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On Blanchard Manifolds

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Introduction

A compact complex manifold M of dimension 3 is called a *Blanchard manifold*, if its universal covering is biholomorphic to the complement of a projective line ℓ in a three dimensional complex projective space $\mathbf{P}^3$. Let Ω denote the complement $\mathbf{P}^3 \setminus \ell$. Then M is given as a quotient space Ω/Γ, where Γ is a group of holomorphic automorphisms of Ω. By [K, Theorem C], we know that

THEOREM A. (1) *The group Γ is a subgroup of the projective general linear group* $\mathrm{PGL}(4, \mathbf{C})$.

(2) *Γ contains a free abelian subgroup Γ_0 of finite index.*

(3) *By a suitable choice of homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ on $\mathbf{P}^3$ with*

$$\ell = \{z_2 = z_3 = 0\},$$

Γ_0 is contained in either

$$\left\{\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\} \qquad \text{Type (A)}, \tag{1}$$

or

$$\left\{\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\} \qquad \text{Type (B)}. \tag{2}$$

When we say that Γ_0 is of type(A), $\mathrm{rank}(I - g) = 3$ for some $g \in \Gamma_0$. Otherwise, we say that Γ_0 is of type(B). It is known that, if Γ_0 of type(B), $\mathrm{rank}(I - g) = 2$ for any $g \in \Gamma_0 \setminus \{I\}$ ([K, Proposition 5.40]). In this short note we shall prove the following

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THEOREM B. *Γ_0 is always of type* (B).

The first author gave an "example" of type (A) in [K, page 387]. Unfortunately, the action of Γ given there was not properly discontinuous on Ω. In this note, we shall show

PROPOSITION. *The action of any Γ_0 of type* (A) *is not properly discontinuous on Ω.*

Theorem B follows from the proposition. Examples of type (B) are well-known.

1. Proof of the Proposition

Assuming that Γ_0 is of type (A) and that its action on Ω is properly discontinuous, we shall derive a contradiction. To derive a contradiction, it is enough to construct a sequence of points $\{p_n\}$ in Ω and an infinite sequence of transformations $\{g_n\}$ in Γ_0 such that both $\lim_{n\to\infty} p_n$ and $\lim_{n\to\infty} g_n(p_n)$ converge to points in Ω.

We put $M_0 = \Omega/\Gamma_0$, which is an finite unramified covering of $M = \Omega/\Gamma$. On $\mathbf{P}^3$, we fix the system of homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ used in Theorem A. We write elements of Γ_0 as if they are in $SL(4, \mathbf{C})$. Let I be the identity matrix of size 4 and put

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

We fix a set of generators

$$G_k = I + a_k N + b_k N^2 + c_k N^3 , \quad k = 1, \ldots, 4$$

of Γ_0. Put

$$S = \{[z_0 : z_1 : z_2 : z_3] \in \Omega : z_3 = 0\} .$$

On $\Omega \setminus S \simeq \mathbf{C}^3$, we consider the following system of coordinates

$$(u_1, u_2, u_3) = (x_1 - x_2 x_3 + x_3^3/3, x_2 - x_3^2/2, x_3) , \tag{3}$$

where

$$(x_1, x_2, x_3) = (z_0/z_3, z_1/z_3, z_2/z_3) .$$

Similarly, on $S \simeq \mathbf{C}^2$, we consider the following system of coordinates

$$(v_1, v_2) = (y_1 - y_2^2/2, y_2) , \tag{4}$$

where

$$(y_1, y_2) = (z_0/z_2 , z_1/z_2) .$$

Define four vectors $\tau_k \in \mathbf{C}^2$ by

$$\tau_k = \begin{pmatrix} e_k \\ a_k \end{pmatrix} , \quad e_k = b_k - a_k^2/2 , \quad k = 1, \ldots, 4 , \tag{5}$$

and

$$f_k = c_k - a_k b_k + a_k^3/3\,, \quad k = 1, \ldots, 4\,. \tag{6}$$

Let $\rho \in \mathbf{C}$ be a root of

$$x^2 + x + 1/3 = 0\,.$$

For $n \in \mathbf{N}$, put

$$\varepsilon_n = -(\rho + 1/2)n^2\,. \tag{7}$$

LEMMA 1.1. *The vectors* τ_k, $k = 1, \ldots, 4$, *are linearly independent over* $\mathbf{R}$.

