TOKYO J. MATH.
VOL. 27, NO. 1, 2004

Chaos and Entropy for Graph Maps

Megumi MIYAZAWA

(Communicated by Y. Ohnita)

Abstract. Our aim is to check that the notions of positive entropy, chaos in the sense of Devaney and ω-chaos are equivalent for the graph maps.

For any continuous map of the compact interval into itself, Li ([Li]) showed that it is chaotic in the sense of Devaney if and only if it has positive entropy. On the other hand, for continuous maps of the circle into itself, we showed that the result due to Li is also true for maps of the circle (see [Miy]). In this paper we generalize these results to graph maps.

First we introduce some notions and definitions. We say that e is an *edge* if there is a continuous surjection $\varphi : [0, 1] \to e$ such that $\varphi|_{(0,1)}$ is a homeomorphism. Here $\varphi|_A$ is the restriction of φ to A. The set $\varphi((0, 1))$ is called the *interior* of e, and $\varphi(0)$ and $\varphi(1)$ are *endpoints* of e. Each endpoint of each edge is called a *vertex*. A *graph* is a connected compact metric space which is a union of finitely many disjoint sets: interiors of edges and the set of vertices (see [LM] for more details).

Let (X, d) be a compact metric space and $C(X)$ denote the set of all continuous maps of X into itself. For a map $f \in C(X)$ we say that a set $A \subset X$ is *f-invariant* if $f(A) \subset A$, and f is called *topologically transitive* if for every pair of non-empty open sets U and V in X, there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$. We say that f is *topologically mixing* if for any non-empty open sets U and V in X, there is a positive integer N such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$. $f \in C(X)$ is said to be *chaotic in the sense of Devaney* if there is an f-invariant closed infinite set $D \subset X$ such that the following conditions hold:

(D1) $f|_D$ is topologically transitive and

(D2) $\mathrm{Per}(f|_D)$ is dense in D,

where $\mathrm{Per}(f|_D)$ is the set of all periodic points of $f|_D$. We call such a set D *chaotic* (see [Dev] and [Li]).

On the other hand, for any continuous map f of the compact interval into itself, Li ([Li]) introduced the following notion of ω-chaos, and showed that f is ω-chaotic if and only if f has positive entropy. We say that a subset S of X is an *ω-scrambled set* for f if, for any $x, y \in S$ with $x \neq y$, the following conditions hold:

Received April 25, 2003; revised November 5, 2003
2000 Mathematics Subject Classification. Primary 37E25, 54H20.

(ω1) $\omega(x, f) \setminus \omega(y, f)$ is uncountable,
(ω2) $\omega(x, f) \cap \omega(y, f) \neq \emptyset$ and
(ω3) $\omega(x, f) \not\subset \mathrm{Per}(f)$,

where the set $\omega(x, f)$ is an ω-limit set of a point $x \in X$. We say that f is *ω-chaotic* if there exists an uncountable ω-scrambled set for f (see [Li]).

In the present paper we shall extend the results in [Li] and [Miy] to the case of continuous graph maps. More precisely, our aim is to show the following:

MAIN THEOREM. *Let f be a continuous map of a graph into itself. The following conditions are equivalent.*

(I) *f has positive topological entropy.*
(II) *There is an uncountable ω-scrambled set S such that $\bigcap_{x \in S} \omega(x, f) \neq \emptyset$.*
(III) *f is ω-chaotic.*
(IV) *There is an ω-scrambled set consisting of exactly two points.*
(V) *f is chaotic in the sense of Devaney.*
(VI) *There is a chaotic set D containing an uncountable ω-scrambled set S.*

For the proof of Main Theorem, implications (II) $\Rightarrow$ (III) $\Rightarrow$ (IV) and (VI) $\Rightarrow$ (V) are obvious. By Propositions 1 and 2 below, we have (IV) $\Rightarrow$ (I) and (V) $\Rightarrow$ (I). Let G be a graph and we denote by $h(f)$ the topological entropy of $f \in C(G)$.

