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Harmonic Sections Normal to Submanifolds and Their Stability

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Abstract. For sphere bundles of the induced bundles of isometric immersions, harmonic sections which are normal to submanifolds and their index are studied. The lower bounds of the index of these sections are given in terms of intrinsic quantities of submanifolds.

1. Introduction

Let M be a Riemannian manifold and TM its tangent bundle with Sasaki metric. The unit tangent sphere bundle is denoted by $U(TM)$. A smooth unit vector field on M is a smooth map from M into $U(TM)$. Therefore, the energy of a unit vector field can be defined. A smooth unit vector field is said to be a harmonic vector field if it is a stationary point of the energy of vector fields with unit length. Note that the volume of a unit vector field and a minimal vector field can be considered. We refer to [3], [5], [11], [12], [13], [15], for example. For a harmonic vector field, its stability or instability is also studied. See [4], [5] and [15], for example. These formulations can be generalized to the case of the sphere bundle of a Riemannian vector bundle with metric connection. A section with unit length is called a harmonic section if it is a stationary point of the energy of sections with unit length relative to the canonical metric. We refer to [8] and [15].

Let E be a Riemannian vector bundle over a Riemannian manifold (M, g) with fiber metric h. The space of smooth sections of a vector bundle E is denoted by $\Gamma(E)$. Set $UE(= U(E)) := \{\eta \in E | h(\eta, \eta) = 1\}$. The set of all sections $\xi \in \Gamma(E)$ satisfying $h(\xi(x), \xi(x)) = 1$ for all $x \in M$ is denoted by $\Gamma(UE)$. Let $f : (M, g) \to (\tilde{M}, \tilde{g})$ be an isometric immersion. Set $n := \dim M$ and $n + p := \dim \tilde{M}$. We have $f^{\#}(T\tilde{M}) = TM \oplus T^{\perp}M$, where $f^{\#}(T\tilde{M})$ is the induced bundle of $T\tilde{M}$ by f and $T^{\perp}M$ is the normal bundle. The inclusion map from $T^{\perp}M$ to $f^{\#}(T\tilde{M})$ is denoted by ι. Let S and $\nabla^{\perp}$ be the shape operator and the normal connection of f, respectively.

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Let $\mathcal{E}$ be the energy functional for maps from M to $U(f^{\#}(T\tilde{M}))$ with canonical metric. For a section $\xi \in \Gamma(U(T^{\perp}M))$, we have

$$\mathcal{E}(\iota \circ \xi) = \frac{n}{2}\mathrm{Vol}(M) + \frac{1}{2}\int_M (\|S_\xi\|^2 + \|\nabla^{\perp}\xi\|^2)dv_g\,.$$

The Hessian at a harmonic section η is denoted by $\mathcal{H}_\eta$. Clearly, if $p = 1$ and f is totally geodesic, then $\iota \circ \xi$ attains a minimum, hence, it is a harmonic section of $\Gamma(U(f^{\#}(T\tilde{M})))$ and $\mathrm{Index}(\mathcal{H}_{\iota\circ\xi}) = 0$, where $\xi \in \Gamma(U(T^{\perp}M))$. On the other hand, if the square of the norm of shape operator S_ξ is "large", we deduced from the second variation formula that $\iota \circ \xi$ is unstable ($\mathrm{Index}(\mathcal{H}_{\iota\circ\xi}) > 0$) for the harmonic section $\iota \circ \xi$. For example, as we will prove in Section 5 (Theorem 5.1), if $\tilde{M} = S^{n+1}(1)$ (the unit sphere with standard metric) and M is a constant mean curvature hypersurface with $\|S_\xi\|^2 > n/(n-2)$ and $n \geq 3$, then $\iota\circ\xi$ is unstable. Hence, the relations between the square of the norm of shape operator and $\mathrm{Index}(\mathcal{H}_{\iota\circ\xi})$ are of interest. If $\xi \in \Gamma(U(T^{\perp}M))$, then variation vector fields of $\iota \circ \xi$ in $\Gamma(U(f^{\#}(T\tilde{M})))$ can be identified with sections of $f^{\#}(T\tilde{M})$ normal to $\iota \circ \xi$. Therefore it seems that the index of the harmonic section $\iota \circ \xi$ also depends on intrinsic quantities of a submanifold. The purpose of this paper is to study harmonic sections of $\Gamma(U(f^{\#}(T\tilde{M})))$ which are normal to submanifolds.

In Section 2, we will prepare the preliminaries. The harmonic sections normal to submanifolds are studied and their examples are given in Section 3. The lower bounds of the index for harmonic sections which are normal to submanifolds are given in terms of intrinsic quantities of submanifolds in Section 4. Finally, in the last section, we study the stability of harmonic sections for constant mean curvature hypersurfaces in the unit spheres.

