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Antisymmetrically Deformed Quantum Homogeneous Spaces

Shingo KAMIMURA

Keio University

Abstract. We construct dual objects for quantum complex projective spaces as quantum homogeneous spaces of quantum unitary groups, in which the deformation parameters are antisymmetric matrices. We prove the splitting formula and the nondegeneracy of the Hochschild dimensions for the quantum complex projective spaces.

1. Introduction

Homogeneous spaces, as quotient spaces of classical Lie groups, are one of the important geometric objects. From the noncommuative geometrical point of view, it would be interesting also as the quantum notion corresponding to the classical homegeneous spaces.

In this paper, we exhibit a noncommutative or quantum generalization of certain homogeneous spaces in the framework of quantum groups. This is a non-formal deformation and called θ-deformation, where θ is a deformation parameter of the quantization of a given classical homogeneous space taken from the parameter set $A(n;\mathbf{R})$. Here $A(n;\mathbf{R})$ is the set of antisymmetric matrices of size n.

In the sense mentioned above θ-deformed quantum tori are the most fundamental objects in the θ-deformation. As it will be seen in the splitting theorem stated in the section 5, one can also expect that θ-deformed quantum spaces will be constructed form θ-deformed quantum tori. The typical $2n$-dimensional θ-deformed quantum torus can be made from the Heisenberg canonical commutation relations and in this case the antisymmetric matrix θ is nothing but the noncommutativity in the Heisenberg canonical commutation relations or the canonical symplectic form on $\mathbf{R}^{2n}$.

We construct quantum complex projective spaces as quantum homogeneous spaces of the quantum unitary groups after defining restriction maps and coactions between the quantum unitary groups. A remarkable feature which distinguishes the θ-deformation from the other deformations is the non-degeneracy of the Hochschild dimension. Namely, the Hochschild dimension of the corresponding coordinate ring of the θ-deformed quantum space remains constant during the deformation. Similar properties have been already proven for certain quantum groups but have not been proven for quantum homogeneous spaces by Connes and Dubois-Viorette [C-DV].

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The first main result of this paper is the splitting formula for th θ-deformed quantum homogeneous spaces (see Theorem 5.1.5). It is proved that the coordinate rings of the θ-deformed quantum homogeneous spaces split into the tensor product of the ordinary coordinate rings of the classical homogeneous spaces and the coordinate rings of the θ-deformed quantum tori with some torus action. This formula gives us another and much simpler definition of the θ-deformed quantum homogeneous spaces.

The second main result is concerned with the nondegeneracy of the Hochschild dimension of the θ-deformed quantum homogeneous spaces (see Theoem 5.2.1). The Hochschild dimension is the homology dimension of the Hochschild homology of algebras, and it coincides with the ordinary dimension in the case of the coordinate rings of the classical spaces. It is proved that the Hochschild dimensions of the θ-deformed quantum homogeneous spaces are equal to the classical dimension. Nondegeneracy of the Hochschild dimension plays an essential role when we naturally extend the classical calculus from classical spaces to quantum spaces.

2. Hochschild homology

First, let us recall the Hochschild homology of **C**-algebras.

DEFINITION 2.0.1 (Hochschild homology). For a **C**-algebra A, the Hochschild chain groups are

$$CH_k(A) := A^{\otimes(k+1)}$$

and its boundary operators

$$\partial_k : CH_k(A) \to CH_{k-1}(A)$$

are given by

$$\partial_k(a_0\otimes\cdots\otimes a_k) := \sum_{i=1}^{k-1}(-1)^i a_0\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_k + (-1)^k a_k a_0\otimes a_1\otimes\cdots\otimes a_{k-1}\,.$$

We denote by $HH_k(A)$ the Hochschild homology of degree k for the **C**-algebra A. The Hochschild dimension of the given **C**-algebra A is the homological dimension of the Hochschild homology of this algebra and is denoted by $\dim_H(A)$.

We note that

EXAMPLE 2.0.2. 1. For an n-dimensional real vector space V,

$$HH_k(S(V)) = S(V) \otimes A^k(V)\,.$$

where $S(V)$ and $A^k(V)$ are the symmetric algebra and the k-th differential forms of V.

2. For a differentiable manifold M,

$$HH_k(C^\infty(M)) = \Omega^k(M)\,.$$

The above examples gives a geometrical meaning of the Hochschild homology. Namely, we have

PROPOSITION 2.0.3. *Let V be an n dimensional real vector space and M a differential manifold. Then, we have*

$$\dim_H(S(V)) = \dim(V)\,, \quad \dim_H(C^\infty(M)) = \dim(M)\,.$$

By the above proposition, the Hochschild dimesnion is viewed as a reasonable concept of dimensions for the given **C**-*algebras A even if it is non-commutative. On the other hand, we know the following*:

PROPOSITION 2.0.4. *Under the identification $SU(2) \cong S^3$,*

$$\begin{array}{ll} \dim_H(C^{alg}(SU_q(2))) < \dim(SU(2)) & \text{([M-N-W1, M-N-W2])} \\ \dim_H(C^{alg}(S^3_\theta)) = \dim(S^3) & \text{([C-DV])} \end{array}$$

Here C^{alg} and θ stands for coordinate ring functor of the quantum groups and an antisymmetric matrix of some size, respectively.

