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Latent Quaternionic Geometry

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Abstract. We discuss the interaction between the geometry of a quaternion-Kähler manifold M and that of the Grassmannian $\mathbb{G}_3(\mathfrak{g})$ of oriented 3-dimensional subspaces of a compact Lie algebra $\mathfrak{g}$. This interplay is described mainly through the moment mapping induced by the action of a group G of quaternionic isometries on M. We give an alternative expression for the imaginary quaternionic endomorphisms I, J, K in terms of the structure of the Grassmannian's tangent space. This relies on a correspondence between the solutions of respective twistor-type equations on M and $\mathbb{G}_3(\mathfrak{g})$.

1. Introduction

This paper is concerned with the action of groups on quaternion-Kähler manifolds, and the geometry arising from associated moment mappings.

Let G be a compact Lie group acting by isometries on a quaternion-Kähler manifold M, with parallel 4-form Ω. In this case, we may assume that each element A in the Lie algebra $\mathfrak{g}$ of G generates a Killing vector field $\tilde{A}$ such that $L_{\tilde{A}}\Omega = 0$. A fundamental result of Galicki–Lawson [14] implies that there is a section μ_A of the standard rank 3 vector bundle over M (whose complexification is often written S^2H and can be identified with a subbundle of 2-forms) that satisfies the equation

$$d\mu_A = i(\tilde{A})\Omega\,. \qquad (1)$$

Letting A range over $\mathfrak{g}$ gives rise to a section $\mu \in \Gamma(M, S^2H \otimes \mathfrak{g}^*)$ that is a close counterpart of the moment mappings induced on symplectic manifolds associated to M (such as the twistor space and hyperkähler cone).

For certain purposes, it is more natural to encode μ into a mapping whose target is a fixed manifold, rather than a section of a bundle. We therefore consider the associated G-equivariant mapping

$$\Psi : M_0 \longrightarrow \mathbb{G}_3(\mathfrak{g})\,,$$

where M_0 is the subset of M on which μ has rank 3, and $\mathbb{G}_3(\mathfrak{g})$ is the Grassmannian of oriented 3-dimensional subspaces of $\mathfrak{g}$. The morphism Ψ was introduced by Swann ([27],

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[28]) to study the unstable manifolds for the gradient flow of the natural functional ψ on this type of Grassmannian. However, little was known about the way in which Ψ embeds the quaternionic structure of M into the distinctive 3-Grassmannian geometry.

The quaternionic structure of M is governed by orthonormal triples of almost complex structures $I_1 = I,\ I_2 = J,\ I_3 = K$ that are local sections of S^2H. The complexified tangent space can be represented in the form

$$T_xM \cong H \otimes E\,, \tag{2}$$

in which I_1, I_2, I_3 act on the standard representation $H \cong \mathbb{C}^2$ of $Sp(1)$. By contrast, the tangent space to the Grassmannian at $V \subset \mathfrak{g}$ is

$$T_V\mathbb{G}_3(\mathfrak{g}) \cong \mathrm{Hom}(V, V^{\perp}) \cong V \otimes V^{\perp}\,. \tag{3}$$

The problem we face is to reconcile these two descriptions, and to compare the roles of the "auxiliary" spaces H and V. It is solved by means of Theorem 4.2, using musical isomorphisms to compare the respective metrics on M and $\mathbb{G}_3(\mathfrak{g})$. We call this result the 'coincidence theorem' as it asserts that the structure of each quaternionic space (2) coincides with a less obvious one arising from the real tensor product in (3).

If $V = \Psi(x)$, we are able to choose a conformal identification of the endomorphisms I_1, I_2, I_3 of (2) with a basis v_1, v_2, v_3 of V in (3). Given $X \in T_xM$, we may then use (3) to write

$$\Psi_*(X) = \sum_{i=1}^{3} v_i \otimes p_i\,, \quad \Psi_*(I_1X) = \sum_{i=1}^{3} v_i \otimes q_i\,.$$

Theorem 4.2 then provides a memorable way of converting tangent vectors of $\mathbb{G}_3(\mathfrak{g})$ to tangent vectors on M, in which $v_i \otimes p_i$ is replaced by $I_i\tilde{p}_i$, where $\tilde{p}_i$ is the value of the Killing vector field induced by p_i. As a consequence (Corollary 4.4), we succeed in expressing the q_i's in terms of the p_i's and a projection operator ρ.

While each homogeneous quaternion-Kähler (Wolf) space $G/(K\,Sp(1))$ can be realized inside $\mathbb{G}_3(\mathfrak{g})$ as an extreme value of ψ, it is best fitted into our theory by reducing to an isometry group that fails to act transitively on M. Indeed, our theory is tailored to the study of *non*-homogeneous quaternion-Kähler manifolds, for which the orbits of G determine a proper subspace of (2) common to (3). One conclusion is that the mapping Ψ is not in general an *isometric* immersion. Although the resulting submanifolds $\Psi(M)$ are best understood when M has positive curvature, it is our hope that there will be future applications to the negative-curvature case.

Here is a brief summary of the contents. In Section 2, we introduce the natural first-order differential operator D on the tautological rank k vector bundle over a Grassmannian $\mathbb{G}_k(\mathbb{R}^n)$, which annihilates projections of constant sections. Indeed, we show that all solutions of D arise in this way (Theorem 2.2). This is a simple example whereby solutions of an overdetermined differential operator may be interpreted as parallel sections of some associated connection ([9]). Although quaternionic geometry and Lie algebras are not yet involved, we present D as an analogue of the more complicated *twistor operator* $\mathcal{D}$ on a quaternion-Kähler manifold.

In Section 3, we recall the definition of $\mathcal{D}$ on sections of S^2H, and explain that it is satisfied by μ_A. We then prove that, under suitable hypotheses, the map Ψ induces the natural isomorphism of $\ker \mathcal{D}$ with $\ker D$, where D now acts on the tautological rank 3 vector bundle V over $\mathbb{G}_3(\mathfrak{g})$ (Proposition 3.2). The main results occur in Section 4, which describes first the action of Ψ^* on simple 1-forms (Lemma 4.1). The correspondence between the v_i's and the I_i's is already evident at this stage, and culminates with Theorem 4.2 and Corollary 4.4 cited above.

In Section 5, we apply the theory to the case of an $Sp(1) \times Sp(1)$ action on $\mathbb{HP}^1$. We identify explicitly the gradient flow of ψ, before passing to other compatible examples. Under some general assumptions, each tangent space $\Psi_*(T_xM)$ contains a distinguished 4-dimensional subspace generated by $\operatorname{grad}\psi$ and the values of the Killing vector fields $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$. It was natural to conjecture that this subspace corresponds to a quaternionic line in T_xM, and we prove this conjecture (Corollary 5.1).

We expect a study of the immersion of other "low-dimensional" quaternion-Kähler manifolds into Grassmannians using the methods of this paper to lead to a further understanding of special geometries and group actions. In particular, the map $\Psi : G_2/SO(4) \to \mathbb{G}_3(\mathfrak{su}(3))$ is relevant to a study of cohomogeneity-one $SU(3)$ actions on 8-manifolds that we pursue elsewhere.