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2. The energy functionals for sections and splitting of vector bundle with connection

Let E be a Riemannian vector bundle over an n-dimensional Riemannian manifold (M, g) with a fiber metric g^E and a metric connection ∇^E. The Levi-Civita connection of g is denoted by ∇^0. Let $K : TE \to E$ be the connection map with respect to ∇^E. The space of cross sections of E is denoted by $\Gamma(E)$. The canonical metric G on E is defined by

$$G(\xi, \xi) = g(p_*(\xi), p_*(\xi)) + g^E(K(\xi), K(\xi))\,,$$

where $\xi \in T(TE)$ and $p : E \to M$ is the bundle projection. Set $UE(= U(E)) := \{\eta \in E \mid g^E(\eta, \eta) = 1\}$. The set of all sections $\xi \in \Gamma(E)$ satisfying $g^E(\xi(x), \xi(x)) = 1$ for all $x \in M$ is denoted by $\Gamma(UE)$. The restriction of G to UE is also denoted by G. Hereafter we assume that M is compact. Let $\mathcal{E}$ be the energy functional defined on the space of smooth maps from

M to UE. For a smooth section $\xi \in \Gamma(UE)$, the energy $\mathcal{E}(\xi)$ is given by

$$\mathcal{E}(\xi) = \frac{n}{2}\mathrm{Vol}(M) + \frac{1}{2}\int_M \|\nabla^E \xi\|^2 dv_g\,,$$

where dv_g denotes the volume element of (M, g). The second term of $\mathcal{E}(\xi)$ is called the vertical energy of ξ. Note that the vertical energy is defined in more general setting (cf. [14]). In the case where $E = TM$, this term is also called the total bending of the vector field up to constant (cf. [13]). The variation vector field of $\xi \in \Gamma(UE)$ can be identified with a smooth section of E orthogonal to ξ. Set

$$V_\xi := \{\eta \in \Gamma(E) \mid g^E(\xi, \eta) = 0\}$$

for $\xi \in \Gamma(UE)$. The rough Laplacian $\bar{\Delta}^{\nabla^E}$ of ∇^E is defined by

$$\bar{\Delta}^{\nabla^E}(\xi) = -\sum_{k=1}^{n}(\nabla^E_{e_i}\nabla^E_{e_i}\xi - \nabla^E_{\nabla^0_{e_i}e_i}\xi)\,,$$

for $\xi \in \Gamma(E)$, where $e_1, \cdots, e_n$ is an orthonormal frame of (M, g). We say that $\xi \in \Gamma(UE)$ is a *harmonic section* of UE if ξ is a stationary point of $\mathcal{E}|_{\Gamma(UE)}$. The following Lemma is proved in [8].

LEMMA 2.1. *A section $\xi \in \Gamma(UE)$ is a harmonic section if and only if the equation*

$$\bar{\Delta}^{\nabla^E}(\xi) = \|\nabla^E\xi\|^2\xi$$

holds.

For a harmonic section $\xi \in \Gamma(UE)$, the Hessian at ξ, which is defined by the second variation formula, is denoted by $\mathcal{H}_\xi$. The following Lemma is given in [15].

LEMMA 2.2. *Let $\xi \in \Gamma(UE)$ be a harmonic section. Then, for $\alpha, \beta \in V_\xi$, the equation*

$$\mathcal{H}_\xi(\alpha, \beta) = \int_M g^E(\bar{\Delta}^{\nabla^E}(\alpha) - \|\nabla^E\xi\|^2\alpha, \beta)dv_g$$

holds.

Next, we consider splitting of vector bundles with connection following the idea of Abe used in [1]. Let E_1 and E_2 be subbundles of E with orthogonal direct sum $E = E_1 \oplus E_2$. For the rest of this section, we assume that $i, j \in \{1, 2\}$ and $i \neq j$ and often omit the symbol of the composition of maps, "$\circ$". Let $\iota_i : E_i \to E$ and $\pi_i : E \to E_i$ be the inclusions and projections, respectively. The following equations hold:

$$\pi_i\iota_i = \mathrm{id}_{E_i}\,, \quad \pi_j\iota_i = 0 \quad \text{and} \quad \iota_j\pi_j + \iota_i\pi_i = \mathrm{id}_E\,.$$

For the connection ∇^E, we set

$$\nabla^i_X := \pi_i\nabla^E_X\iota_i \quad \text{and} \quad B^i_X := \pi_j\nabla^E_X\iota_i\,,$$

where $X \in \Gamma(TM)$. We define $(\nabla^i_X B^i)_Y$ by

$$(\nabla^i_X B^i)_Y \xi = \nabla^j_X (B^i_Y \xi) - B^i_{\nabla^0_X Y} \xi - B^i_Y (\nabla^i_X \xi)\,,$$

for $\xi \in \Gamma(E_i)$ and $X, Y \in \Gamma(TM)$. For the subbundle E_i, the induced fiber metric is denoted by g^E_i. Note that ∇^i is a metric connection with respect to g^E_i. We have