PROPOSITION 1. *Let $f \in C(G)$. If $h(f) = 0$ then f is not chaotic in the sense of Devaney.*

PROPOSITION 2. *Let $f \in C(G)$. If $h(f) = 0$ then there is no ω-scrambled set consisting of exactly two points.*

By Proposition 3 below, we have (I)$\Rightarrow$(II) and (I)$\Rightarrow$(VI).

PROPOSITION 3. *Let $f \in C(G)$. If $h(f) > 0$ then there are a chaotic set D and an uncountable ω-scrambled set $S \subset D$ such that $\bigcap_{x \in S} \omega(x, f) \neq \emptyset$.*

We prepare some lemmas that are used in proving the propositions.

SPECTRAL DECOMPOSITION AND HORSESHOES.

For $f \in C(G)$, $x \in G$ and $A \subset G$, we put $Orb(x, f) = \bigcup_{i=0}^{\infty} f^i(x)$, $Orb(A, f) = \bigcup_{i=0}^{\infty} f^i(A)$, $\Lambda(f) = \bigcup_{x \in G} \omega(x, f)$ and $PP(f) = \{p \geq 1 : f$ has a periodic point of period $p\}$.

A connected closed set $Y \subset G$ is called a *subgraph*. For a map $f \in C(G)$, a subgraph Y is called *periodic of period n* if $Y, \cdots, f^{n-1}(Y)$ are pairwise disjoint and $f^n(Y) = Y$.

For a periodic subgraph Y of period n we put $M = \mathrm{Orb}(Y, f)$ and

$$E(M) = \{x \in M : \overline{\mathrm{Orb}(U, f)} = M \text{ for any neighbourhood } U \text{ of } x \text{ in } M\}.$$

Here $\bar{A}$ is the closure of A. If $\mathrm{Per}(f|_M) \neq \emptyset$ and $\sharp E(M) = \infty$ ($\sharp A$ denotes the cardinality of A), then we call the set $E(M)$ a *basic set* and denote it by $B(M)$. On the other hand, if

$\mathrm{Per}(f|_M) = \emptyset$ and $\sharp E(M) = \infty$, then we call it a *circle-like set* $C(M)$.

THEOREM 4 [Bl2, Theorem 2]. *Let* n, $E(M)$, $B(M)$, $C(M)$ *be as above. If* $\sharp E(M) = \infty$, *then the following conditions hold.*

(1) *There exist a set* $K = \bigcup_{i=1}^{n} Z_i$ *(*Z_i *is a connected graph), a topologically transitive map* $g : K \to K$ *and a continuous surjection* $\varphi : M \to K$ *such that* (a) g *permutes* $\{Z_i\}_{i=1}^{n}$ *cyclically,* (b) $\varphi(E(M)) = K$, (c) *for any* $y \in K$ *the set* $\varphi^{-1}(\{y\})$ *is connected and* $\varphi^{-1}(\{y\}) \cap E(M) \subset \partial(\varphi^{-1}(\{y\}))$. *Here* ∂A *is the boundary of* A.

(2) *In the case when* $E(M) = C(M)$ *we can obtain the above connected graph* Z_i *as the unit circle and* $g^n|_{Z_i}$ *is an irrational rotation* ($i = 1, \cdots, n$)*, which implies* $h(f|_{C(M)}) = 0$.

(3) *In the case when* $E(M) = B(M)$ *there exist a number* $k \geq 1$ *and a subset* $D \subset B(M)$ *such that the set* $f^i(D) \cap f^j(D)$ *is finite* ($0 \leq i < j < kn$), $f^{kn}(D) = D$, $\bigcup_{i=0}^{kn-1} f^i(D) = B(M)$ *and* $f^{kn}|_D$ *is topologically mixing. Moreover there exists* $m \geq 1$ *such that* $\{mi : i \geq 1\} \subset \mathrm{PP}(f|_{B(M)})$.

Let $Y_0, Y_1, \cdots$ be periodic subgraphs of periods $m_0, m_1, \cdots$ with $Y_i \supset Y_{i+1}$ and m_i is a divisor of m_{i+1} for any $i \geq 0$. In the case when $m_i \to \infty$, for $Q = \bigcap_j \mathrm{Orb}(Y_j)$ we call any invariant closed set $S' \subset Q$ a *solenoidal set*, and put $S(Q) = Q \cap \Lambda(f)$. Let $\mathbf{Z}_m$ be the quotient group of $\mathbf{Z}$ modulo $m(\geq 1)$.