2. Operators on Grassmannians

Consider an n-dimensional real vector space $\mathbb{R}^n$ equipped with an inner product $\langle\, , \rangle$; we can construct the Grassmannian of oriented k-planes $\mathbb{G}_k(\mathbb{R}^n)$, whose tangent space at a k-plane V can be identified with the linear space

$$\operatorname{Hom}(V, V^\perp) \cong V^* \otimes V^\perp .$$

If $v_1, \ldots, v_k$ is an orthonormal basis for V and $w_1, \ldots, w_{n-k}$ is an orthonormal basis for $V^\perp$, then each homomorphism T_{ij} defined as $T_{ij}(v_k) = \delta^i_k w_j$, corresponds to an independent tangent direction; more explicitly, the curve

$$\alpha_{ij}(r) := \operatorname{span}\{v_1, \ldots, (\cos r)v_i + (\sin r)w_j, \ldots, v_k\} \tag{4}$$

satisfies $\alpha_{ij}(0) = V$ and $\alpha'_{ij}(0) = T_{ij}$. The presence of a metric on V, induced from the ambient space $\mathbb{R}^n$, will allow us to write $V \otimes V^\perp$, using the metric to define the isomorphism $V \cong V^*$.

We will be interested in studying differential operators on sections of vector bundles on $\mathbb{G}_k(\mathbb{R}^n)$, so we start by describing some induced objects. Given the metric, we have the splitting of the trivial bundle $\mathbb{G}_k(\mathbb{R}^n) \times \mathbb{R}^n$ into two subbundles: the tautological one $\mathbf{V}$ and its orthogonal complement:

$$\begin{tikzcd}
\mathbf{V} \oplus \mathbf{V}^{\perp} \arrow[r, "\cong"] \arrow[dr, "p"'] & \mathbb{G}_k(\mathbb{R}^n) \times \mathbb{R}^n \arrow[d, "p'"] \\
& \mathbb{G}_k(\mathbb{R}^n)
\end{tikzcd} .$$

The presence of this metric also allows us to define connections on these two subbundles merely by composing d with the two projections π and $\pi^{\perp}$. This connection is compatible with the metric induced on the fibres of $\mathbf{V}$ from $\mathbb{R}^n$: in fact if $s, t \in \Gamma(\mathbf{V})$ and $X \in T_V \mathbb{G}_k(\mathbb{R}^n)$ we have

$$\begin{aligned} X\langle s\,,\, t\rangle = \langle Xs\,,\, t\rangle + \langle s\,,\, Xt\rangle &= \langle \pi Xs\,,\, t\rangle + \langle s\,,\, \pi Xt\rangle \\ &= \langle \nabla^{\mathbf{V}}_X s\,,\, t\rangle + \langle s\,,\, \nabla^{\mathbf{V}}_X t\rangle\,. \end{aligned}$$

We obtain the corresponding second fundamental form by projecting in the opposite way:

$$\Gamma(\mathbf{V}) \longrightarrow \Gamma(T^*\mathbb{G}_k(\mathbb{R}^n) \otimes \mathbf{V}^{\perp})\,.$$

This sends s to $\pi^{\perp} ds$; analogously $II^{\perp}$ sends $s \in \Gamma(\mathbf{V}^{\perp})$ to πds. Both II and $II^{\perp}$ are tensors, and we may regard $II^{\perp}$ as a section of the bundle

$$\mathrm{Hom}\big(\mathbf{V}^{\perp}\,,\, T^*\mathbb{G}_k(\mathbb{R}^n) \otimes \mathbf{V}\big) \cong \mathbf{V}^{\perp} \otimes \big(T^*\mathbb{G}_k(\mathbb{R}^n) \otimes \mathbf{V}\big)\,,$$

identifying $\mathbf{V}^{\perp} \cong (\mathbf{V}^{\perp})^*$ as usual. It turns out that this section determines an immersion of $\mathbf{V}^{\perp}$ as a subbundle of $T^*\mathbb{G}_k(\mathbb{R}^n) \otimes \mathbf{V}$; we shall return to this question shortly.

We use the standard objects introduced above in order to construct new differential operators on the tautological bundle $\mathbf{V}$ and on its orthogonal complement $\mathbf{V}^{\perp}$. Similar techniques are used in the quaternionic context of [1]. First of all, given $A \in \mathbb{R}^n$, we can associate two sections of the bundles $\mathbf{V}$ and $\mathbf{V}^{\perp}$ just using the projections: $s_A = \pi A$ and $s_A^{\perp} = \pi^{\perp} A$ with $A = s_A + s_A^{\perp}$. Since A is constant,

$$0 = dA = ds_A + ds_A^{\perp}$$

so that

$$ds_A = -ds_A^{\perp}\,,$$

and in our notation,

$$\nabla^{\mathbf{V}} s_A = \pi ds_A = -\pi ds_A^{\perp} = -II^{\perp} s_A^{\perp}\,.$$

These equations imply that

$$d\,s_A = -II^{\perp} s_A^{\perp} + II s_A\,. \tag{5}$$

For convenience we shall combine the homomorphisms II and $II^{\perp}$ to act upon any $\mathbb{R}^n$-valued function on $\mathbb{G}_3(\mathbb{R}^n)$, giving a mapping

$$i:\ C^{\infty}(\mathbb{G}_3(\mathbb{R}^n), \mathbb{R}^n) \longrightarrow \Gamma(T^* \otimes \mathbb{R}^n\ )$$

defined by

$$i(S) = II(\pi S) - II^{\perp}(\pi^{\perp} S)\,. \tag{6}$$

in a way which is consistent with equation (5). Thus we have

$$ds_A = i(A) \tag{7}$$

and

$$ds_A^{\perp} = -i(A)\,. \tag{8}$$

The image of $II^{\perp}$ corresponds to elements of the type

$$\sum_{i=1}^{k} \lambda\, y \otimes v_i \otimes v_i \tag{9}$$

with $y \in \mathbf{V}^{\perp}$ and $\lambda \in \mathbb{R}$; this can be shown with the following argument. Consider the decomposition as $SO(k) \times SO(n-k)$ modules of the bundles

$$\mathbf{V}^{\perp} \otimes \mathbf{V} \otimes \mathbf{V} \cong \mathbf{V}^{\perp} \otimes \mathbb{R} + \mathbf{V}^{\perp} \otimes (\mathbf{V} \otimes \mathbf{V})_0 \tag{10}$$

where $(\mathbf{V} \otimes \mathbf{V})_0$ is the tracefree part of the tensor product; Schur's Lemma guarantees that the second summand cannot contain any submodule isomorphic to $\mathbf{V}^{\perp}$, so the first summand consists of the unique submodule of this type in the right side term of (10). Therefore, as expression (9) provides an $SO(k) \times SO(n-k)$-equivariant copy of $\mathbf{V}^{\perp}$ inside this bundle, it must coincide with $II^{\perp}(\mathbf{V}^{\perp})$. The same argument shows that

$$II(u) = \sum_{i=1}^{n-k} \lambda\, u \otimes w_i \otimes w_i$$

with $u \in \mathbf{V}$, $\lambda \in \mathbb{R}$. We want now to be more precise about these statements, and calculate explicitly the value of λ. This is done in the next proposition (in which tensor product symbols are omitted).

