \exp^{\perp}(s\tilde{\xi}(t))$ and set $J := \frac{\partial}{\partial t}|_{t=0}$, which is a Jacobi field along γ_ξ. It is clear that $J(0) = e_i$, $J'(0) = -A_\xi e_i$ and $J(1) = \exp_*^{\perp}(\widetilde{(e_i)}_\xi)$. Hence we have

$$\exp_*^{\perp}(\widetilde{(e_i)}_\xi) = P_{\gamma_\xi}(D_\xi^{co} e_i - (D_\xi^{si} \circ A_\xi)e_i)\,.$$

Fix $i \in \{1, \cdots, m-n\}$. Define a two-parameter map $\bar{\delta}$ to N by $\bar{\delta}(t, s) := \exp^{\perp}(s(\xi + t\xi_i^0))$ and $\bar{J} := \frac{\partial}{\partial t}|_{t=0}$, which is a Jacobi field along γ_ξ. It is clear that $\bar{J}(0) = 0$, $\bar{J}'(0) = \xi_i^0$ and $\bar{J}(1) = \exp_*^{\perp}\xi_i$. Hence we have $\exp_*^{\perp}\xi_i = P_{\gamma_\xi} D_\xi^{si}\xi_i^0$. Therefore, we obtain

$$\begin{aligned}
&((\exp^{\perp})^*\tilde{\omega})_\xi(\widetilde{(e_1)}_\xi, \cdots, \widetilde{(e_n)}_\xi, \xi_1, \cdots, \xi_{m-n})\\
&= \tilde{\omega}_{\pi(\xi)}(D_\xi^{co}e_1 - (D_\xi^{si} \circ A_\xi)e_1, \cdots, D_\xi^{co}e_n - (D_\xi^{si} \circ A_\xi)e_n, D_\xi^{si}\xi_1^0, \cdots, D_\xi^{si}\xi_r^0)\\
&= \det D_\xi^{si}\tilde{\omega}_{\pi(\xi)}((D_\xi^{ct} - A_\xi)e_1, \cdots, (D_\xi^{ct} - A_\xi)e_n, \xi_1^0, \cdots, \xi_r^0)\\
&= \det D_\xi^{si}\det(\mathrm{pr}_T \circ D_\xi^{ct}|_{T_{\pi(\xi)}M} - A_\xi)\,.
\end{aligned}$$

On the other hand, we have $(\omega_{T^{\perp}M})_\xi(\widetilde{(e_1)}_\xi, \cdots, \widetilde{(e_n)}_\xi, \xi_1, \cdots, \xi_{m-n}) = 1$. Therefore, we can obtain the desired relation. □

This lemma together with Lemma 3.1 deduces the following fact.

LEMMA 3.3. *A point p of N is not a focal point of M if and only if d_p^2 is non-degenerate* (*i.e., a Morse function*).

PROOF. Let x be a critical point of d_p^2. According to Lemmas 3.1 and 3.2, $(\mathrm{Hess}\, d_p^2)_x$ is degenerate if and only if $\overrightarrow{xp}$ is a critical point of $\exp^{\perp}$, that is, $p = \exp^{\perp}(\overrightarrow{xp})$ is a focal point of (M, x), where we also use $\det D_{\overrightarrow{xp}}^{si} > 0$. This fact deduces the statement. □

In terms of Lemmas 3.2 and 3.3, we obtain the following relations.

PROPOSITION 3.4. *Assume that M is compact. Then, for any bounded closed domain D in N, the following relations hold*:

$$\frac{1}{\mathrm{Vol}(D)}\int_{\xi \in \exp^{\perp -1}(D)} \det(\mathrm{pr}_T \circ D_\xi^{ct}|_{T_{\pi(\xi)}M} - A_\xi)\det D_\xi^{si}\,\omega_{T^{\perp}M} = \chi(M)\,,$$

$$\frac{1}{\mathrm{Vol}(D)}\int_{\xi\in\exp^{\perp-1}(D)}|\det(\mathrm{pr}_T\circ D^{ct}_\xi|_{T_{\pi(\xi)}M}-A_\xi)|\det D^{si}_\xi\,\omega_{T^\perp M}\geq\sum_{k=0}^n b_k(M,\mathbf{F})\,.$$

The equality sign holds in this inequality if M is taut.