LEMMA 2.3. *For $\xi_i \in \Gamma(E_i)$, the equation*

$$\bar{\triangle}^{\nabla^E}(\iota_i \xi_i) = \iota_i \bar{\triangle}^{\nabla^i} \xi_i - \iota_i \sum_{k=1}^n B^j{}_{e_k}(B^i{}_{e_k} \xi_i) - \iota_j \sum_{k=1}^n \{(\nabla^i_{e_k} B^i)_{e_k} \xi_i + 2B^i{}_{e_k}(\nabla^i_{e_k} \xi_i)\}$$

holds, where $e_1, \cdots, e_n$ is an orthonormal frame of (M, g).

PROOF. We have

$$\begin{aligned}
\bar{\triangle}^{\nabla^E}(\iota_i \xi_i) &= -\sum_{k=1}^n (\nabla^E_{e_k} \nabla^E_{e_k} \iota_i \xi_i - \nabla^E_{\nabla^0_{e_k} e_k} \iota_i \xi_i) \\
&= -\sum_{k=1}^n \{\nabla^E_{e_k} \iota_i (\nabla^i_{e_k} \xi_i) + \nabla^E_{e_k} \iota_j (B^i_{e_k} \xi_i) - \iota_i \nabla^i_{\nabla^0_{e_k} e_k} \xi_i - \iota_j B^i_{\nabla^0_{e_k} e_k} \xi_i\} \\
&= \iota_i \bar{\triangle}^{\nabla^i}(\xi_i) - \sum_{k=1}^n \iota_i B^j_{e_k} B^i_{e_k} \xi_i - \sum_{k=1}^n \{\iota_j (\nabla^i_{e_k} B^i)_{e_k} \xi_i + \iota_j 2B^i_{e_k} (\nabla^i_{e_k} \xi_i)\}\,.
\end{aligned}$$

□

By this lemma, we have

LEMMA 2.4. *Let $\xi_i \in \Gamma(U(E_i))$. Then $\iota_i \xi_i \in \Gamma(UE)$ is a harmonic section if and only if the equations*

$$\bar{\triangle}^{\nabla^i} \xi_i - \sum_{k=1}^n B^j{}_{e_k}(B^i{}_{e_k} \xi_i) = (\|\nabla^i \xi_i\|^2 + \|B^i \xi_i\|^2)\xi_i$$

and

$$\sum_{k=1}^n \{(\nabla_{e_k} B^i)_{e_k} \xi_i + 2B^i{}_{e_k}(\nabla^i_{e_k} \xi_i)\} = 0$$

hold, where $e_1, \cdots, e_n$ is an orthonormal frame of (M, g).

From Lemmas 2.2 and 2.3, we obtain

LEMMA 2.5. *Let $\xi \in \Gamma(U(E))$ be a harmonic section. For $\alpha_i \in \Gamma(E_i) \cap V_\xi$, we have*

$$\mathcal{H}_\xi(\alpha_i, \alpha_i) = \int_M \{g^E_i(\bar{\triangle}^{\nabla^i}(\alpha_i), \alpha_i) - \sum_{k=1}^n g^E_i(B^j_{e_k} B^i_{e_k} \alpha_i, \alpha_i) - \|\nabla\xi\|^2 g^E_i(\alpha_i, \alpha_i)\} dv_g$$

and

$$\mathcal{H}_\xi(\alpha_i, \alpha_j) = -\int_M \sum_{k=1}^n \{g_j^E((\nabla_{e_k} B^i)_{e_k}\alpha_i, \alpha_j) + 2g_j^E(B^i_{e_k}(\nabla^i_{e_k}\alpha_i), \alpha_j)\} dv_g,$$

where $e_1, \cdots, e_n$ *is an orthonormal frame of* (M, g).

3. Harmonic sections normal to submanifolds

Let $f : (M, g) \to (\tilde{M}, \tilde{g})$ be an isometric immersion. Set $n := \dim M$ and $n + p := \dim \tilde{M}$. We have $f^\#(T\tilde{M}) = TM \oplus T^\perp M$, where $f^\#(T\tilde{M})$ is the induced bundle of $T\tilde{M}$ by f and $T^\perp M$ is the normal bundle. The inclusion map from $T^\perp M$ to $f^\#(T\tilde{M})$ is denoted by ι. Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections of g and $\tilde{g}$, respectively. The induced connection of $\tilde{\nabla}$ by f is denoted by $f^\#\tilde{\nabla}$. Let h, S and $\nabla^\perp$ be the second fundamental form, the shape operator and the normal connection of f, respectively. The mean curvature vector is denoted by H. We obtain

$$(f^\#\tilde{\nabla})_X Y = f_*(\nabla_X Y) + h(X, Y)$$

and

$$(f^\#\tilde{\nabla})_X \iota \circ \xi = -f_*(S_\xi(Y)) + \nabla^\perp_X \xi\ ,$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$. In this section, we study harmonic sections relative to the canonical metric on $U(f^\#(T\tilde{M}))$, which are normal to submanifolds. From Lemma 2.3, we have