PROPOSITION 2.1. *Let $A \in \mathbb{R}^n$ so that $A = u + y$ with $u \in V$ and $y \in V^{\perp}$ at the point V; let v_j and w_i denote the elements of orthonormal bases of V and $V^{\perp}$ at V; then*

$$II(u) = \sum_j u\, w_j w_j \tag{11}$$

and

$$II^{\perp}(y) = -\sum_i y\, v_i v_i\,. \tag{12}$$

PROOF. We differentiate the section s_A along the curve $\alpha_{ij}(t)$ passing through V and with tangent vector $v_i w_j$ as in (4). Let $u = \sum_{i=1}^{k} a_i v_i$ and $y = \sum_{j=1}^{n-k} b_j w_j$; then

$$\begin{aligned} s_A(\alpha_{ij})(t) &= a_1 v_1 + \cdots + \langle A, \cos r\, v_i + \sin r\, w_j\rangle(\cos r\, v_i + \sin r\, w_j) + \cdots + v_k \\ &= a_1 v_1 + \cdots + (a_i \cos r + b_j \sin r)(\cos r\, v_i + \sin r\, w_j) + \cdots + v_k \end{aligned}$$

so that

$$\frac{d}{dr} s_A(\alpha_{ij})(r)_{|r=0} = d\, s_A \cdot v_i w_j = b_j v_i + a_i w_j\, ;$$

therefore, as an $\mathbb{R}^n$-valued 1-form,

$$\begin{aligned} d\, s_A &= \sum_{ij} b_j\, v_i v_i w_j + a_i\, w_j v_i w_j \\ &= \sum_i y\, v_i v_i + \sum_j u\, w_j w_j\, , \end{aligned}$$

where the second summand belongs to $\mathbf{V} \otimes \mathbf{V}^{\perp} \otimes \mathbf{V}^{\perp}$ and coincides with $II(u)$ as claimed. An analogous calculation for $s_A^{\perp}$ gives

$$d\, s_A^{\perp} = -\sum_i y v_i v_i - \sum_j u w_j w_j$$

as expected from equation (8). ∎

OBSERVATION. The opposite signs in (11) and (12) are consistent with the equation

$$0 = d\langle s_A\, ,\, s_A^{\perp}\rangle|_V = \langle II(u)\, ,\, y\rangle + \langle u\, ,\, II^{\perp}(y)\rangle$$

that expresses the fact that II and $II^{\perp}$ are adjoint linear operators.

Proposition 2.1 shows that $\nabla^{\mathbf{V}} s_A$ is of the form seen in (9), or alternatively that if we denote by π_2 the projection on the second summand in the decomposition (10) and define $D \equiv \pi_2 \circ \nabla^{\mathbf{V}}$, the section s_A satisfies the equation

$$D\, s_A = 0\, . \tag{13}$$

We shall call (13) the *twistor equation* on the Grassmannian $\mathbb{G}_3(\mathbb{R}^n)$.

A converse of this result is provided by

THEOREM 2.2. *A section $s \in \Gamma(\mathbf{V})$ satisfies the twistor equation $Ds = 0$ if and only if there exists another section $s' \in \Gamma(\mathbf{V}^{\perp})$ such that $s + s' = A$ is a constant section of $\mathbb{R}^n$, provided $k > 1$ and $n - k > 1$.*

PROOF. Let us choose an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, every section S of the flat bundle $\mathbb{G}_k(\mathbb{R}^n) \times \mathbb{R}^n$ is an n-tuple of functions

$$f_j : \mathbb{G}_k(\mathbb{R}^n) \longrightarrow \mathbb{R}^n$$

so that

$$S = \sum f_j e_j \,.$$

Applying the exterior derivative on $\mathbb{R}^n$ (which is a connection on the flat bundle) we obtain

$$dS = \sum df_j \otimes e_j$$

and if $1 \wedge i$ denotes an element in

$$\mathrm{Hom}\left(T^* \otimes \mathbb{R}^n, \left(\bigotimes^2 T^*\right) \otimes \mathbb{R}^n\right)$$

(where $T^* = T^*\mathbb{G}_k(\mathbb{R}^n)$) acting in the obvious way, we obtain

$$1 \wedge i\,(dS) = \sum df_j \wedge i\,(e_j)\,.$$

On the other hand

$$d \sum f_j\, i(e_j) = \sum df_j \wedge i(e_j) + f_j\, di(e_j)\,,$$

so if we can show that

$$di(e_j) = 0 \quad \forall j$$

we obtain the commutativity of the following diagram:

$$\begin{array}{ccccc} & & \mathbb{R}^n & \xrightarrow{d} & T^* \otimes \mathbb{R}^n \\ & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle 1\wedge i} \\ \mathbb{R}^n & \xrightarrow{d} & T^* \otimes \mathbb{R}^n & \xrightarrow{d} & \Lambda^2 T^* \otimes \mathbb{R}^n \end{array} \tag{14}$$

Now (7) implies:

$$di(e_j) = dds_{e_j} = 0\,,$$

because the e_j are constant. A consequence of Proposition 2.1 is that i is an injective map (because II and $II^{\perp}$ are). But we claim moreover that

The map $1 \wedge i$ is injective, provided $k > 1$ and $n - k > 1$.

The proof of this fact is straightforward, and we omit it.

Referring to diagram (14), we can deduce the following facts: if $s \in \Gamma(\mathbf{V})$ satisfies $Ds = 0$, then $ds = i(s + s')$ for some $s' \in \Gamma(\mathbf{V}^{\perp})$; this follows by comparing

$$ds = \nabla s + II(s)$$

with (6) and noting that $\pi\, s = s$ in this case: then $s' = -(II^{\perp})^{-1}(\nabla s)$. Obviously $dds = 0$, so $d(s + s') = 0$ too. Hence $A = s + s'$ is a constant element in A. ∎

3. The two twistor equations

Let us consider a compact Lie group G acting by isometries on a quaternion-Kähler manifold M; then its moment map μ can be described locally as

$$\mu = \sum_{i=1}^{3} \omega_i \otimes B_i \tag{15}$$

with ω_i a local orthonormal basis for S^2H and B_i belonging to $\mathfrak{g}$. Suppose that $V := \operatorname{span}\{B_1, B_2, B_3\}$ is a 3-dimensional subspace of $\mathfrak{g}$: then V is independent of the trivialization, as the structure group of S^2H is $SO(3)$. We obtain a well defined map

$$\Psi : M_0 \longrightarrow \mathbb{G}_3(\mathfrak{g})$$

where $M_0 \subset M$ is defined as the subset where $V(x)$ is 3-dimensional.

It turns out that M_0 is an open dense subset of the union $\bigcup S$ of G-orbits S on M such that $\dim S \geq 3$ ([28, Proposition 3.5]). Therefore if the dimension of the maximal G orbits in M is "big enough", then M_0 is an open dense subset of M.

From now on we will assume that

$$B_i = \lambda(x) v_i \tag{16}$$

for v_i an orthonormal basis of V.