The shadows of $C^{alg}(SU_q(2))$'s are degenerate, but those of $C^{alg}(S^3_\theta)$'s are nondegenerate. These degeneracy phenomena in q-deformation must have some singularities, but in many cases they are not so clear and the explanations of such phenomena have not been so successful. Anyway, the above results suggest to us that the θ-deformation seems to be more natural quantum object than q-deformation, and this is the reason we adopted θ-deformation in defining quantum homogeneous spaces in this paper.

3. Quantum matrix algebras via θ-deformation

In this section, we construct several quantum objects based on the quantum tori by using the antisymmetric deformation. Most of this section is quoted from [C-DV] with small rearrangements and some remarks.

3.1. Quantum tori. The most important example of θ-deformation is quantum torus, which is usually called noncommutative torus. There are several ways to define quantum torus. We now construct a cannoical quantum tori from the Heisenberg's cannonical commutation relations, which seems the most natural physical context.

DEFINITION 3.1.1. For $2n$ selfadjoint elements $(q_1, \cdots, q_n, p_1, \cdots, p_n)$, the Heisenberg CCR is given by

$$[q_i, p_i] = \sqrt{-1}\hbar\delta_{ij}1\,, \quad [q_i, q_j] = [p_i, p_j] = 0$$

Since selfadjoint elements in such rlations will be represented as unbounded operators, this causes a functional-analytical difficulty. For this reason, based on Stone's theorem, we choose the following description:

DEFINITION 3.1.2. The Weyl's CCR is given by

$$v_j(t)u_i(t) = \exp(\sqrt{-1}st\hbar\delta_{ij})u_i(s)v_i(t)\,,$$

where

$$u_i(s) := \exp(\sqrt{-1}sq_i)\,, \quad v_j(t) := \exp(\sqrt{-1}tp_j)\,, \quad r, s \in \mathbf{R}$$

are one-parameter unitary groups.

We can consider the discrete version of the Weyl's CCR:

DEFINITION 3.1.3. We set

$$v_j(n)u_i(m) = \exp(\sqrt{-1}mn\hbar\delta_{ij})u_i(m)v_i(n)\,, \quad m, n \in \mathbf{Z}\,.$$

In particular,

$$v_j u_i = \exp(\sqrt{-1}\hbar\delta_{ij})u_i v_j\,,$$

where we set

$$u_i := u_i(1)\,, \quad v_j := v_j(1)\,.$$

We note that the second relation in Definition 3.1.3 gives nothing but a quantum torus.

3.2. Quantum Euclidian spaces. We review the general definition of quantum tori, T^n_θ with $\theta \in A(n, R)$.

DEFINITION 3.2.1. Let $C^{alg}(T^n_\theta)$ be the unital $*$-algebra generated by n unitary elements

$$\bar{u}^i u^i = u^i \bar{u}^i = 1 \quad (1 \leq i \leq n)$$

with commutation relations

$$u^i u^j = \lambda^{ij} u^j u^i\,, \quad u^i \bar{u}^j = \bar{\lambda}^{ij} \bar{u}^j u^i$$

$$\bar{u}^i u^j = \bar{\lambda}^{ij} u^j \bar{u}^i\,, \quad \bar{u}^i \bar{u}^j = \lambda^{ij} \bar{u}^j \bar{u}^i\,.$$

Here

$$\lambda^{ij} = \exp(\sqrt{-1}\theta^{ij})\,, \quad \theta = (\theta^{ij}) \in A(n; \mathbf{R}) = o(n) = Lie(O(n))\,,$$

and we use $\bar{\ }$ instead of $*$-operation.

If we take θ to be

$$\omega_0 := \begin{pmatrix} 0_n & (\delta_{ij}) \\ (-\delta_{ij}) & 0_n \end{pmatrix} \in A(2n; \mathbf{R})\,, \quad (\delta_{ij}) \in A(n; \mathbf{R})$$

the resulting quantum torus is the canonical quantum torus which appeared in Definition 3.1.3.

REMARK 3.2.2. The canonical quantum tori are defined for even unitary elements. However $C^{alg}(T^n_\theta)$ can be defined for not only for even but also for odd unitary elements.