THEOREM 5 [Bl2, Theorem 1]. *Let* $\{m_j\}, Q, S(Q)$ *be as above. Define* $H(Q)$= $\{(r_0, r_1, \cdots) \in \mathbf{Z}_{m_0} \times \mathbf{Z}_{m_1} \times \cdots : r_{i+1} \equiv r_i \pmod{m_i}\}$, *and* $\tau : H(Q) \to H(Q)$ *by* $\tau((r_0, r_1, \cdots)) = (r_0 + 1, r_1 + 1, \cdots)$. *Then the following conditions hold*:

(1) *there exists a continuous surjection* $\varphi : Q \to H(Q)$ *such that* $\tau \circ \varphi = \varphi \circ f|_Q$;

(2) *for any* $r \in H(Q)$ *the set* $\varphi^{-1}(\{r\})$ *is a connected component of* Q *and* $\varphi|_{S(Q)}$ *is at most* 2*-to-*1;

(3) $\mathrm{Per}(f|_Q) = \emptyset$ *and* $h(f|_Q) = 0$.

We call a periodic orbit Z a *maximal-cycle* if there is no point $x \in G$ with $\omega(x, f) \supsetneq Z$.

THEOREM 6 [Bl2, Theorem 3]. *Let* $f \in C(G)$. *Then there exist a family of maximal-cycles* $\mathcal{Z}_f = \{Z_\alpha\}$, *a finite number of circle-like sets* $\mathcal{C}_f = \{C_i\}_{i=1}^{k}$, *a finite or countable family of basic sets* $\mathcal{B}_f = \{B_j\}$ *and a family of solenoidal sets* $\mathcal{S}_f = \{S_\beta\}$ *such that*

$$\Lambda(f) = \left(\bigcup_\alpha Z_\alpha\right) \cup \left(\bigcup_{i=1}^{k} C_i\right) \cup \left(\bigcup_j B_j\right) \cup \left(\bigcup_\beta S_\beta\right).$$

Moreover, *if we set* $\mathcal{D}_f = \mathcal{Z}_f \cup \mathcal{C}_f \cup \mathcal{B}_f \cup \mathcal{S}_f$, *then the following holds*: *for two different elements* $P, P' \in \mathcal{D}_f$ *with non-empty intersection*, *we have* $P, P' \in \mathcal{B}_f$.

LEMMA 7 [Bl1]. *Let* $\mathcal{D}_f$ *be as in Theorem* 6. *For* $x \in G$ *there exists* $P_x \in \mathcal{D}_f$ *such that* $P_x \supset \omega(x, f)$.

LEMMA 8 [Bl1, Theorems 1 and 3]. *Let $\mathcal{C}_f$ and $\mathcal{S}_f$ be as in Theorem* 6. *For $x \in G$ the following conditions hold*:

(1) *if $\omega(x, f) \subset C$ for $C \in \mathcal{C}_f$ then $\omega(x, f) = C$ and*

(2) *if $\omega(x, f) \subset S$ for $S \in \mathcal{S}_f$ then $S \setminus \omega(x, f)$ is at most countable set.*

A set $I \subset G$ is called an *interval* if there is a homeomorphism $h : J \to I$, where J is $[0, 1]$, $(0, 1]$, $[0, 1)$ or $(0, 1)$. A set $h((0, 1))$ is called the *interior* of I. If $I = h([0, 1])$, the interval I is called a *closed interval*.

Let I be a closed interval of G and $J_1, \cdots, J_s$ be closed subintervals of I, where $s \geq 2$. If $f(J_j) = I$ $(j = 1, \cdots, s)$, $J_1 \cup \cdots \cup J_s \subset \mathrm{Int}(I)$ and $J_1, \cdots, J_s$ are pairwise disjoint, the finite sequence $(I; J_1, \cdots, J_s)$ is called a *strong s-horseshoe*, where $\mathrm{Int}(I)$ is the interior of I.