This hypothesis is not excessively restrictive, in the sense that it is compatible with the existence of open $G_{\mathbb{C}}$ orbits on the twistor space $\mathcal{Z} = \mathbb{P}(\mathcal{U})$: in fact the projectivization of the complex-contact moment map f induced on $\mathcal{Z}$ satisfies

$$(\mathbb{P}f)(\omega_1) = \operatorname{span}_{\mathbb{C}}\{B_2 + \imath B_3\},$$

and in this case this turns out to be a ray of nilpotent elements in $\mathfrak{g}_{\mathbb{C}}$ (see ([28, §3]). Nilpotent elements belong to the zero set of any invariant symmetric tensor over $\mathfrak{g}_{\mathbb{C}}$, in particular with respect to the Killing form. In fact by Engel's Theorem their adjoint representation can be given in terms of strictly upper triangular matrices, with respect to a suitable basis; the product of such matrices is still strictly upper triangular and hence traceless. In other words

$$\begin{aligned} 0 = \operatorname{Tr}(ad_{B_2+\imath B_3} \circ ad_{B_2+\imath B_3}) &= \langle B_2 + \imath B_3\,,\ B_2 + \imath B_3 \rangle \\ &= \|B_2\|^2 - \|B_3\|^2 + 2\imath \langle B_2\,,\ B_3 \rangle\,, \end{aligned}$$

which implies $B_2 \perp B_3$ and $\|B_2\| = \|B_3\|$. These conditions are equivalent to the assumption, permuting cyclically the indices. Therefore condition (16) holds for all unstable manifolds described in [28], as in that case the twistor bundle $\mathcal{Z}$ is $G_{\mathbb{C}}$-homogeneous.

Using the map Ψ, we can construct on M_0 the pullback bundle $\Psi^*\mathbf{V}$; the latter is unique up to isomorphism of bundles (see [29, Chap. I, Prop. 2.15]). More precisely, any vector bundle $W \longrightarrow M_0$ for which there exists a map of bundles $\hat{\Phi} : W \longrightarrow \mathbf{V}$ which is injective on the fibres, and a commutative diagram

$$\begin{tikzcd} W \arrow[r, "\hat{\Phi}"] \arrow[d, "p_V^*"'] & \mathbf{V} \arrow[d, "p_V"] \\ M_0 \arrow[r, "\Psi"'] & \mathbb{G}_3(\mathfrak{g}), \end{tikzcd} \tag{17}$$

is necessarily isomorphic to $\Psi^*\mathbf{V}$.

LEMMA 3.1. *On M_0, we have an isomorphism: $S^2H \cong \Psi^*\mathbf{V}$.*

PROOF. To complete the commutative diagram (17), define the morphism of bundles

$$\hat{\Phi} : S^2H \longrightarrow \mathbf{V}$$

by

$$\big(x, \omega_i(x)\big) \longmapsto \big(\mathrm{span}\{B_1(x), B_2(x), B_3(x)\}, B_i(x)\big)$$

(see (15)), extending linearly on the fibres. This corresponds to the contraction of a vector $v \in S^2H_x$ with the S^2H component of $\mu(x)$ using the metric, so it does not depend on the trivialization (the structure group preserves the metric) and is injective on the fibres by definition of M_0. ■

We should point out that $\hat{\Phi}$ is not a bundle isometry in general, when we equip S^2H and $\mathbf{V}$ with the natural metrics coming respectively from M and from $\mathbb{G}_3(\mathfrak{g})$. Nevertheless, under the hypotheses discussed above, we can assume that $\hat{\Phi}$ is a conformal map on each fibre.

Let us now recall the definition of the *quaternion-Kähler twistor operator*. It is defined as the composition

$$\mathcal{D} : S^2H \xrightarrow{\nabla} E \otimes H \otimes S^2H \xrightarrow{\mathrm{sym}} E \otimes S^3H\ ,$$

of covariant differentiation with a symmetrization on the $Sp(1)$ factor. (The symbol Γ denoting "space of sections" has been omitted.) Under the assumption of nonzero scalar curvature, Salamon proved in [24, Lemma 6.5] that sections of S^2H belonging to ker $\mathcal{D}$ are in bijection with the elements in the space $\mathcal{K}$ of Killing vector fields preserving the quaternion-Kähler structure. More explicitly, consider the composition

$$\delta : S^2H \xrightarrow{\nabla} E \otimes H \otimes S^2H \hookrightarrow (E \otimes \underline{H}) \otimes (H \otimes \underline{H}^*) \longrightarrow T^*$$

where the underlined terms are contracted and $T^* = E \otimes H$. If ν is in ker $\mathcal{D}$, then $\delta(\nu)$ is dual to a Killing vector field $\tilde{A} \in \mathcal{K}$ and, on the other hand, $\nu = \mu_A$ or in other words

$$\mathcal{D}\mu_A = 0 \tag{18}$$

and all elements in ker $\mathcal{D}$ are of this form.

Recall now the Grassmannian discussion in Section 2: there is another differential operator D on the tautological bundle $\mathbf{V}$ over $\mathbb{G}_3(\mathfrak{g})$, and the elements in its kernel are precisely the sections s_A obtained by projection from the trivial bundle with fibre $\mathfrak{g}$ (see Theorem 2.2). We wish to relate the kernels of $\mathcal{D}$ and D through the map Ψ induced by μ. Recall that the bundle homomorphism $\hat{\Phi}$ is defined up to a bundle automorphism of S^2H; we can for instance introduce a dilation

$$\xi(x, w) = \left(x, \frac{w}{\|B_i\|}\right), \tag{19}$$

which is independent of the trivialization. In this way

$$\hat{\Xi}(\omega_i) := \hat{\Phi} \circ \xi(\omega_i) = \frac{B_i}{\|B_i\|},$$

and so an orthonormal basis is sent to another orthonormal basis: this yields an isometry of the two bundles compatible with the map Ψ induced by μ.

We can now state the main result of this section. Let us denote by $\mathcal{K}_\mathfrak{g}$ the subspace of Killing vector fields induced by $\mathfrak{g}$ and by $(\ker \mathcal{D})_\mathfrak{g}$ the space of the corresponding twistor sections; then

PROPOSITION 3.2. *There exists a lift $\hat{\Psi}$ of the map Ψ such that*

$$\hat{\Psi} \circ \mu_A = s_A \circ \Psi,$$

inducing the natural isomorphism $(\ker \mathcal{D})_\mathfrak{g} \cong \ker D$.