Replacement of the above unitary conditions in Definition 3.2.1 to the corresponding normal conditions leads us to a natural definition of the unital $*$-algebra $C^{alg}(\mathbf{R}^{2n}_\theta)$. That is,

DEFINITION 3.2.3. $C^{alg}(\mathbf{R}^{2n}_\theta)$ is generated by n normal elements

$$\bar{z}^i z^i = z^i \bar{z}^i \quad (1 \le i \le n)$$

with the same commutation relations as above,

$$z^i z^j = \lambda^{ij} z^j z^i \,, \quad z^i \bar{z}^j = \bar{\lambda}^{ij} \bar{z}^j z^i$$

$$\bar{z}^i z^j = \bar{\lambda}^{ij} z^j \bar{z}^i \,, \quad \bar{z}^i \bar{z}^j = \lambda^{ij} \bar{z}^j \bar{z}^i .$$

Here also

$$\lambda^{ij} = \exp(\sqrt{-1}\theta^{ij}) \,, \quad \theta = (\theta^{ij}) \in A(n; \mathbf{R}) \,.$$

In order to check the correspondence between the the above quantum formulation and the following classical formulation,

$$T^n \subset \mathbf{R}^{2n} \cong \mathbf{C}^n$$

$$u^i_{cl} = \exp(\sqrt{-1}t^i_{cl}) = \cos t^i_{cl} + \sqrt{-1}\sin t^i_{cl}$$

$$z^i_{cl} = x^i_{cl} + \sqrt{-1}y^i_{cl} \,,$$

it is helpful to take the Descartes decompositions of the unitary and normal generators,

$$u^i = v^i + \sqrt{-1}w^i = \frac{u^i + \bar{u}^i}{2} + \sqrt{-1}\frac{u^i - \bar{u}^i}{2\sqrt{-1}}$$

$$z^i = x^i + \sqrt{-1}y^i = \frac{z^i + \bar{z}^i}{2} + \sqrt{-1}\frac{z^i - \bar{z}^i}{2\sqrt{-1}} \,.$$

We can easily verify

$$\bar{v}^i = v^i \,, \quad \bar{w}^i = w^i \,, \quad [v^i, w^i] = 0 \,, \quad (v^i)^2 + (w^i)^2 = 1$$

$$\bar{x}^i = x^i \,, \quad \bar{y}^i = y^i \,, \quad [x^i, y^i] = 0 \,,$$

and recover $C^{alg}(T^n_\theta)$ from $C^{alg}(\mathbf{R}^{2n}_\theta)$ as follows:

PROPOSITION 3.2.4.

$$C^{alg}(T^n_\theta) \cong C^{alg}(\mathbf{R}^{2n}_\theta)/(z^1\bar{z}^1 - 1, \cdots, z^n\bar{z}^n - 1) \,.$$

3.3. Quantum matrix algebras. We recall the notion of quantum matrix algebras, the unital $*$-algebra $M_\theta(2n;\mathbf{R})$ which has been proposed by [C-DV]. We represent it here briefly. The elementary isomorphisms

$$M(2n;\mathbf{R})\cong\mathbf{R}^{4n^2}$$

and

$$M(2n;\mathbf{R})\cong End(\mathbf{R}^{2n})\cong(\mathbf{R}^{2n})^*\otimes\mathbf{R}^{2n}\cong\mathbf{R}^{2n}\otimes(\mathbf{R}^{2n})^*$$

would justify the following inclusion

$$\iota : C^{alg}(M_\theta(2n;\mathbf{R}))\cong C^{alg}(\mathbf{R}^{4n^2}_\Theta)\hookrightarrow C^{alg}(\mathbf{R}^{2n}_\theta)\otimes C^{alg}(\mathbf{R}^{2n}_{-\theta})\,.$$

Here

$$\Theta\in A(2n^2;\mathbf{R})$$

is determined by

$$\theta\in A(n;\mathbf{R})$$

as follows.

$$\Theta=\left(\begin{array}{cccc|cccc} & & & & & & & \theta \\ & & & & & & \cdot & \\ & & & & & \cdot & & \\ & & & & \cdot & & & \\ & & & & \theta & & & \\ \hline & & & -\theta & & & & \\ & & \cdot & & & & & \\ & \cdot & & & & & & \\ \cdot & & & & & & & \\ -\theta & & & & & & & \end{array}\right)$$

The following is the alternative definition of the quantum matrix algebra $M_\theta(2n;\mathbf{R})$ to the original one given by [C-DV] (see p. 561):

DEFINITION 3.3.1. $M_\theta(2n;\mathbf{R})$ is the unital associative $\mathbf{C}$-algebra generated by $2n^2$ normal elements

$$a^i_j\,,\quad b^i_j\quad(1\le i,\,j\le n)$$

such that

$$\iota(a^i_j)=z^i\otimes z_j\,,\quad \iota(b^i_j)=z^i\otimes\bar{z}_j\,,$$

$$z^i\in C^{alg}(\mathbf{R}^{2n}_\theta)\,,\quad z_j\in C^{alg}(\mathbf{R}^{2n}_{-\theta})\,,\quad z_i:=z^i$$

with commutation relations

$$a^i_j a^k_l = \lambda^{ik}\lambda_{jl} a^k_l a^i_j\,, \quad a^i_j \bar{a}^k_l = \lambda^{ik}\lambda_{jl}\bar{a}^k_l a^i_j\,,$$
$$a^i_j b^k_l = \lambda^{ik}\lambda_{jl} b^k_l a^i_j\,, \quad a^i_j \bar{b}^k_l = \lambda^{ik}\lambda_{jl}\bar{b}^k_l a^i_j\,,$$
$$b^i_j b^k_l = \lambda^{ik}\lambda_{jl} b^k_l b^i_j\,, \quad b^i_j \bar{b}^k_l = \lambda^{ik}\lambda_{jl}\bar{b}^k_l b^i_j\,,$$

plus th relation obtained by hermitian conjugation, where we have used the notation λ_{ij} for λ^{ij} to indicate that there is no summation in the above fomulas, and we also set $\lambda^{ik} := \lambda_{ik}$.