PROOF. First we shall show the first relation. Denote by F the set of all focal points of M, which is of measure zero by Sard's theorem. Let $p\in D\setminus F$ and $\xi\in\exp^{\perp-1}(p)$. Set $x:=\pi(\xi)$. According to Lemma 3.2, $(\exp^\perp_*)_\xi$ preserves (resp. reverses) the orientation if and only if $\det(\mathrm{pr}_T\circ D^{ct}_\xi|_{T_xM}-A_\xi)>0$ (resp. <0), where we use $\det D^{si}_\xi>0$. On the other hand, it follows from Lemma 3.1 that the index of the critical point x of d_p^2 is even (resp. odd) if and only if $\det(\mathrm{pr}_T\circ D^{ct}_\xi|_{T_xM}-A_\xi)>0$ (resp. <0). By using these facts, we obtain

$$\begin{aligned}
&\int_{\xi\in\exp^{\perp-1}(D)}\det(\mathrm{pr}_T\circ D^{ct}_\xi|_{T_{\pi(\xi)}M}-A_\xi)\det D^{si}_\xi\,\omega_{T^\perp M}\\
&=\int_{\xi\in\exp^{\perp-1}(D)}((\exp^\perp)^*\tilde\omega)_\xi=\int_{\xi\in\exp^{\perp-1}(D\setminus F)}((\exp^\perp)^*\tilde\omega)_\xi\\
&=\int_{p\in D\setminus F}(\beta_{even}(d_p^2)-\beta_{odd}(d_p^2))\tilde\omega_p\\
&=\int_{D\setminus F}\chi(M)\tilde\omega=\chi(M)\mathrm{Vol}(D)\,,
\end{aligned}$$

where $\beta_{even}(d_p^2)$ (resp. $\beta_{odd}(d_p^2)$) is the number of critical points of even (resp. odd) index of d_p^2. Thus we obtain the first relation. Similarly, we can obtain the second relation in terms of the Morse inequality. Further, it is shown that the equality sign holds in the second relation if M is taut. □

Now we shall calculate the volume $\mathrm{vol}_{N,r}$ of a geodesic sphere of radius r in an m-dimensional simply connected symmetric space $N=G/K$ of non-positive curvature. Let $S_p(r)$ be the geodesic sphere in N of center p and radius r. Let $S(r)$ be the hypersphere in T_pN of center 0 and radius r, where 0 is the origin of T_pN. Denote by exp the exponential map of N, which diffeomorphically maps $S(r)$ onto $S_p(r)$. Also, denote by $\omega_{S_p(r)}$ (resp. $\omega_{S(r)}$) the volume element of $S_p(r)$ (resp. $S(r)$). Let $\xi\in S(r)$ and X be a unit tangent vector of $S(r)$ at ξ and X_0 be the element of T_pN corresponding to X under the natural identification of $T_\xi(T_pN)$ and T_pN. Define a two-parameter map δ to N by $\delta(t,s)=\exp s(\cos t\cdot\xi+r\sin t\cdot X_0)$. For simplicity, denote $\delta_*(\frac{\partial}{\partial t})$ (resp. $\delta_*(\frac{\partial}{\partial s})$) by $\frac{\partial}{\partial t}$ (resp. $\frac{\partial}{\partial s}$). Set $J:=\frac{\partial}{\partial t}|_{t=0}$, which is a Jacobi field along the geodesic γ_ξ. It is clear that $J(0)=0$ and $J'(0)=rX_0$. Hence, it follows from (3.1) that $J(s)=rsP_{\gamma_\xi|_{[0,s]}}D^{si}_{s\xi}X_0$. On the other hand, we have $J(1)=r\exp_*X$. Thus we obtain $\exp_*X=P_{\gamma_\xi}D^{si}_\xi X_0$. This deduces $(\exp^*\omega_{S_p(r)})_\xi=\det(D^{si}_\xi|_{\mathrm{Span}\{\xi\}^\perp})(\omega_{S(r)})_\xi=\det D^{si}_\xi(\omega_{S(r)})_\xi$ and hence