PROOF. We are looking for a lift $\hat{\Psi}$ such that the diagram

$$\begin{array}{ccc} S^2H & \xrightarrow{\hat{\Psi}} & \mathbf{V} \\ \uparrow{\scriptstyle \mu_A} & & \uparrow{\scriptstyle s_A} \\ M_0 & \xrightarrow[\Psi]{} & \mathbb{G}_3(\mathfrak{g}). \end{array}$$

commutes; recall the usual local description (15) of μ, and let us define $\hat{\Psi}$ so that

$$\hat{\Psi}(\omega_i) = \frac{B_i}{\|B_i\|^2},$$

obtained by composing $\hat{\Phi}$ with the dilation ξ^2 (see (19)); this is again a lift of Ψ; consider as usual $\mu_A \in \Gamma(S^2H)$ satisfying the twistor equation; then

$$\begin{aligned} \hat{\Psi}(\mu_A) &= \hat{\Psi}\left(\sum_i \omega_i \langle B_i, A\rangle\right) \\ &= \sum_i \frac{B_i}{\|B_i\|^2} \langle B_i, A\rangle \end{aligned}$$

$$= \pi_V A = s_A \,,$$

as required. As the lift $\hat{\Psi}$ is injective on the fibres, and as

$$\dim(\ker \mathcal{D})_{\mathfrak{g}} = \dim \mathcal{K}_{\mathfrak{g}} = \dim \mathfrak{g} = \dim \ker D\,,$$

the last assertion follows. ∎

4. The coincidence theorem

Another way of expressing the *twistor equation* (1) is given by

$$\nabla^{S^2H}\mu_A = k\sum_{i=1}^{3} I_i \tilde{A}^\flat \otimes I_i \tag{20}$$

(see [14], [6] and, in a more general context, [17]). Here ∇^{S^2H} is the induced $Sp(1)$ connection, $\tilde{A}$ is the Killing vector field generated by A in $\mathfrak{g}$, the symbol $\flat$ means Riemannian conversion to the dual 1-form, and k is the scalar curvature. The latter is constant as the metric is Einstein (for simplicity we can put $k = 1$). On the other hand on $\mathbf{V}$, we have defined the sections s_A and the natural connection $\nabla^{\mathbf{V}}$ so that

$$\nabla^{\mathbf{V}} s_A = \sum_{i=1}^{3} s_A^{\perp} \otimes v_i \otimes v_i \,.$$

(see (9) and Proposition 2.1).

In general, given a differentiable map $\Psi : M \to N$ of manifolds, and an isomorphism $\hat{\Phi}$ between vector bundles $E \to F$ on the manifold M and N respectively, the second one equipped with a connection ∇^F, we can define the *pullback connection* $\hat{\Psi}^*\nabla^F$ acting in the following way on elements s of $\Gamma(E)$:

$$(\Psi^*\nabla^F)_Y(s) := \hat{\Psi}^*(\nabla^F_{\Psi_* Y}(\hat{\Psi} \circ s))$$

where $Y \in T_xM$ and the right-hand $\hat{\Psi}^*$ is the appropriate pullback operator.

We want to apply this construction to the map $\Psi : M \to \mathbb{G}_3(\mathfrak{g})$ induced by μ, $N = \mathbb{G}_3(\mathfrak{g})$, $E = S^2H$, $F = \mathbf{V}$. Our aim is to relate, at a fixed point $x \in M$, the action of the quaternionic structure on certain 1-forms (the duals of the Killing vector fields) with special cotangent vectors on the Grassmannian $\mathbb{G}_3(\mathfrak{g})$:

LEMMA 4.1. *Let M, $\mathfrak{g}$, $\mathbb{G}_3(\mathfrak{g})$, μ, Ψ be defined as usual, so that*

$$\mu = \sum_{i=1}^{3} I_i \otimes B_i \,,$$

where $B_i = \lambda v_i$ with λ a differentiable G-invariant function on M and v_i an orthonormal basis of a point $V \in \mathbb{G}_3(\mathfrak{g})$. Choose $A \in V^\perp \subset \mathfrak{g}$; then at the point x such that $\Psi(x) = V$, *we have*

$$\frac{1}{\lambda} I_i \tilde{A}^\flat = \Psi^*(A \otimes v_i)^\flat \,, \tag{21}$$

where $A \otimes v_i \in T_x\mathbb{G}_3(\mathfrak{g})$. *Moreover,* $\|\mu\|^2 = 3\lambda^2$.

PROOF. Let Ψ denote the conformal lift of the map μ so that

$$\Psi(I_i) = \frac{1}{\lambda^2} B_i \,. \tag{22}$$

Hence, as seen in Proposition 3.2, $\Psi(\mu_A) = s_A \circ \Psi$. Applying the pulled-back connection $\Psi^*\nabla^{\mathbf{V}}$ of S^2H, we obtain

$$\begin{aligned}(\Psi^*\nabla^{\mathbf{V}})\mu_A &= \Psi^*(\nabla^{\mathbf{V}}(\Psi(\mu_A)))\\ &= \Psi^*(\nabla^{\mathbf{V}} s_A)\\ &= \Psi^*\Bigg(\sum_{i=1}^{3} s_A^\perp \otimes v_i \otimes v_i\Bigg)\\ &= \lambda \sum_{i=1}^{3} \Psi^*(s_A^\perp \otimes v_i) \otimes I_i \,;\end{aligned} \tag{23}$$

on the other hand the difference of two connections on the same vector bundle is a tensor, so given any section $s \in S^2H$ which vanishes at a point $x \in M$

$$(\nabla^{S^2H} - \Psi^*\nabla^{\mathbf{V}})s_{\,|_x} = 0 \,.$$

This is precisely what happens for the section μ_A at the point x for which $\Psi(S^2H_x) = V$, because $A \in V^\perp$ by hypothesis; in other words

$$\nabla^{S^2H}\mu_{A\,|_x} = (\Psi^*\nabla^{\mathbf{V}})\mu_{A\,|_x} \,.$$

In the light of the calculations leading to (23) and the twistor equation (20), we deduce

$$\sum_{i=1}^{3} I_i \tilde{A}^\flat \otimes I_i = \lambda \sum_{i=1}^{3} \Psi^*(s_A^\perp \otimes v_i) \otimes I_i \,;$$

the result follows as $s_A^\perp = A$ at V. ■

Lemma 4.1 leads to various ways of relating elements in the spaces T_xM and $T_V\mathbb{G}_3(\mathfrak{g})$ and the quaternionic elements I_i; nevertheless it is stated merely in terms of 1-forms, whereas we are interested in involving the two metrics. To this aim, let us define a linear transformation $\natural$ of T_xM by

$$X^\natural := (\Psi^*(\Psi_*X)^\flat)^\sharp \tag{24}$$

in $\mathrm{End}(T_x M)$. This corresponds to moving in a counterclockwise sense around the following diagram, starting from bottom left:

$$\begin{array}{ccc} T_x^* M & \xleftarrow{\Psi^*} & T_V^* \mathbb{G}_3 \\ \sharp \downarrow & & \uparrow \flat \\ T_x M & \xrightarrow[\Psi_*]{} & T_V \mathbb{G}_3 \end{array} . \tag{25}$$

Thus the linear endomorphism $(\cdot)^\natural$ measures the noncommutativity of the diagram (25), and the difference between the pullbacked Grassmannian metric from the quaternionic one.