Since $M(2n;\mathbf{R})$ fails to have a group structure with respect to the ordinary multiplication of matrices, we cannot expect $C^{alg}(M_\theta(2n;\mathbf{R}))$ to have the corresponding Hopf algebra structure. But the essential obstruction is nothing but the antipode map. That is, we have no obstructions to define the corresponding bialgebra structure on it.

$$\begin{array}{rccc}
\Delta: & C^{alg}(M_\theta(2n;\mathbf{R})) & \longrightarrow & (M_\theta(2n;\mathbf{R}))\otimes(M_\theta(2n;\mathbf{R})) \\
& a^i_j & \longmapsto & a^i_k \otimes a^k_j + b^i_k \otimes \bar{b}^k_j \\
& b^i_j & \longmapsto & a^i_k \otimes b^k_j + b^i_k \otimes \bar{a}^k_j \\
\varepsilon: & C^{alg}(M_\theta(2n;\mathbf{R})) & \longrightarrow & \mathbf{C} \\
& a^i_j & \longmapsto & \delta^i_j \\
& b^i_j & \longmapsto & 0
\end{array}$$

$M(2n;\mathbf{R})$ has a natural action on $\mathbf{R}^{2n}$. It is not so hard to define the corresponding coaction of $C^{alg}(M_\theta(2n;\mathbf{R}))$ on $C^{alg}(\mathbf{R}^{2n}_\theta)$.

$$\begin{array}{rccc}
\alpha: & M(2n;\mathbf{R})\times\mathbf{R}^{2n} & \longrightarrow & \mathbf{R}^{2n} \\
\beta: & C^{alg}(\mathbf{R}^{2n}_\theta) & \longrightarrow & C^{alg}(M_\theta(2n;\mathbf{R}))\otimes C^{alg}(\mathbf{R}^{2n}_\theta) \\
& z^i & \longmapsto & a^i_j \otimes z^j + b^i_j \otimes \bar{z}^j
\end{array}$$

3.4. Quantum orthogonal groups and unitary groups. The dual objects of quantum linear Lie groups is defined as the quotient algebras of $C^{alg}(M_\theta(2n;\mathbf{R}))$ by appropriate sided ideal:

$$C^{alg}(G_\theta) := C^{alg}(M_\theta(2n;\mathbf{R}))/\mathcal{I}$$

We recall that $O(2n)$ is defined as a quotient of $M(2n;\mathbf{R})$ such that each of its elements preserves the quadratic form $\sum_{i=1}^{n} z^i \bar{z}^i$. Thus it is quite natural to characterize $C^{alg}(O_\theta(2n))$ by the following proposition, which is given by [C-DV].

PROPOSITION 3.4.1. *Let I be the two-sided ideal of $C^{alg}(M_\theta(2n;\mathbf{R}))$ generated by*

$$\sum_{i=1}^{n}(\bar{a}^i_j a^i_k + b^i_j \bar{b}^i_k) - \delta_{jk}$$

$$\sum_{i=1}^{n}(\bar{a}^i_j b^i_k + b^i_j \bar{a}^i_k)\,, \quad \sum_{i=1}^{n}(\bar{b}^i_j a^i_k + a^i_j \bar{b}^i_k)\,.$$

Then, we have

$$\pi : \quad C^{alg}(M_\theta(2n;\mathbf{R})) \longrightarrow C^{alg}(O_\theta(2n)) := C^{alg}(M_\theta(2n;\mathbf{R}))/\,{}^{\exists}I$$

with the coaction

$$\begin{array}{cccc} \beta' : & C^{alg}(\mathbf{R}^{2n}_\theta) & \longrightarrow & C^{alg}(O_\theta(2n)) \otimes C^{alg}(\mathbf{R}^{2n}_\theta) \\ & \sum_{i=1}^{n} z^i \bar{z}^i & \longmapsto & 1 \otimes \sum_{i=1}^{n} z^i \bar{z}^i \end{array}$$

Recalling the unitary group $U(n)$ is defined as

$$U(n) = \{g \in O(2n) \mid Jg = gJ\}\,,$$

we can translate this into the following dual formulation.

$$C^{alg}(U_\theta(n)) := C^{alg}(O_\theta(2n))/(\pi(b^i_j)\,, \quad \pi(\bar{b}^i_j))$$

4. Quantum complex projective spaces

In this section we will construct quantum complex projective spaces as quantum homogeneous spaces of the quantum unitary groups which appeared in previous section.