$$\mathrm{vol}_{N,r}=\int_{\xi\in S(r)}\det D^{si}_\xi\,\omega_{S(r)}=r^{m-1}\int_{\xi\in S(1)}\det D^{si}_{r\xi}\,\omega_{S(1)}\,,\tag{3.3}$$

where $S(1)$ is the unit hypersphere in T_pN centered 0.

4. Proofs of Theorems A and Corollaries C and D

In this section, we shall prove Theorem A and Corollaries C and D. First we shall prove Theorem A in terms of Proposition 3.4.

PROOF OF THEOREM A. Fix $o \in N$. Let $B_o(r)$ be the geodeisc ball of center o and radius r. Set $\phi_r(\xi) := \sup\{s \mid \gamma_\xi([0,s]) \subset B_o(r)\}$ for $\xi \in U^\perp M$ and a sufficiently big positive number r. Then we have

$$\begin{aligned}&\int_{\xi\in\exp^{\perp-1}(B_o(r))} \det(\mathrm{pr}_T \circ D^{ct}_\xi|_{T_{\pi(\xi)}M} - A_\xi)\det D^{si}_\xi \omega_{T^\perp M}\\ &= \int_{\xi\in U^\perp M}\left(\int_0^{\phi_r(\xi)} \det\left(\frac{1}{s}\mathrm{pr}_T \circ D^{ct}_{s\xi}|_{T_{\pi(\xi)}M} - A_\xi\right)\det D^{si}_{s\xi} s^{m-1} ds\right)\omega_{U^\perp M}\,.\end{aligned}$$

Also, we can show $\lim_{s\to\infty} \frac{1}{s} D^{ct}_{s\xi} = D_\xi$. These relations together with $\mathrm{Vol}(B_o(r)) = \int_0^r \mathrm{vol}_{N,s} ds$ deduce

$$\begin{aligned}&\lim_{r\to\infty} \frac{1}{\mathrm{Vol}(B_o(r))}\int_{\xi\in\exp^{\perp-1}(B_o(r))} \det(\mathrm{pr}_T \circ D^{ct}_\xi|_{T_{\pi(\xi)}M} - A_\xi)\det D^{si}_\xi \omega_{T^\perp M}\\ &= \lim_{r\to\infty}\frac{1}{\mathrm{vol}_{N,r}}\int_{\xi\in U^\perp M}\det\left(\frac{1}{\phi_r(\xi)}\mathrm{pr}_T \circ D^{ct}_{\phi_r(\xi)\xi}|_{T_{\pi(\xi)}M} - A_\xi\right)\\ &\qquad\qquad \times \det D^{si}_{\phi_r(\xi)\xi}\phi_r(\xi)^{m-1}\frac{d\phi_r(\xi)}{dr}\omega_{U^\perp M}\\ &= \int_{\xi\in U^\perp M}\det(\mathrm{pr}_T \circ D_\xi|_{T_{\pi(\xi)}M} - A_\xi)\lim_{r\to\infty}\left(\frac{\phi_r(\xi)^{m-1}}{\mathrm{vol}_{N,r}}\det D^{si}_{\phi_r(\xi)\xi}\frac{d\phi_r(\xi)}{dr}\right)\omega_{U^\perp M}\,,\end{aligned}$$