We are in position now to prove the following *coincidence theorem*:

THEOREM 4.2. *Let $Y \in T_x M$ such that*

$$\Psi_* Y = \sum v_i \otimes p_i ;$$

for $p_i \in V^\perp$ with $V = \Psi(x)$; then

$$Y^\natural = \frac{1}{\lambda} \sum_i I_i \tilde{p}_i .$$

PROOF. Using the definitions and (21) we obtain

$$\begin{aligned} (\Psi_* Y)^\flat (\Psi_* Z) &= \Big\langle \sum v_i \otimes p_i \, , \, \Psi_* Z \Big\rangle_{\mathbb{G}_3} \\ &= \frac{1}{\lambda} \Big\langle \sum I_i \tilde{p}_i \, , \, Z \Big\rangle_M \end{aligned}$$

for any $Z \in T_x M$, hence the conclusion. ∎

The equivariance of the moment map μ implies that Killing vector fields on M map to Killing vector fields on $\mathbb{G}_3(\mathfrak{g})$: in other words if $\tilde{A}$ is induced by $A \in \mathfrak{g}$ on M, then

$$\Psi_* \tilde{A} = \sum_{i=1}^{3} v_i \otimes [A \, , \, v_i]^\perp .$$

Set $\alpha = (\sum_{i=1}^3 v_i \otimes p_i)^\flat \in T_x^* \mathbb{G}_3(\mathfrak{g})$, and let A_r be an orthonormal basis of $V^\perp$. Then

$$\begin{aligned} \sum_{r=1}^{n-3} \langle \Psi^* \alpha, \, \tilde{A}_r \rangle A_r &= \sum_{r=1}^{n-3} \langle \alpha, \, \Psi_* \tilde{A}_r \rangle A_r = \sum_{i,r} \langle p_i , \, [v_i \, , \, A_r]^\perp \rangle A_r \\ &= \sum_{i,r} \langle \, p_i , \, [v_i \, , \, A_r] \, \rangle A_r = \sum_{i,r} \langle [p_i , \, v_i] \, , \, A_r \rangle A_r \\ &= \sum_i [p_i , \, v_i]^\perp . \end{aligned}$$

We can therefore define a mapping

$$\rho : \ T_x^*M \longrightarrow V^{\perp} \tag{26}$$

by $\rho(\zeta) = \sum_r \langle \zeta \, , \, \tilde{A}_r \rangle A_r$. So if $\alpha \in T_x^*\mathbb{G}_3(\mathfrak{g})$, then $\Psi^*\alpha \in T_x^*M$, and the composition $\tilde{\gamma} = \rho \circ \Psi^*$ is a map

$$\tilde{\gamma} : \ T_x^*\mathbb{G}_3(\mathfrak{g}) \longrightarrow V^{\perp}$$

defined by $\tilde{\gamma}(\alpha) = \sum_i [v_i \, , \ p_i]^{\perp}$. This operator can be described as

$$\tilde{\gamma} = \pi^{\perp} \circ \gamma$$

where $\gamma(\alpha) = \sum_i [v_i \, , \ p_i]$ is the obstruction to the orthogonality of α to the G-orbit. In fact

LEMMA 4.3. *A tangent vector* $P = \sum_{i=1}^{3} v_i \otimes p_i \in T_V\mathbb{G}_3(\mathfrak{g})$ *is orthogonal to the* G*-orbit through the point* V *if and only if* $\gamma(P) = 0$.

PROOF. For any $A \in \mathfrak{g}$ let us consider the Killing vector field $\tilde{A}$ on $\mathbb{G}_3(\mathfrak{g})$. The condition of orthogonality of P is expressed by

$$\begin{aligned} 0 = \langle \, \tilde{A} \, , \ P \, \rangle &= \sum_{i=1}^{3} \langle \, [A \, , \ v_i]^{\perp} \, , \ p_i \, \rangle \\ &= \sum_{i=1}^{3} \langle \, [A \, , \ v_i] \, , \ p_i \, \rangle = \sum_{i=1}^{3} \langle \, A \, , [\, v_i \, , \ p_i] \, \rangle \\ &= \langle \, A \, , \ \gamma(P) \, \rangle \, , \end{aligned}$$

and the result follows. ■

We give now a more explicit description of the quaternionic endomorphisms:

COROLLARY 4.4. *Let* $Y \in T_xM$ *so that*

$$\Psi_* Y = v_1 \otimes p_1 + v_2 \otimes p_2 + v_3 \otimes p_3 \, .$$

Then

$$\Psi_*(I_1 Y) = \frac{1}{\lambda} \, v_1 \otimes \rho(Y^{\flat}) - \, v_2 \otimes p_3 + v_3 \otimes p_2 \, . \tag{27}$$

PROOF. Consider any $A \in V^{\perp}$, then

$$\begin{aligned} \langle p_1 \, , \ A \rangle_K = \langle \Psi_* Y \, , \ A \otimes v_1 \rangle_{\mathbb{G}_3} &= \frac{1}{\lambda} \langle I_1 \tilde{A}^{\flat} \, , \ Y \rangle \\ &= \frac{1}{\lambda} \langle I_1 \tilde{A} \, , \ Y \rangle_M = -\frac{1}{\lambda} \langle \tilde{A} \, , \ I_1 Y \rangle_M \\ &= -\frac{1}{\lambda} \langle I_1 Y^{\flat} \, , \ \tilde{A} \rangle \, . \end{aligned} \tag{28}$$

Here $\langle\ ,\ \rangle_{M,\mathbb{G}}$ denote the respective Riemannian metrics, $\langle\ ,\ \rangle_K$ minus the Killing form on $\mathfrak{g}$ and $\langle\ ,\ \rangle$ without subscript is merely the contraction of a cotangent and tangent vector. Then considering (28) and (26)

$$\begin{aligned} p_1 &= \sum_r \langle p_1\,,\ A_r\rangle_K\, A_r = -\frac{1}{\lambda}\sum_r \langle I_1 Y^\flat\,,\ \tilde{A}_r\rangle\, A_r \\ &= -\frac{1}{\lambda}\rho(I_1 Y^\flat)\,, \end{aligned}$$

and similarly

$$p_i = -\frac{1}{\lambda}\rho(I_i Y^\flat)\,, \quad i = 2, 3\,.$$

In consequence

$$\begin{aligned} \Psi_* I_1 Y &= \frac{1}{\lambda}\, v_1 \otimes \rho(Y^\flat) - \frac{1}{\lambda}\, v_2 \otimes \rho(I_3 Y^\flat) + \frac{1}{\lambda}\, v_3 \otimes \rho(I_2 Y^\flat) \\ &= \frac{1}{\lambda}\, v_1 \otimes \rho(Y^\flat) - v_2 \otimes p_3 + v_3 \otimes p_2\,. \end{aligned}$$ ■

Analogous assertions are clearly valid for I_2 and I_3.

REMARK. A striking feature of (27) is that the first term on the right-hand side (the one involving v_1) is independent of I_1. The operators ρ, γ appear as the essential ingredient to reconstruct the quaternionic action; the complementary summand $-v_2 \otimes p_3 + v_3 \otimes p_2$ is obtained from the adjoint representation of $\mathfrak{sp}(1)$ and is not sufficient. Nevertheless, Corollary 4.4 predicts that if Y is perpendicular to the G-orbit on M, then

$$\rho(Y^\flat) = 0\,,$$

thanks to the definition of ρ (see Lemma 4.3); in that case

$$\Psi_*(I_1 Y) = -\,v_2 \otimes p_3 + v_3 \otimes p_2$$

which coincides with the irreducible representation of $\mathfrak{sp}(1)$ on $V = \mathbb{R}^3$.