PROOF. The group Γ_0 acts on S and the quotient S/Γ_0 is a closed non-singular surface in M_0, which is compact. Each G_k sends (y_1, y_2) to $(y_1 + a_k y_2 + b_k, y_2 + a_k)$. Hence it sends (v_1, v_2) to $(v_1 + b_k - a_k^2/2, v_2 + a_k)$. Thus Γ_0 acts on $S \simeq \mathbf{C}^2$ as a translation group generated by G_k. Hence S/Γ_0 is compact torus which is the quotient of $\mathbf{C}^2$ by the lattice generated by τ_k, $k = 1, \ldots, 4$. Therefore τ_k's are linearly independent over $\mathbf{R}$. ∎

Consider the matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ e_1 & e_2 & e_3 & e_4 \\ \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} \\ \overline{e_1} & \overline{e_2} & \overline{e_3} & \overline{e_4} \end{pmatrix}. \tag{8}$$

By Lemma 1.1, A is a non-singular matrix. Therefore, for each $n \in \mathbf{N}$, we have an unique solution $(r_1, r_2, r_3, r_4) \in \mathbf{R}^4$ such that

$$\begin{pmatrix} n \\ \varepsilon_n \\ n \\ \overline{\varepsilon_n} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ e_1 & e_2 & e_3 & e_4 \\ \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} \\ \overline{e_1} & \overline{e_2} & \overline{e_3} & \overline{e_4} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix}. \tag{9}$$

For each $n \in \mathbf{N}$, we choose a set of integers $N_n = (n_1, n_2, n_3, n_4) \in \mathbf{Z}^4$ so that

$$|n_k - r_k| \le 1/2\,, \quad \text{for} \quad k = 1, \ldots, 4\,. \tag{10}$$

Thus we have defined a sequence $\{N_n\}_{n=1}^{\infty}$ in $\mathbf{Z}^4$. Define

$$a(n) = \sum_{k=1}^{4} n_k a_k\,, \quad e(n) = \sum_{k=1}^{4} n_k e_k\,, \quad f(n) = \sum_{k=1}^{4} n_k f_k\,. \tag{11}$$

Let $\mathcal{L}$ be the set of $\mathbf{C}$-valued functions $\delta(n)$ on $\mathbf{N}$ satisfying

$$|\delta(n)| \le Kn \quad \text{for any} \quad n \in \mathbf{N}\,,$$

where $K > 0$ is some constant independent of n. The norm $\|X\|$ of a matrix $X = (x_{ij})$ is defined by $\|X\| = \max_{i,j}\{|x_{ij}|\}$.

LEMMA 1.2. *There is a constant K independent of n such that*

$$|a(n)-n| \le K \quad \text{for all} \quad n \in \mathbf{N}\,. \tag{12}$$

PROOF. By (11), (9) and (10), we have

$$|a(n)-n| = \left|\sum_{k=1}^{4} n_k a_k - \sum_{k=1}^{4} r_k a_k\right| \le \sum_{k=1}^{4} |n_k - r_k|\,|a_k| \le \sum_{k=1}^{4} \frac{|a_k|}{2} \le 2\|A\|\,. \quad ∎$$

LEMMA 1.3. *There is a function $\delta_1 \in \mathcal{L}$ such that*

$$e(n) = -\,(\rho+1/2)\,a(n)^2 + \delta_1(n)\,. \tag{13}$$

PROOF. By (11), (9), (10) and (7), we have

$$\begin{aligned}
\left|e(n) + \left(\rho+\frac{1}{2}\right)a(n)^2\right| &= \left|\sum_{k=1}^{4} n_k e_k - \sum_{k=1}^{4} r_k e_k + \varepsilon_n + \left(\rho+\frac{1}{2}\right)a(n)^2\right| \\
&\le \sum_{k=1}^{4}\frac{|e_k|}{2} + \left|\varepsilon_n + \left(\rho+\frac{1}{2}\right)a(n)^2\right| \le 2\|A\| + \left|\left(\rho+\frac{1}{2}\right)(a(n)^2-n^2)\right| \\
&\le 2\|A\| + K_1|a(n)^2-n^2| \le Kn
\end{aligned}$$

for some constants K_1, K independent of n. Here we have used Lemma 1.2 to derive the last inequality. Thus, we have the lemma. ∎