which is equal to $\chi(M)$ by Proposition 3.4. On the other hand, it is easy to show that $\lim_{r\to\infty}(\phi_r(\xi) - r) = b_\xi(o)$ and that $\phi_r(\xi)$ is concave. Hence we have $\lim_{r\to\infty}\frac{\phi_r(\xi)}{r} = \lim_{r\to\infty}\frac{d\phi_r(\xi)}{dr} = 1$. Also, we can show $\lim_{r\to\infty}\det(D^{si}_{\phi_r(\xi)\xi} \circ (D^{si}_{r\xi})^{-1}) = e^{b_\xi(o)\mathrm{Tr}D_\xi}$. Therefore, we have

$$\lim_{r\to\infty}\left(\frac{\phi_r(\xi)^{m-1}}{\mathrm{vol}_{N,r}}\det D^{si}_{\phi_r(\xi)\xi}\frac{d\phi_r(\xi)}{dr}\right) = \frac{1}{v(\xi)}e^{b_\xi(o)\mathrm{Tr}D_\xi}\,.$$

Thus we obtain the relation (1.5). Similarly, we obtain the relation (1.6). Assume that M is contained in a geodesic ball of radius r_0. Let p_0 be the center of the geodesic ball. Then we have $|b_\xi(p_0)| \le r_0$ ($\xi \in U^\perp M$). Hence the relation (1.6) for $o = p_0$ deduces (1.7). □

Next we shall prove Corollary C.

PROOF OF COROLLARY C. Since the ambient space is $H^m(c)$, we have

$$D_\xi = \sqrt{-c}\,\mathrm{pr}^\perp_\xi \quad\text{and}\quad D^{si}_{s\xi} = \mathrm{pr}_\xi + \frac{\sinh(\sqrt{-c}s)}{\sqrt{-c}s}\mathrm{pr}^\perp_\xi \quad (\xi \in U^\perp M)\,,$$

where pr_ξ (resp. $\mathrm{pr}_\xi^\perp$) is the orthogonal projection of $T_{\pi(\xi)}H^m(c)$ onto $\mathrm{Span}\{\xi\}$ (resp. $\mathrm{Span}\{\xi\}^\perp$). These relations deduce $\mathrm{Tr}D_\xi = (m-1)\sqrt{-c}$ and $\det D^{si}_{s\xi} = (\frac{\sinh(\sqrt{-c}s)}{\sqrt{-c}s})^{m-1}$. The second relation together with (3.3) deduces $\upsilon(\xi) = \mathrm{Vol}(S^{m-1}(1))$. Hence, the statement of this corollary is deduced from Theorem A. □

Next we shall prove Corollary D.

PROOF OF COROLLARY D. Since the ambient space is $\mathbf{F}H^m(c)$, we have $D_\xi = \sqrt{-c}\mathrm{pr}_1^\xi + \frac{\sqrt{-c}}{2}\mathrm{pr}_2^\xi$ and $D^{si}_{s\xi} = \mathrm{pr}_0^\xi + \frac{\sinh(\sqrt{-c}s)}{\sqrt{-c}s}\mathrm{pr}_1^\xi + \frac{2\sinh\frac{\sqrt{-c}s}{2}}{\sqrt{-c}s}\mathrm{pr}_2^\xi$, where pr_0^ξ is the orthogonal projection of $T_{\pi(\xi)}\mathbf{F}H^m(c)$ onto $\mathrm{Span}\{\xi\}$ and pr_i^ξ $(i = 1, 2)$ are as in the statement of this corollary. These relations deduce $\mathrm{Tr}D_\xi = \frac{(qm+q-2)\sqrt{-c}}{2}$ and $\det(rD^{si}_{r\xi}) = (\frac{\sinh(\sqrt{-c}r)}{\sqrt{-c}})^{q-1}(\frac{2\sinh\frac{\sqrt{-c}r}{2}}{\sqrt{-c}})^{qm-q}\times r$. The second relation together with (3.3) deduces $\upsilon(\xi) = \mathrm{Vol}(S^{qm-1}(1))$. Hence the statement of this corollary is deduced from Theorem A. □