5. Examples and applications

We shall first illustrate some key aspects of the theory we have described with reference to the simplest of all Wolf spaces, namely

$$\mathbb{HP}^1 \cong \frac{Sp(2)}{Sp(1)\times Sp(1)} \cong \frac{SO(5)}{SO(4)} \cong S^4\,.$$

The stabilizer $Sp(1)\times Sp(1)$ has Lie algebra $\mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_- = \mathfrak{so}(4)$. It acts with cohomogeneity one, and generic orbits are isomorphic to

$$S^3 \cong \frac{Sp(1)\times Sp(1)}{Sp(1)_\Delta}$$

where $Sp(1)_\Delta$ is the diagonal subgroup, and there are 2 singular orbits corresponding to two antipodal points N, S. Let us choose at the point N any closed geodesic $\beta(t)$ connecting N to S: this will be orthogonal to any $Sp(1) \times Sp(1)$ orbit, and will intersect all of them (a *normal geodesic* in the language of [5], which in higher cohomogeneity is generalized by submanifolds called *sections*, see [15]). For instance, we can choose $N = e\,Sp(1) \times Sp(1)$, and take the geodesic corresponding to following copy of $U(1) \subset Sp(2)$:

$$g(t) = \begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix} = \exp \begin{pmatrix} 0 & t & 0 & 0 \\ -t & 0 & 0 & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & -t & 0 \end{pmatrix}, \tag{29}$$

where the matrix on the right is denoted by $t\,u$. This subgroup generates a geodesic $\beta(t)$ connecting N $(t = 0)$ with the south pole S $(t = \pi/2)$ passing through the equator $(t = \pi/4)$, and then backwards to N $(t = \pi)$. The stabilizer of the $Sp(1) \times Sp(1)$ action is constant along $\beta(t)$ on points that are different from N and S, and coincides with $Sp(1)_\Delta$, both along $\beta(t)$ in $\mathbb{HP}^1$ and along $\mathfrak{u}(1)$ for the isotropy representation.

Now let e_i and f_i denote orthonormal bases of $\mathfrak{sp}(1)_+$ and $\mathfrak{sp}(1)_-$ respectively. As $\mathfrak{so}(4)$ is a subalgebra of $\mathfrak{sp}(2)$ corresponding to the longest root, the elements of the two copies of $\mathfrak{sp}(1)$ correspond to the following matrices:

$$e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \imath & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\imath & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \imath & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\imath \end{pmatrix}, \tag{30}$$

$$e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{31}$$

and

$$e_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \imath & 0 \\ 0 & 0 & 0 & 0 \\ \imath & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \imath \\ 0 & 0 & 0 & 0 \\ 0 & \imath & 0 & 0 \end{pmatrix}. \tag{32}$$

Then if $e_i(t)$ and $f_i(t)$ denote an orthonormal basis of the isotropy subalgebra at $\beta(t)$ (given by $Ad_{g(t)}\mathfrak{so}(4)$), we get via the Killing metric:

$$\langle e_i\,,\ f_j(t)\rangle = \delta^i_j \sin^2 t$$
$$\langle e_i\,,\ e_j(t)\rangle = \delta^i_j \cos^2 t$$

$$\langle f_i\,,\ e_j(t)\rangle = \delta^i_j \sin^2 t$$
$$\langle f_i\,,\ f_j(t)\rangle = \delta^i_j \cos^2 t\,.$$

In terms of Killing vector fields this implies

$$\pi_{S^2H}(\nabla\tilde{e_i}) = \sin^2 t\ f_i(t)\,,\quad \pi_{S^2H}(\nabla\tilde{f_i}) = \cos^2 t\ f_i(t)\,.$$

if we identify $S^2H \cong Ad_{g(t)}\mathfrak{sp}(1)_-$.

The conclusion is that along $\beta(t)$, the moment map for the action of the group $Sp(1)\times Sp(1)$ on $\mathbb{HP}^1$ is given by

$$\mu(\beta(t)) = \sum_i \omega_i \otimes\ (\cos^2 t\ f_i + \sin^2 t\ e_i)\,, \tag{33}$$

up to a constant. This is the only information that we need to reconstruct the moment map on the whole $\mathbb{HP}^1$, as $\beta(t)$ intersects all the orbits and the moment map is equivariant.

We can now interpret these facts in terms of the induced map

$$\Psi : \mathbb{HP}^1 \longrightarrow \mathbb{G}_3(\mathfrak{so}(4))\,;$$

first of all we note that in this case $M_0 = M$, as the three vectors

$$B_i(t) = \cos^2 t\ f_i + \sin^2 t\ e_i \tag{34}$$

are linearly independent for all t; moreover we observe that $\hat{\Phi}$ is a conformal mapping of bundles, as asked in the general hypotheses discussed in Section 3.

Recall from [28] that the critical manifolds for the gradient flow of the functional

$$\psi = \langle [v_1, v_2], v_3\rangle$$

defined on $\mathbb{G}_3(\mathfrak{so}(4))$ are given by the maximal points $\mathfrak{sp}(1)_+$, $\mathfrak{sp}(1)_-$ and the submanifold

$$C_\Delta = \mathbb{RP}^3 \cong \frac{Sp(1)\times Sp(1)}{\mathbb{Z}_2\times Sp(1)_\Delta}$$

corresponding to the 3-dimensional subalgebra $\mathfrak{sp}(1)_\Delta$, for $\psi > 0$; the unstable manifold M_Δ emanating from this last one is 4-dimensional and isomorphic to

$$\frac{\mathbb{HP}^1 \setminus \{N, S\}}{\mathbb{Z}_2}\,.$$

A trajectory for the flow of $\nabla\psi$ is given by

$$V(x, y) = \mathrm{span}\{xe_i + yf_i \mid x^2 + y^2 = 1\,,\ i = 1\ldots 3\}\,, \tag{35}$$

therefore, comparing (35) with (34) we obtain that $\Psi(\mathbb{HP}^1) = M_\Delta \cup \mathfrak{sp}(1)_+ \cup \mathfrak{sp}(1)_-$; in particular:

$$\Psi(N) = \mathfrak{sp}(1)_- \tag{36}$$

$$\Psi(S) = \mathfrak{sp}(1)_+ \tag{37}$$

$$\Psi(\beta(\pi/4)) = \mathfrak{sp}(1)_\Delta . \tag{38}$$

OBSERVATION. The map Ψ is not injective. The points corresponding to t and $\pi - t$ are sent to the same 3-plane; so the principal orbits of type S^3 in $\mathbb{HP}^1$ are sent to the orbits of type $\mathbb{RP}^3$ in M_Δ. The map Ψ becomes injective on the orbifold $\mathbb{HP}^1/\mathbb{Z}_2$, and its differential Ψ_* is injective away from $N,\ S$.

The $Sp(1) \times Sp(1)$ orbit through $x_\Delta = \beta(\pi/4)$ is sent by Ψ to the critical orbit C_Δ. An analogous situation holds for appropriate orbits in the following cases, which are all cohomogeneity-one actions on classical Wolf spaces:

- $Sp(n)Sp(1)$ acting on $\mathbb{HP}^n$;
- $Sp(n)$ acting on $\mathbb{G}_2(\mathbb{C}^{2n})$;
- $SO(n-1)$ acting on $\mathbb{G}_4(\mathbb{R}^n)$.