LEMMA 3.1. *For* $\xi \in \Gamma(T^\perp M)$, *the equation*

$$\bar{\Delta}^{(f^\#\tilde{\nabla})}(\iota \circ \xi) = \bar{\Delta}^{\nabla^\perp}(\xi) + \sum_{k=1}^n h(e_k, S_\xi(e_k)) + \sum_{k=1}^n \{(\nabla_{e_k} S)_\xi(e_k) + 2S_{\nabla^\perp{}_{e_k}\xi}(e_k)\}$$

holds, where we omit the inclusion maps from TM *and* $T^\perp M$ *to* $f^\#(T\tilde{M})$ *and* $e_1, \cdots, e_n$ *is an orthonormal frame of* M.

The following proposition can be obtained immediately.

PROPOSITION 3.2. *For* $\xi \in \Gamma(U(T^\perp M))$, $\iota \circ \xi \in \Gamma(U(f^\#(T\tilde{M})))$ *is a harmonic section if and only if the equations*

$$\bar{\Delta}^{\nabla^\perp}(\xi) + \sum_{k=1}^n h(e_k, S_\xi(e_k)) = (\|S_\xi\|^2 + \|\nabla^\perp\xi\|^2)\xi$$

and

$$\sum_{k=1}^n \{(\nabla_{e_k} S)_\xi(e_k) + 2S_{\nabla^\perp_{e_k}\xi}(e_k)\} = 0$$

hold, where $e_1, \cdots, e_n$ *is an orthonormal frame of* M.

Note that the inclusion map ι and its restriction map to $U(T^\perp M)$ are called the normal map and the spherical Gauss map, respectively. The harmonicity of the spherical Gauss map is studied in [7].

To the end of this section, we give some examples. Let $\tilde{R}$ be the curvature tensor of $\tilde{M}$.

COROLLARY 3.3. *Let* M *be a orientable immersed hypersurface in the space form of a constant curvature, and* $\xi \in \Gamma(U(T^\perp M))$. *Then* $\iota \circ \xi \in \Gamma(U(f^\#(T\tilde{M})))$ *is a harmonic section if and only if* M *has a constant mean curvature.*

PROOF. Since $p = 1$ and $\nabla^\perp \xi = 0$, it it sufficient to prove that

$$\sum_{k=1}^{n} (\nabla_{e_k} S)_\xi (e_k) = 0$$

if and only if $\nabla^\perp H = 0$. It is clear from the Codazzi equation. □

The following notion is defined in [9]. A section $X \in \Gamma(f^\#(T\tilde{M}))$ is called of $\bar{\triangle}^{(f^\#\tilde{\nabla})}$-type k if X admits a finite spectral decomposition

$$X = \sum_{i=1}^{k} X_i, \quad \bar{\triangle}^{(f^\#\tilde{\nabla})} X_i = \lambda_i X_i \quad (i = 1, \cdots, k).$$

We also refer to [2].

COROLLARY 3.4. *Let* M *be a submanifold in* $\tilde{M}$ *with nonzero parallel mean curvature* H *immersed by* f. *Then* $\iota \circ H$ *is of* $\bar{\triangle}^{(f^\#\tilde{\nabla})}$*-type* 1 *if and only if* $\iota \circ (1/\|H\|)H$ *is a harmonic section and* $\|S_H\|$ *is constant. Especially, if* M *is an extrinsic sphere, then* $\iota \circ (1/\|H\|)H$ *is a harmonic section.*

PROOF. Set $\bar{H} := (1/\|H\|)H$. At first, we assume that H is of $\bar{\triangle}^{(f^\#\tilde{\nabla})}$-type 1. Then we have $\bar{\triangle}^{(f^\#\tilde{\nabla})}\bar{H} = \lambda \bar{H}$. Hence $\iota \circ \bar{H}$ is a harmonic section and $\|S_H\|^2$ equals to a constant $\lambda\|H\|^2$. Conversely, Since $\bar{\triangle}^{(f^\#\tilde{\nabla})}\bar{H} = \|S_{\bar{H}}\|^2 \bar{H}$, it follows that H is of $\bar{\triangle}^{(f^\#\tilde{\nabla})}$-type 1. □