4.1. Restrictions and coactions of quantum unitary groups. Recall that the quotient space of the action

$$\alpha : U(n) \times (U(1) \times U(n-1)) \to U(n)$$

is the complex projective space

$$P^{n-1}(\mathbf{C}) = U(n)/(U(1) \times U(n-1))\,.$$

It is natural to consider the dual object for quantum complex projective space as invariant subalgebra of such a coaction as

$$\beta : C^{alg}(U_{\theta_n}(n)) \to C^{alg}(U_{\theta_n}(n)) \otimes (C^{alg}(U_{\theta_1}(1)) \otimes C^{alg}(U_{\theta_{n-1}}(n-1))) .$$

$$C^{alg}(P^{n-1}_{\theta_n}(\mathbf{C})) := C^{alg}(U_{\theta_n}(n))^{(C^{alg}(U_{\theta_1}(1)) \otimes C^{alg}(U_{\theta_{n-1}}(n-1)))}$$

$$\theta_n \in A(n; \mathbf{R}) , \quad \theta_1 = 0 \in A(1; \mathbf{R})$$

First we have to construct such a restriction as

$$\rho : C^{alg}(U_{\theta_k}(k)) \to C^{alg}(U_{\theta_{k-1}}(k-1))$$

for the standard inclusion

$$\iota : U(k-1) \to U(k) .$$

LEMMA 4.1.1. *For the normal generators*

$$a^i_j, b^i_j \in C^{alg}(M_{\theta_k}(2k; \mathbf{R})) ,$$

let I and J to be ideals generated by

$$a^1_1 - 1 , \; b^1_1 , \; a^1_j , \; a^i_1 , \; b^1_j , \; b^i_1 \quad \textit{for} \quad 2 \le i , \quad j \le k$$

and

$$a^k_k - 1 , \; b^k_k , \; a^k_j , \; a^i_k , \; b^k_j , \; b^i_k \quad \textit{for} \quad 1 \le i, \; j \le k-1 ,$$

respectively. Then the images of quotient maps

$$\rho_k : C^{alg}(M_{\theta_k}(2k; \mathbf{R})) \to C^{alg}(M_{\theta_k}(2k; \mathbf{R}))/I$$

and

$$\rho'_k : C^{alg}(M_{\theta_k}(2k; \mathbf{R})) \to C^{alg}(M_{\theta_k}(2k; \mathbf{R}))/J$$

coinside with

$$C^{alg}(M_{\theta_{k-1}}(2(k-1); \mathbf{R})) \quad \textit{and} \quad C^{alg}(M_{\theta'_{k-1}}(2(k-1); \mathbf{R})) ,$$

respectively. Here, there exists for a given antisymmetric martix θ_k of size k,

$$^{\exists}\theta_{k-1} , \; \theta'_{k-1} \in A(k-1; \mathbf{R})$$

such that

$$\theta_k = \begin{pmatrix} \theta'_{k-1} & * \\ * & 0 \end{pmatrix} , \begin{pmatrix} 0 & * \\ * & \theta_{k-1} \end{pmatrix} \in A(k; \mathbf{R}) .$$

Moreover, the induced restrictions on C^{alg} of quantum unitary groups, denoted by the same notations above,

$$\rho_k : C^{alg}(U_{\theta_k}(k)) \to C^{alg}(U_{\theta_{k-1}}(k-1))$$

and

$$\rho_k' : C^{alg}(U_{\theta_k}(k)) \to C^{alg}(U_{\theta_{k-1}}(k-1))$$

are Hopf algebra homomorphisms.

Using these restrictions, we can get the restrictions corresponding to the standard inclusion

$$U(1)\times U(n-1) \hookrightarrow U(n)\,.$$

PROPOSITION 4.1.2. *Let $\rho_{n,1}$ be*

$$\rho_{n,1} := ((\rho_2\circ\cdots\circ\rho_n)\otimes\rho_n')\circ\Delta\,.$$

Then

$$\rho_{n,1} : C^{alg}(U_{\theta_n}(n)) \to (C^{alg}(U_{\theta_1}(1)) \otimes C^{alg}(U_{\theta_{n-1}}(n-1))$$

is a surjective $*$-Hopf algebra morphism, and

$$(id\otimes\rho_{n,1})\circ\Delta : C^{alg}(U_{\theta_n}(n)) \to C^{alg}(U_{\theta_n}(n)) \otimes (C^{alg}(U_{\theta_1}(1)) \otimes C^{alg}(U_{\theta_{n-1}}(n-1))$$

is a right coaction.