LEMMA 1.4. *There is a function $\delta_2 \in \mathcal{L}$ such that*

$$f(n) = \delta_2(n)a(n)\,. \tag{14}$$

PROOF. By (11), there are constants $\lambda_k \in \mathbf{C}$, $k=1,\dots,4$, which are determined by the entries of A and f_k, $k=1,\dots,4$, such that

$$f(n) = \lambda_1 a(n) + \lambda_2 e(n) + \lambda_3\overline{a(n)} + \lambda_4\overline{e(n)}\,.$$

Hence, by Lemmas 1.2 and 1.3, we have the lemma easily. ∎

Now we define a sequence of points $\{p_n\}_n$ in Ω by

$$p_n \;:\; [0 : -(\rho+1)\delta_1(n) - \delta_2(n) : \rho a(n) : 1] \tag{15}$$

and a sequence of transformations g_n of Γ_0 by

$$g_n = G_1^{n_1} G_2^{n_2} G_3^{n_3} G_4^{n_4}\,. \tag{16}$$

Note that, in terms of coordinates (u_1, u_2, u_3) on $\Omega \setminus S$, G_k acts as

$$G_k \;:\; (u_1,\ u_2,\ u_3) \mapsto (u_1 + f_k,\ u_2 + e_k,\ u_3 + a_k)\,.$$

Thus g_n acts as

$$g_n \;:\; (u_1,\ u_2,\ u_3) \mapsto (u_1', u_2', u_3') = (u_1 + f(n),\ u_2 + e(n),\ u_3 + a(n))\,.$$

Put $q_n = g_n(p_n)$. In terms of coordinates (u_1, u_2, u_3), p_n is given by

$$u_1 = \rho^3 a(n)^3/3 + ((\rho+1)\delta_1(n) + \delta_2(n))\,\rho a(n) \tag{17}$$

$$u_2 = -\rho^2 a(n)^2/2 - (\rho+1)\delta_1(n) - \delta_2(n) \tag{18}$$

$$u_3 = \rho a(n)\,. \tag{19}$$

By simple calculations using (17), (18), (19), and Lemmas 1.3, 1.4, we can verify that $q_n = (u_1 + f(n), u_2 + e(n), u_3 + a(n))$ is given in terms of coordinates (x_1, x_2, x_3) by

$$\begin{aligned} x_1' &= (u_1 + f(n)) + (u_2 + e(n))(u_3 + a(n)) + (u_3 + a(n))^3/6 = 0 \\ x_2' &= (u_2 + e(n)) + (u_3 + a(n))^2/2 = -\rho\delta_1(n) - \delta_2(n) \\ x_3' &= u_3 + a(n) = (\rho+1)a(n)\,. \end{aligned}$$

Thus in homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$, the sequence $\{q_n\}_n \subset \Omega$ is given by

$$q_n \;:\; [0 : -\rho\delta_1(n) - \delta_2(n) : (\rho+1)a(n) : 1]\,. \tag{20}$$

By Lemma 1.2, and since $\delta_1, \delta_2 \in \mathcal{L}$, we can choose convergent subsequences of $\{p_n\}_n$ and $\{q_n\}_n$ to points in Ω,

$$\lim_{n\to\infty} p_n = [0 : * : \rho : 0]\,, \quad \lim_{n\to\infty} q_n = [0 : * : \rho+1 : 0]\,. \quad ∎$$

As a corollary, we have

THEOREM 1.1. *The algebraic dimension of any Blanchard manifold is equal to one.*

PROOF. If Γ_0 is of type (B), it is easy to see that Γ_0-invariant homogeneous polynomial is of the form $f(z_2, z_3)$. Thus the function field of M_0 is $\mathbf{C}(z_2/z_3)$. Hence the theorem follows. ∎

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