We recall the definition of Sasakian manifolds. Let $\tilde{M}$ be a $(2n+1)$-dimensional manifold and φ, V, η be a $(1,1)$-tensor field, a vector field, 1-form on $\tilde{M}$, respectively, such that

$$\varphi^2(X) = -X + \eta(X)V, \quad \varphi(V) = 0, \quad \eta(\varphi(X)) = 0 \quad \text{and} \quad \eta(V) = 1$$

for any vector field X on $\tilde{M}$. Then $\tilde{M}$ is said to have an almost contact structure (φ, V, η) and is called an almost contact manifold. If a Riemannian metric tensor field $\tilde{g}$ is given on an almost contact manifold $\tilde{M}$ and satisfies

$$\tilde{g}(\varphi(X), \varphi(Y)) = \tilde{g}(X, Y) - \eta(X)\eta(Y) \quad \text{and} \quad \eta(X) = g(V, X)$$

for any vector fields X and Y on $\tilde{M}$, then $(\varphi, V, \eta, \tilde{g})$ is called an almost contact metric structure and $\tilde{M}$ is called an almost contact metric manifold. If $d\eta(X, Y) = \tilde{g}(X, \varphi(Y))$ for any vector fields X and Y on $\tilde{M}$, then an almost contact metric structure is called a contact metric structure. If moreover the structure is normal, that is, $\mathcal{N} + d\eta \otimes V = 0$, then a contact metric structure is called Sasakian structure and $\tilde{M}$ is called a Sasakian manifold, where $\mathcal{N}$ is the Nijenhuis torsion for φ. An n-dimensional Riemannian submanifold M in a Sasakian manifold $\tilde{M}$ is called Legendrian if $f^*\eta = 0$.

COROLLARY 3.5. *Let M be a Legendrian submanifold in a $(2n+1)$-dimensional Sasakian manifold $(\tilde{M}, \eta, V, \varphi, \tilde{g})$ immersed by f. Then $f^{\#}(V)$ is a harmonic section if and only if M is minimal, where $f^{\#}(V)$ is the pull back section of V by f.*

PROOF. For simplicity, we use V instead of $f^{\#}(V)$. We have

$$\bar{\Delta}^{\nabla^{\perp}}(V) = nV\,.$$

Since $S_V = 0$, it follows that

$$\sum_{k=1}^{n} h(e_k, S_V(e_k)) = 0\,.$$

Since $\nabla^{\perp}_X V = -\varphi(X)$ holds for all $X \in \Gamma(TM)$, we obtain

$$\sum_{k=1}^{n} (\nabla_{e_k} S)_V(e_k) + 2S_{\nabla^{\perp}_{e_k} V} e_k = -\sum_{k=1}^{n} S_{\varphi(e_k)} e_k\,.$$

Moreover, we get

$$g\Bigg(\sum_{k=1}^{n} S_{\varphi(e_k)} e_k, X\Bigg) = \tilde{g}(h(e_k, X), \varphi(e_k)) = \tilde{g}(h(e_k, e_k), \varphi(X)) = n\tilde{g}(H, \varphi(X))$$

for all tangent vectors X on M, and which implies the conclusion. □

4. The index of harmonic sections normal to submanifolds

In this section, we study the index of harmonic sections normal to submanifolds. Let $f : (M, g) \to (\tilde{M}, \tilde{g})$ be an isometric immersion with $\dim M = n$ and $\dim \tilde{M} = n + p$. Set $r_x(X, X) = \sum_{i=1}^{n} \tilde{g}(\tilde{R}(f_*(e_i), f_*(X)) f_*(X), f_*(e_i))$ for $x \in M$ and $X \in T_x M$, where $e_1, \cdots, e_n$ is an orthonormal basis of $T_x M$. The Ricci tensor of M is denoted by Ric_M. We start with a second variation formula.

LEMMA 4.1. *Let $\xi \in \Gamma(U(T^{\perp}M))$ and assume that M is compact. If $\iota \circ \xi \in \Gamma(U(f^{\#}(T\tilde{M})))$ is a harmonic section, then we have*

$$\mathcal{H}_{\iota\circ\xi}(X, X) = \int_M \{g(\nabla X, \nabla X) - Ric_M(X, X) + r(X, X)$$

$$+ ng(S_H X, X) - \|S_\xi\|^2 g(X, X) - \|\nabla^\perp \xi\|^2 g(X, X)\} dv_g$$

and

$$\mathcal{H}_{\iota\circ\xi}(\eta, \eta) = \int_M \{\|S_\eta\|^2 + \tilde{g}(\nabla^\perp \eta, \nabla^\perp \eta) - \|S_\xi\|^2 \tilde{g}(\eta, \eta) - \|\nabla^\perp \xi\|^2 \tilde{g}(\eta, \eta)\} dv_g$$

for $X \in \Gamma(TM)$ *and* $\eta \in \Gamma(T^\perp M) \cap V_{\iota\circ\xi}$.