We now define the following:

DEFINITION 4.1.3. As the right comodule algebra $C^{alg}(U_{\theta_n}(n))$ is over

$$(C^{alg}(U_{\theta_1}(1))\otimes C^{alg}(U_{\theta_{n-1}}(n-1)))\,,$$

we define $C^{alg}(P^{n-1}_{\theta_n}(\mathbf{C}))$ to be the invariant subalgebra of its right coaction.

$$C^{alg}(P^{n-1}_{\theta_n}(\mathbf{C})) := C^{alg}(U_{\theta_n}(n))^{(C^{alg}(U_{\theta_1}(1))\otimes C^{alg}(U_{\theta_{n-1}}(n-1)))}$$

$$:= \{f\in C^{alg}(U_{\theta_n}(n)) \mid ((id\otimes\rho_{n,1})\circ\Delta)(f) = f\otimes 1\}$$

The algebra $C^{alg}(P^{n-1}_{\theta_n}(\mathbf{C}))$ is called *quantum complex projective space.*

REMARK 4.1.4. By a routine procedure as above construction, we can easily define

$$C^{alg}(Gr^{n,k}_\theta(\mathbf{C}))\,,\quad C^{alg}(F^{n,k_1,\dots,k_d}_\theta(\mathbf{C}))\quad\text{and}\quad C^{alg}(St^{n,k}_\theta(\mathbf{C}))\,,$$

which are dual of quantum complex Grassmannian manifold, complex flag manifold and complex Stiefel manifold, respectively. The real versions corresponding to these homogeneous spaces can also be obtained along this idea.

REMARK 4.1.5. We can also define odd dimensional quantum spheres as quantum homogeneous spaces of quantum unitary groups.

$$(id \otimes \rho_n) \circ \Delta : C^{alg}(U_{\theta_n}(n)) \to C^{alg}(U_{\theta_n}(n)) \otimes C^{alg}(U_{\theta_{n-1}}(n-1))$$

$$C^{alg}(S^{2n-1}_{\theta_n}) := \{ f \in C^{alg}(U_{\theta_n}(n)) \mid ((id \otimes \rho_n) \circ \Delta)(f) = f \otimes 1 \}$$

On the other hand, we can find another definition of them in [C-DV], which has the following expression:

$$C^{alg}(S^{2n-1}_{\theta_n}) := C^{alg}(\mathbf{R}^{2n}_{\theta_n}) \Big/ \left(\sum_{i=1}^{n} z^i \bar{z}^i - id \right)$$

Since $S^3 \cong SU(2)$ classically, it is expected that $C^{alg}(S^3_{\theta_2}) \cong C^{alg}(SU_{\theta_2}(2))$. But we cannot quantize $SU(n)$ in the context of θ-deformation ([C-DV]). In spite of this it is still interesting to consider the Hopf algebra structure on $C^{alg}(S^3_{\theta_2})$ with respect to the above two expressions.

5. Main theorems

In this section, we show two remarkable properties for the quantum complex projective spaces: the splitting formula and the nondegeneracy of the Hochschild dimension. These type of theorems have already been shown for the case of some quantum groups in [C-DV]. However, the restrictions and coactions defined in this paper require a little longer proofs than those shown in [C-DV].

The first is a splitting formura, which justifies the importance of the quantum tori in θ-deformation.

5.1. Splitting formula. We first show the splitting formura, which justifies the importance of the quantum tori in θ-deformation. We define the splitting homomorphisms mapping the coordinate rings on the quantum $2n$-Euclidian spaces $\mathbf{R}^{2n}_{\theta}$ into the coordinate rings on the product of the classical Euclidian spaces $\mathbf{R}^{2n}$ with the quantum n-torus T^n_{θ}.

We first consider two natural actions σ and τ of T^n on $C^{alg}(\mathbf{R}^{2n}_{\theta})$ and $C^{alg}(T^n_{\theta})$, respectively.

DEFINITION 5.1.1.

$$\begin{array}{rcl} \sigma : \quad T^n & \longrightarrow & \mathrm{Aut}(C^{alg}(\mathbf{R}^{2n}_{\theta})) \\ s & \longmapsto & \sigma_s \\ \sigma_s : C^{alg}(\mathbf{R}^{2n}_{\theta}) & \longrightarrow & C^{alg}(\mathbf{R}^{2n}_{\theta}) \\ z^i & \longmapsto & \exp(2\pi\sqrt{-1}s_i) z^i \end{array}$$

DEFINITION 5.1.2.

$$\begin{aligned} \tau :\quad T^n &\longrightarrow \mathrm{Aut}(C^{alg}(T^n_\theta)) \\ t &\longmapsto \tau_t \\ \tau_t : C^{alg}(T^n_\theta) &\longrightarrow C^{alg}(T^n_\theta) \\ : \quad u^i &\longmapsto \exp(2\pi\sqrt{-1}t_i)u^i \end{aligned}$$

This yields two actions σ and τ of T^n on $\mathbf{R}^{2n} \times T^n_\theta$ given by the group-homomorphisms $s \mapsto \sigma_s \otimes I$ and $s \mapsto I \otimes \tau_s$ of T^n into $\mathrm{Aut}(C^{alg}(\mathbf{R}^{2n}) \otimes C^{alg}(T^n_\theta))$ with obvious notations. The noncommutative space $\mathbf{R}^{2n} \times T^n_\theta$ is here defined by duality by writing $C^{alg}(\mathbf{R}^{2n} \times T^n_\theta) = C^{alg}(\mathbf{R}^{2n}) \otimes C^{alg}(T^n_\theta)$. We shall use the actions σ and the diagonal action $\sigma \times \tau^{-1}$ of T^n on $\mathbf{R}^{2n} \times T^n_\theta$, where $\sigma \times \tau^{-1}$ is defined by $s \mapsto \sigma_s \otimes \tau_{-s} = (\sigma \times \tau^{-1})_s$ (as group homomorphism of T^n into $\mathrm{Aut}(C^{alg}(\mathbf{R}^{2n} \times T^n_\theta))$).