THEOREM 9 [LM], [Bl2]. *Let $f \in C(G)$. Then the following conditions are equivalent.*

(1) $h(f) > 0$.

(2) *There exist integers $k > 0$ and $s \geq 2$ such that f^k has a strong s-horseshoe.*

(3) *There exists an integer $m \geq 1$ such that $\{mn : n \geq 1\} \subset \mathrm{PP}(f)$.*

This theorem is proved in [LM, Theorems B and E, and Lemmas 3.3 and 3.4] and [Bl2, Main Theorem].

LEMMA 10. *Let $f \in C(G)$. If f has a basic set then $h(f) > 0$.*

This lemma is concluded from Theorem 4 (3) and Theorem 9.

THE PROOF OF THE PROPOSITIONS.

PROOF OF PROPOSITION 1. Suppose f has a chaotic set D. By $(D1)$ there exists $a \in D$ such that $\omega(a, f) = \overline{\mathrm{Orb}(a, f)} = D$. By Lemma 7, there exists $P_a \in \mathcal{D}_f = \mathcal{Z}_f \cup \mathcal{C}_f \cup \mathcal{B}_f \cup \mathcal{S}_f$ such that $D = \omega(a, f) \subset P_a$. Since D is infinite and $h(f) = 0$, by Lemma 10 we have $P_a \in \mathcal{C}_f \cup \mathcal{S}_f$. By the definition of circle-like sets and Theorem 5 (3), we have $P_a \cap \mathrm{Per}(f) = \emptyset$, this contradicts the condition (D2). Therefore f is not chaotic in the sense of Devaney. □

PROOF OF PROPOSITION 2. Suppose $\{y, z\}$ $(y \neq z)$ is an ω-scrambled set. By Lemma 7, there exist $P_y, P_z \in \mathcal{D}_f = \mathcal{Z}_f \cup \mathcal{C}_f \cup \mathcal{B}_f \cup \mathcal{S}_f$ such that $\omega(y, f) \subset P_y$ and $\omega(z, f) \subset P_z$. Since $\{y, z\}$ satisfies the condition $(\omega 1)$ and $h(f) = 0$, by Lemma 10 we have $P_y, P_z \in \mathcal{C}_f \cup \mathcal{S}_f$. In the case when $P_y \neq P_z$, Theorem 6 implies $P_y \cap P_z = \emptyset$, so $\omega(y, f) \cap \omega(z, f) = \emptyset$, this contradicts the condition $(\omega 2)$. Let $P_y = P_z = P$. In the case when $P \in \mathcal{C}_f$, Lemma 8 (1) implies $\omega(y, f) = \omega(z, f)$, this contradicts the condition $(\omega 1)$. In the case when $P \in \mathcal{S}_f$, by Lemma 8 (2), $\omega(y, f) \setminus \omega(z, f)$ is at most countable set, this contradicts the condition $(\omega 1)$. Therefore there is no ω-scrambled set consisting of exactly two points. □

PROOF OF PROPOSITION 3. Since $h(f) > 0$, by Theorem 9 there exists an integer $n > 0$ such that f^n has a strong 2-horseshoe. Let Σ_2 be the compact metric space of all infinite sequences $(a_1, a_2, \cdots)$ $(a_k = 0$ or $1)$, and $\sigma : \Sigma_2 \to \Sigma_2$ be the map defined by $\sigma((a_1, a_2, \cdots)) = (a_2, a_3, \cdots)$. In the same way as in the proof of [BC, Proposition II.15], we can easily obtain a closed set Y and a continuous surjection $q : Y \to \Sigma_2$ such that $f^n(Y) = Y$, $q \circ f^n|_Y = \sigma \circ q$ and q is at most 2-to-1 map. Furthermore, there are only countably many points in Σ_2 which have 2 preimages for q, and if one of the preimages for q is periodic, then so is the other. Applying the result of Li [(Li, Chapter 4)], we can obtain a chaotic set D and an uncountable ω-scrambled set $S \subset D$ such that $\bigcap_{x \in S} \omega(x, f) \neq \emptyset$. $\square$

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Present Address:
KAMITSURUMA, SAGAMIHARA-SHI, KANAGAWA, 228–0802 JAPAN.