PROOF. By Lemma 2.5 and the Gauss equation, we have

$$\begin{aligned}
\mathcal{H}_\xi(X, X) &= \int_M \{g(\nabla X, \nabla X) + \sum_{i=1}^n \tilde{g}(\tilde{R}(e_i, X)X, e_i) \\
&\quad - \sum_{i=1}^n g(R(e_i, X)X, e_i) + n\tilde{g}(H, h(X, X)) \\
&\quad - g(S_\xi, S_\xi) g(X, X) - \tilde{g}(\nabla^\perp \xi, \nabla^\perp \xi) g(X, X)\} dv_g \\
&= \int_M \{g(\nabla X, \nabla X) - Ric_M(X, X) + r(X, X) + n\tilde{g}(H, h(X, X)) \\
&\quad - g(S_\xi, S_\xi) g(X, X) - \tilde{g}(\nabla^\perp \xi, \nabla^\perp \xi) g(X, X)\} dv_g\,.
\end{aligned}$$

The second equation can also be obtained immediately from Lemma 2.5. □

It is well-known that the identity map of a Riemannian manifold M is a harmonic map. The index and nullity of id_M are denoted by $\mathrm{Index}(\mathrm{id}_M)$ and $\mathrm{Null}(\mathrm{id}_M)$ as a harmonic map. The Hessian H_{id_M} at the identity map id_M is given by

$$H_{\mathrm{id}_M}(X, X) = \int_M (g(\nabla X, \nabla X) - Ric_M(X, X)) dv_g$$

for $X \in \Gamma(TM)$ (cf. [10]).

By Lemma 4.1, we have

PROPOSITION 4.2. *Let* $\xi \in \Gamma(U(T^\perp M))$. *Assume that* M *is compact and* $\iota \circ \xi \in \Gamma(U(f^\#(T\tilde{M})))$ *is a harmonic section. If* $(\|S_\xi\|^2 + \|\nabla^\perp \xi\|^2) g(X, X) \ge r(X, X) + ng(S_H(X), X)$ *for all* $X \in TM$, *then we have*

$$\mathrm{Index}(\mathcal{H}_{\iota\circ\xi}) \ge \mathrm{Index}(\mathrm{id}_M)\,.$$

If $(\|S_\xi\|^2 + \|\nabla^\perp \xi\|^2) g(X, X) > r(X, X) + ng(S_H(X), X)$ *for all non zero* $X \in TM$, *then*

$$\mathrm{Index}(\mathcal{H}_{\iota\circ\xi}) \ge \mathrm{Index}(\mathrm{id}_M) + \mathrm{Null}(\mathrm{id}_M)\,.$$

Note that the term $\|S_\xi\|^2 + \|\nabla^\perp \xi\|^2$ is the integrand of a part of the energy functional $\mathcal{E}$. Proposition 4.2 implies that the index of $\mathcal{H}_{\iota\circ\xi}$ depends on the intrinsic quantities $\mathrm{Index}(\mathrm{id}_M)$ and $\mathrm{Null}(\mathrm{id}_M)$ and the extrinsic quantities $\|S_\xi\|^2 + \|\nabla^\perp \xi\|^2$.

From Proposition 4.2, we can obtain the following corollaries.

COROLLARY 4.3. *Let M be an n-dimensional compact orientable constant mean curvature immersed hypersurface in the space form of a constant curvature c, and $\xi \in \Gamma(U(T^{\perp}M))$. If $\|S_{\xi}\|^2 g(X,X) \geq c(n-1)g(X,X)+ng(S_H(X),X)$ for all $X \in TM$, then we have*

$$\mathrm{Index}(\mathcal{H}_{\iota\circ\xi}) \geq \mathrm{Index}(\mathrm{id}_M)\,.$$

If $\|S_{\xi}\|^2 g(X,X) > c(n-1)g(X,X)+ng(S_H(X),X)$ for all non zero $X \in TM$, then

$$\mathrm{Index}(\mathcal{H}_{\iota\circ\xi}) \geq \mathrm{Index}(\mathrm{id}_M)+\mathrm{Null}(\mathrm{id}_M)\,.$$

PROOF. It is clear from $\nabla^{\perp}\xi = 0$. □

COROLLARY 4.4. *Let M be a compact extrinsic sphere in $\tilde{M}$. If $0 \geq r(X,X)$ for all $X \in TM$, then we have*

$$\mathrm{Index}(\mathcal{H}_{\iota\circ\bar{H}}) \geq \mathrm{Index}(\mathrm{id}_M)\,.$$

If $0 > r(X,X)$ for all non zero $X \in TM$, then

$$\mathrm{Index}(\mathcal{H}_{\iota\circ\bar{H}}) \geq \mathrm{Index}(\mathrm{id}_M)+\mathrm{Null}(\mathrm{id}_M)\,,$$

where $\bar{H} = (1/\|H\|)H$.