In the following statement, $z^i_{(0)}$ denotes the classical coordinates of $\mathbf{C}^n$ corresponding to z^μ for $\theta = 0$.

PROPOSITION 5.1.3. a) *There is a unique homomorphism of unital $*$-algebra*

$$st : C^{alg}(\mathbf{R}^{2n}_\theta) \to C^{alg}(\mathbf{R}^{2n}) \otimes C^{alg}(T^n_\theta)$$

such that $st(z^i) = z^i_{(0)} \otimes u^i$ for $i = 1, \cdots, n$.

b) *The homomorphism st induces an isomorphism of $C^{alg}(\mathbf{R}^{2n}_\theta)$ onto the subalgebra $C^{alg}(\mathbf{R}^{2n} \times T^n_\theta)^{\sigma\times\tau^{-1}}$ of $C^{alg}(\mathbf{R}^{2n} \times T^n_\theta)$ of fixed points of the diagonal action of T^n.*

It is obvious that the $st(z^i)$ are invariant by the diagonal action of T^n. Thus the only non-trivial parts of the statement, which are not difficult to show, are the injectivity of st and the fact that $C^{alg}(\mathbf{R}^{2n} \times T^n_\theta)^{\sigma\times\tau^{-1}}$ is generated by the z^i as unital $*$-algebra.

Let us consider the homomorphism

$$\begin{aligned} r_{23} \circ (st \otimes st) : C^{alg}(\mathbf{R}^{2n}_\theta) \otimes C^{alg}(\mathbf{R}^{2n}_{-\theta}) \\ \to C^{alg}(\mathbf{R}^{2n}) \otimes C^{alg}(\mathbf{R}^{2n}) \otimes C^{alg}(T^n_\theta) \otimes C^{alg}(T^n_{-\theta}) \end{aligned}$$

where r_{23} is the transposition of the second and the third factors in the tensor product, (i.e. $C^{alg}(T^n_\theta) \otimes C^{alg}(\mathbf{R}^{2n})$ is replaced by $C^{alg}(\mathbf{R}^{2n}) \otimes C^{alg}(T^n_\theta)$ there). This $*$-homomorphism restricts to give a homomorphism, again denoted by st

$$st : M_\theta(2n, \mathbf{R}) \to M(2n, \mathbf{R}) \otimes C^{alg}(T^n_\theta) \otimes C^{alg}(T^n_{-\theta})$$

which is again a homomorphism of unital $*$-algebras and will be also refered to as splitting homomorphism. It is the unique unital $*$-homomorphism such that

$$st(a^i_j) = \widetilde{a^i_j} \otimes u^i \otimes u_j \tag{1}$$

$$st(b^i_j) = \widetilde{b^i_j} \otimes u^i \otimes u^*_j \tag{2}$$

for $i, j = 1, \cdots, n$ where $\widetilde{a^i_j}$ and $\widetilde{b^i_j}$ are the classical coordinates corresponding to a^i_j and b^i_j for $\theta = 0$. The counterpart of $b)$ in Proposition 5.1.3 is that st induces here an isomorphism of $M_\theta(2n, \mathbf{R})$ onto the subalgebra of elements x of $M(2n, \mathbf{R}) \otimes C^{alg}(T^n_\theta) \otimes C^{alg}(T^n_{-\theta})$ which are invariant by the diagonal action $(\sigma \otimes \sigma) \times (\tau \otimes \tau)^{-1}$ of $T^n \times T^n$ i.e. which satisfy $(\sigma_s \otimes \sigma_t)(\tau_{-s} \otimes \tau_{-t})(x) = x, \forall (s, t) \in T^n \times T^n$. Here,

DEFINITION 5.1.4.

$$\begin{array}{rcl} \sigma \times \sigma \ : \quad T^n \times T^n & \longrightarrow & \mathrm{Aut}(C^{alg}(M_\theta(2n; \mathbf{R}))) \\ (s, t) & \longmapsto & \sigma_s \otimes \sigma_t \\ \sigma_s \otimes \sigma_t : C^{alg}(M_\theta(2n; \mathbf{R})) & \longrightarrow & C^{alg}(M_\theta(2n; \mathbf{R})) \\ a^i_j & \longmapsto & \exp(2\pi\sqrt{-1}(s_i + t_j))\, a^i_j \\ b^i_j & \longmapsto & \exp(2\pi\sqrt{-1}(s_i - t_j))\, b^i_j \end{array}$$

One has

$$st \circ (\sigma_s \otimes \sigma_t) = ((\sigma_s \otimes \sigma_t) \otimes I \otimes I) \circ st$$

which implies that st induces an isomorphism of $M_\theta(2n, \mathbf{R})^{\sigma\otimes\sigma}$ onto $M(2n, \mathbf{R})^{\sigma\otimes\sigma} \otimes 1 \otimes 1$ where $M_\theta(2n, \mathbf{R})^{\sigma\otimes\sigma}$ denotes the subalgebra of elements which are invariant by the action of $T^n \times T^n$, (the same for $\theta = 0$ on the right-hand side).