5. Examples

In this section, for a geodesic sphere in $H^m(c)$ and $\mathbf{F}H^m(c)$, we shall calculate the integral quantities in Corollaries C and D and confirm that the equality and the inequality in those corollaries hold. First we shall consider a geodesic sphere $S_o(r)$ of center o and radius r in $H^m(c)$. Let ξ be the inward unit normal vector field of $S_o(r)$. Then we can show $b_{\pm\xi_x}(o) = \pm r$ $(x \in M)$ and $A_\xi = \frac{\sqrt{-c}}{\tanh(\sqrt{-c}r)}\mathrm{id}$. Hence we see that the integral quantity in the first relation of Corollary C is equal to $\frac{\sqrt{-c}^{m-1}\{(-1)^{m-1}+1\}}{\sinh^{m-1}(\sqrt{-c}r)}\mathrm{Vol}(S_o(r))$, which is further equal to $\{(-1)^{m-1}+1\}\mathrm{Vol}(S^{m-1}(1))$. Thus it is confirmed that the first relation of Corollary C holds. Similarly, we can show that the integral quantity in the second relation of Corollary C is equal to $\frac{2\sqrt{-c}^{m-1}}{\sinh^{m-1}(\sqrt{-c}r)}\mathrm{Vol}(S_o(r))$, which is further equal to $2\mathrm{Vol}(S^{m-1}(1))$. Thus it is confirmed that the equality sign holds in the second relation of Corollary C. This is compatible with the fact that $S_o(r)$ is taut.

Next we shall consider a geodesic sphere $S_o(r)$ of center o and radius r in $\mathbf{F}H^m(c)$. Let ξ be the inward unit normal vector field of $S_o(r)$. Then we can show $b_{\pm\xi_x}(o) = \pm r$ $(x \in M)$ and $A_{\xi_x} = \frac{\sqrt{-c}}{\tanh(\sqrt{-c}r)}\mathrm{pr}_{\xi_x} \oplus \frac{\sqrt{-c}}{2\tanh\frac{\sqrt{-c}r}{2}}\mathrm{pr}_{\xi_x}^\perp$ $(x \in M)$. Hence we see that the integral quantity in the first relation of Corollary D is equal to

$$\sum_{i=1}^{2} e^{\frac{(-1)^i (qm+q-2)\sqrt{-c}r}{2}} \left(1 - \frac{(-1)^i}{\tanh(\sqrt{-c}r)}\right)^{q-1} \left(1 - \frac{(-1)^i}{\tanh\frac{\sqrt{-c}r}{2}}\right)^{qm-q} \times \frac{\sqrt{-c}^{qm-1}}{2^{qm-q}} \mathrm{Vol}(S_o(r)),$$

which is equal to 0. Thus it is confirmed that the first relation of Corollary D holds. Similarly, we can show that the integral quantity in the second relation of Corollary D is equal to

$$\sum_{i=1}^{2} e^{\frac{(-1)^i (qm+q-2)\sqrt{-c}r}{2}} \left|1 - \frac{(-1)^i}{\tanh(\sqrt{-c}r)}\right|^{q-1} \times \left|1 - \frac{(-1)^i}{\tanh\frac{\sqrt{-c}r}{2}}\right|^{qm-q} \times \frac{\sqrt{-c}^{qm-1}}{2^{qm-q}} \mathrm{Vol}(S_o(r)),$$

which is further equal to

$$\frac{\sqrt{-c}^{qm-1}}{2^{qm-q}\sinh^{q-1}(\sqrt{-c}r)\sinh^{qm-q}\frac{\sqrt{-c}r}{2}} \times 2\mathrm{Vol}(S_o(r)) \quad (= 2\mathrm{Vol}(S^{qm-1}(1))).$$

Thus it is confirmed that the equality sign holds in the second relation of Corollary D. This is compatible with the fact that $S_o(r)$ is taut.

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