PROOF. From $\nabla^{\perp}H = 0$, $S_H = \|H\|^2\mathrm{id}_{TM}$ and $\|S_{\bar{H}}\|^2 = n\|H\|^2$, we have the desired conclusion. □

COROLLARY 4.5. *Let M be a compact Legendrian minimal submanifold in a $(2n+1)$-dimensional Sasakian manifold $(\tilde{M},\eta,V,\varphi,\tilde{g})$. If $n \geq r(X,X)$ for all $X \in TM$, then we have*

$$\mathrm{Index}(\mathcal{H}_{f^{\#}(V)}) \geq \mathrm{Index}(\mathrm{id}_M)\,.$$

If $n > r(X,X)$ for all non zero $X \in TM$, then

$$\mathrm{Index}(\mathcal{H}_{f^{\#}(V)}) \geq \mathrm{Index}(\mathrm{id}_M)+\mathrm{Null}(\mathrm{id}_M)\,.$$

PROOF. It is clear from $S_V = 0$ and $\nabla^{\perp}_X V = -\varphi(X)$ for all $X \in \Gamma(TM)$, where we used V instead of $f^{\#}(V)$. □

A harmonic section $\xi \in \Gamma(UE)$ is called weakly stable if $\mathrm{Index}(\mathcal{H}_{\xi}) = 0$. In the case where M is a hypersurface in $\tilde{M}$, since $V_{\iota\circ\xi} = \Gamma(TM)$, we obtain a stability theorem.

THEOREM 4.6. *Let $p = 1$ and $\xi \in \Gamma(U(T^{\perp}M))$. Assume that M is compact and $\iota\circ\xi$ is a harmonic section. If id_M is weakly stable as a harmonic map and*

$$r(X,X)+ng(S_H(X),X) \geq \|S_{\xi}\|^2 g(X,X)$$

for all $X \in TM$, then $\iota\circ\xi$ is weakly stable, that is, $\mathrm{Index}(\mathcal{H}_{\iota\circ\xi}) = 0$.

5. Stability for constant mean curvature hypersurfaces of the unit spheres

Let $f : (M, g) \to (\tilde{M}, \tilde{g})$ be an isometric immersion with $\dim M = n$ and $\dim \tilde{M} = n + 1$. Let $\xi \in \Gamma(U(T^{\perp}M))$. Clearly, if f is totally geodesic, then $\iota \circ \xi$ attains a minimum, hence, it is a harmonic section of $\Gamma(U(f^{\#}(T\tilde{M})))$ and $\mathrm{Index}(\mathcal{H}_{\iota\circ\xi}) = 0$. But, there exist non-totally geodesic hypersurfaces with weakly stable harmonic sections $\iota \circ \xi$. For example, the n-dimensional unit sphere $S^n(1) \subset \mathbf{R}^{n+1}$ has the harmonic section $\iota \circ \xi$, and $\mathrm{Index}(\mathcal{H}_{\iota\circ\xi}) = \mathrm{Index}(\mathrm{id}_{S^n(1)})$ holds by Lemma 4.1. Therefore $S^1(1)$ and $S^2(1)$ are non-totally geodesic hypersurfaces in the Euclidean spaces with weakly stable harmonic sections $\iota\circ\xi$. It is important to study hypersurfaces with weakly stable harmonic section $\iota\circ\xi \in \Gamma(U(f^{\#}(T\tilde{M})))$. In this section, we study hypersurfaces of constant mean curvature in $S^{n+1}(1)$ such that $\mathrm{Index}(\mathcal{H}_{\iota\circ\xi}) = 0$.

THEOREM 5.1. *Let M be a compact orientable connected Riemannian manifold of $\dim M = n$ and $f : M \to S^{n+1}(1)$ an isometric immersion with constant mean curvature. If $n \geq 3$ and $\|S_\xi\|^2 > n/(n-2)$ at each point of M, then $\iota\circ\xi$ is unstable, that is,* $\mathrm{Index}(\mathcal{H}_{\iota\circ\xi}) > 0$, *where* $\xi \in \Gamma(U(T^{\perp}M))$.