In the first case the orbit sent through Ψ to a critical submanifold of type C_Δ in the corresponding Grassmannian is one of the principal orbits S^{4n-1}, in the second and third case it is one of the singular orbits, more precisely

$$\frac{Sp(n)}{Sp(n-2) \times U(2)} \quad \text{and} \quad \mathbb{G}_3(\mathbb{R}^{n-1}) \cong \frac{SO(n-1)}{SO(n-4) \times SO(3)}$$

respectively.

In general, the presence of the G-action allows us to single out a quaternionic line of $T_x M$: this determines a quaternionic 1-dimensional distribution $\mathcal{N}_{\mathbb{H}}$ on M, or a section $\tau : M \longrightarrow \mathbb{HP}(TM)$ of the associated $\mathbb{HP}^{n-1}$-bundle.

The distribution $\mathcal{N}_{\mathbb{H}}$ arises in the following way: recall that at a point $V \in \mathbb{G}_3(\mathfrak{g})$ with $v_1,\ v_2,\ v_3$ orthonormal basis, we have

$$\operatorname{grad} \psi = v_1 \otimes [v_2, v_3]^\perp + v_2 \otimes [v_3, v_1]^\perp + v_3 \otimes [v_1, v_2]^\perp .$$

Maintaining the general hypotheses considered in Sections 3 and 4, and assuming that Ψ_* is injective, let us define $X := \Psi_*^{-1}(\operatorname{grad} \psi)$; then we have:

COROLLARY 5.1. *Suppose that $\Psi(x) = V$. Then the subspaces*

$$\operatorname{span}\{\operatorname{grad} \psi,\ \tilde{v_1},\ \tilde{v_2},\ \tilde{v_3}\} \subset T_V \mathbb{G}_3(\mathfrak{g})$$
$$\operatorname{span}\{X,\ \tilde{v_1},\ \tilde{v_2},\ \tilde{v_3}\} \subset T_x M$$

are $Sp(1)$ invariant, hence quaternionic.

PROOF. We need to prove that the endomorphisms of $S^2 H$ over x (or equivalently those of $\mathbf{V}$ over V) preserve the respective subspaces; let us recall the description of $I_1,\ I_2,\ I_3$ given in Corollary 4.4, then

$$I_1(\operatorname{grad} \psi) = \frac{1}{\lambda}\, v_1 \otimes \rho\big((\operatorname{grad} \psi)^\flat\big) - v_2 \otimes [v_1, v_2]^\perp + v_3 \otimes [v_3, v_1]^\perp$$

$$= -\, v_2 \otimes [v_1, v_2]^\perp + v_3 \otimes [v_3, v_1]^\perp$$
$$= -\tilde{v_1}\,, \tag{39}$$

where the first summand vanishes thanks to the G-invariance of ψ, which implies that grad ψ is orthogonal to the G orbits. Analogously, $I_2(\operatorname{grad}\psi) = -\tilde{v_2}$ and $I_3(\operatorname{grad}\psi) = -\tilde{v_3}$, and the quaternionic identities imply that the whole of span$\{\operatorname{grad}\psi,\ \tilde{v_1},\ \tilde{v_2},\ \tilde{v_3}\}$ is preserved; the second inclusion follows from the injectivity and equivariance of Ψ. ■

In all the examples discussed above the distribution $\mathcal{N}_{\mathbb{H}}$ turns out to be integrable, with integral manifolds isomorphic to $\mathbb{HP}^1$ embedded quaternionically in $\mathbb{HP}^n$, $\mathbb{G}_2(\mathbb{C}^{2n})$ or $\mathbb{G}_4(\mathbb{R}^n)$ respectively.

For $Sp(1) \times Sp(1)$ acting on $\mathbb{HP}^1$ the distribution $\mathcal{N}_{\mathbb{H}}$ clearly coincides with the tangent bundle; in this case it is possible to describe the relationship between the two metrics and the $(\cdot)^\natural$ endomorphism:

PROPOSITION 5.2. *Let $M = \mathbb{HP}^1 \setminus \{N, S\}$; consider the decomposition*

$$\begin{aligned} T_x M &\cong \operatorname{span}\{\tilde{v_1}, \tilde{v_2}, \tilde{v_3}\} \oplus \operatorname{span}\{X\} \\ &=: C_1 \oplus C_2 \end{aligned} \tag{40}$$

induced by the $Sp(1) \times Sp(1)$ action; then the map $\Psi : M \longrightarrow \mathbb{G}_3(\mathfrak{so}(4))$ satisfies the condition

$$\Psi^* \langle\, ,\, \rangle_{\mathbb{G}_3}|_{C_i} = \eta_i(x) \langle\, ,\, \rangle_M \quad i = 1,\, 2 \tag{41}$$

where $\eta_i(x)$ two real-valued $Sp(1) \times Sp(1)$ invariant functions defined on M. The endomorphism (24) *is just the multiplication by $\eta_i(x)$ on C_i.*

PROOF. The tangent space $T_V \mathbb{G}_3(\mathfrak{so}(4))$ along the unstable manifold can be seen as an irreducible $Sp(1)_\Delta$-module, and Ψ_* as a morphism of $Sp(1)$-modules. Schur's Lemma guarantees the uniqueness of an invariant bilinear form (up to a constant), for every irreducible submodule. Since

$$T_x M \cong \Sigma^2 \oplus \Sigma^0$$

as $Sp(1)_\Delta$ representations, corresponding to the splitting (40): therefore equation (41) holds, as both metrics are $Sp(1)_\Delta$ invariant. For the second assertion, let $Y \in C_i$:

$$\begin{aligned} Y^\natural &= \left(\Psi^*(\Psi_* Y)^\flat\right)^\sharp \\ &= \left(\Psi^*\left(\langle \Psi_* Y,\ \cdot\,\rangle_{\mathbb{G}_3}\right)\right)^\sharp \\ &= \eta_i(x)\left(\langle Y,\ \cdot\,\rangle_M\right)^\sharp \\ &= \eta_i(x) Y \end{aligned}$$

as required. ■

Equation (39) together with the equality $\|\mathrm{grad}\,\psi\| = 3\|\tilde{v_i}\|/2$ confirms that the endomorphisms I_i are *not* orthogonal relative to the Grassmannian metric; hence $\Psi^*\langle\, ,\, \rangle_{\mathbb{G}_3}$ and $\langle\, ,\, \rangle_M$ cannot coincide. Indeed,

$$\|\mathrm{grad}\ \psi\|^2_{\mathbb{G}_3} = \tfrac{3}{2}\ \|\tilde{v_1}\|^2_{\mathbb{G}_3} = \tfrac{3}{2}\ \eta_2\ \|\tilde{v_1}\|^2_M\ ;$$

moreover

$$\|\mathrm{grad}\,\psi\|^2_{\mathbb{G}_3} = \eta_1\ \|X\|^2_M$$

and $\|X\|_M = \|I_1 X\|_M = \|\tilde{v_1}\|_M$. Thus $\eta_1/\eta_2 = 3/2$. An analogous result is expected to hold in general.

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