PROOF. For $W \in \mathbf{R}^{n+2}$, the parallel vector field induced from W is denoted by $\bar{W}$. Using the inclusion $i : S^{n+1}(1) \to \mathbf{R}^{n+2}$, we consider that M is a submanifold in $\mathbf{R}^{n+2}$. Let $\bar{W}^T$ (resp. $\bar{W}^N$) be the tangential (resp. normal) part of $\bar{W}$. We define a quadratic form Q on $\mathbf{R}^{n+2}$ by $Q(W, W) := \mathcal{H}_{\iota\circ\xi}(\bar{W}^T, \bar{W}^T)$. Let $v_1, \cdots, v_{n+2}$ be an orthonormal basis of $\mathbf{R}^{n+2}$. Let $\bar{S}$ be the shape operator of M in $\mathbf{R}^{n+2}$. Since $\nabla_X \bar{W}^T = \bar{S}_{\bar{W}^N} X$, we have

$$
\begin{aligned}
\mathrm{Tr} Q &= \sum_{i=1}^{n+2} Q(\bar{v}_i, \bar{v}_i) \\
&= \int_M \{\|S_\xi\|^2 + n - \rho_M + n^2\|H\|^2 - n\|S_\xi\|^2 + n(n-1)\} dv_g \\
&= \int_M \{\|S_\xi\|^2 + n - (n^2\|H\|^2 - \|S_\xi\|^2 + n(n-1)) \\
&\quad + n^2\|H\|^2 - n\|S_\xi\|^2 + n(n-1)\} dv_g \\
&= \int_M \{(2-n)\|S_\xi\|^2 + n\} dv_g ,
\end{aligned}
$$

where ρ_M is the scalar curvature of (M, g). From $n \geq 3$ and $\|S_\xi\|^2 > n/(n-2)$, it follows that $\mathrm{Tr} Q < 0$. Therefore $\iota \circ \xi$ is unstable. □

For any unit vector $a \in \mathbf{R}^{n+2}$ and for any s, $0 \leq s < 1$, let

$$
\Sigma^n(s) = \{x \in S^{n+1}(1) | \langle x, a \rangle = s\} ,
$$

where $< \cdot , \cdot >$ is the standard inner product of $\mathbf{R}^{n+2}$. Then $\Sigma^n(s)$ are constant mean curvature hypersurfaces in $S^{n+1}(1)$.

LEMMA 5.2. *Let ξ be the unit normal vector field on $\Sigma^n(s)$ in $S^{n+1}(1)$. Then* $\mathrm{Index}(\mathcal{H}_{\iota\circ\xi}) = 0$ *if and only if $(n-1)s^2 \leq 1$. Especially, in the case where $n \geq 3$, $\iota \circ \xi$ is weakly stable if and only if $\|S_\xi\|^2 \leq n/(n-2)$.*

PROOF. It holds that

$$\mathcal{H}_{\iota\circ\xi}(X, X) = H_{\mathrm{id}_{\Sigma^n(s)}}(X, X) + (n-1)\int_{\Sigma^n(s)} g(X, X)dv_g .$$

Then $\mathcal{H}_{\iota\circ\xi}(X, X) \geq 0$ for all vector field X on $\Sigma^n(s)$ if and only if

$$\lambda_1 \geq \frac{2(n-1)}{1-s^2} - (n-1) = \frac{(n-1)(1+s^2)}{1-s^2} ,$$

where λ_1 is the first eigenvalue of the Laplacian acting on functions. Since $\Sigma^n(s)$ is of constant curvature of $1/(1-s^2)$, we have $\lambda_1 = n/(1-s^2)$, which implies the conclusion. From $s^2 = \|S_\xi\|^2/(n + \|S_\xi\|^2)$, it follows that $\iota \circ \xi$ is stable if and only if $\|S_\xi\|^2 \leq n/(n-2)$ in the case where $n \geq 3$. □

In this paper, we say that $\Sigma^n(s)$ is a *stable small sphere* if $(n-1)s^2 \leq 1$, that is, $\mathrm{Index}(\mathcal{H}_{\iota\circ\xi}) = 0$. In [6], the following theorem is proved.

THEOREM 5.3. *Let M be a constant mean curvature hypersurface with constant length of the second fundamental form in $S^{n+1}(1)$. If $\|h\|^2 < 2\sqrt{n-1}$, then M is locally a piece of small sphere $\Sigma^n(s)$, where $s = \sqrt{\|h\|^2/(n + \|h\|^2)}$.*

Finally, we have the following theorem.

THEOREM 5.4. *Let M be a compact orientable connected constant mean curvature hypersurface with constant length of the second fundamental form in $S^{n+1}(1)$ and $\xi \in \Gamma(U(T^\perp M))$. If* $\mathrm{Index}(\mathcal{H}_{\iota\circ\xi}) = 0$ *and $n \geq 4$, then M is a stable small sphere $\Sigma^n(s)$, where $s = \sqrt{\|S_\xi\|^2/(n + \|S_\xi\|^2)}$.*

PROOF. From Theorem 5.1, we obtain $\|S_\xi\|^2 \leq n/(n-2)$, and $n/(n-2) < 2\sqrt{n-1}$ holds if $n \geq 4$. By Lemma 5.2 and Theorem 5.3, M is a stable small sphere $\Sigma^n(s)$ with $s = \sqrt{\|S_\xi\|^2/(n + \|S_\xi\|^2)}$. □

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