The above homomorphism passes to the quotient to define homomorphisms

$$st : C^{alg}(G_\theta) \to C^{alg}(G) \otimes C^{alg}(T^n_\theta) \otimes C^{alg}(T^n_{-\theta})$$

where G is $O(2n)$ and $U(n)$. These homomorphisms st which will still be refered to as the splitting homomorphisms, have the property that they induce isomorphisms of $C^{alg}(G_\theta)$ onto $(C^{alg}(G) \otimes C^{alg}(T^n_\theta) \otimes C^{alg}(T^n_{-\theta}))^{(\sigma\otimes\sigma)\times(\tau\otimes\tau)^{-1}}$ for these groups G.

By the above discussion and the definition of coaction $\rho_{n,1}$, we obtain the following theorem.

THEOREM 5.1.5. *The noncommutativity between the generators of $C^{alg}(P^{n-1}_{\theta_n}(\mathbf{C}))$ is absorbed in quantum tori. That is*:

$$C^{alg}(P^{n-1}_{\theta_n}(\mathbf{C})) = (C^{alg}(P^{n-1}(\mathbf{C})) \otimes C^{alg}(T^n_\theta) \otimes C^{alg}(T^n_{-\theta}))^{(\sigma\otimes\sigma)\,\times\,(\tau\otimes\tau)^{-1}}.$$

5.2. Nondegeneracy of the Hochschild dimensions. We show the nondegeneracy of dimension of quantum projective spaces, which also implies the reasonableness of the θ-deformation. Namely, we have

THEOREM 5.2.1.

$$\dim_H(C^{alg}(P^{n-1}_{\theta_n}(\mathbf{C}))) = \dim(P^{n-1}(\mathbf{C}))$$

Here, $\dim_H$ *denotes the Hochschild dimension of algebras, which is the last degree of non-trivial Hochschild homology of algebras.*

Recalling that the Hochschild homology of the coordinate ring of a given manifold is nothing but the de Rham algebra of the manifold, we are led to the natural direction of the proof of the above theorem. That is we should consider the differential graded algebras associated to the θ-deformation of classical spaces, especially quantum tori.

We first give the definition of $C^\infty(T^n_\theta)$. The locally convex $*$-algebra $C^\infty(T^n_\theta)$ of smooth functions on the quantum torus T^n_θ is defined as follows. It is the completion of $C^{alg}(T^n_\theta)$ equipped with the locally convex topology generated by the seminorms

$$|u|_r = \sup_{r_1+\cdots+r_n\le r} \|X_1^{r_1}\cdots X_n^{r_n}(u)\|$$

where $\|\cdot\|$ is the C^*-norm (which is the sup of the C^*-seminorms) and where the X_i are the infinitesimal generators of the action $s \mapsto \tau_s$ of T^n on T^n_θ. They are the unique derivations of $C^{alg}(T^n_\theta)$ satisfying

$$X_i(u^j) = 2\pi i \delta_i^j u^j \tag{3}$$

for $i, j = 1, \cdots, n$.

The above definition with the splitting formula for quantum tori leads us the following proposition.

PROPOSITION 5.2.2.

$$C^\infty(T^n_\theta) = (C^\infty(T^n)\,\widehat{\otimes}\,C^\infty(T^n_\theta))^{\sigma\times\tau^{-1}}$$

Let $\Omega(T^n_\theta)$ be the graded-involutive subalgebra $(\Omega(T^n)\,\widehat{\otimes}\,C^\infty(T^n_\theta))^{\sigma\times\tau^{-1}}$ of $\Omega(T^n)\,\widehat{\otimes}\,C^\infty(T^n_\theta)$ consisting of elements which are invariant by the diagonal action $\sigma\times\tau^{-1}$ of T^n. This subalgebra is stable by $d\otimes I$ so $\Omega(T^n_\theta)$ is a locally convex graded-involutive differential algebra which is a deformation of $\Omega(T^n)$ with $\Omega^0(T^n_\theta) = C^\infty(T^n_\theta)$ and which will be referred to as the *algebra of smooth differential forms on* T^n_θ. The action $s\mapsto\sigma_s$ of T^n on $\Omega(T^n)$ induces $s\mapsto\sigma_s\otimes I$ on $\Omega(T^n)\widehat{\otimes}C^\infty(T^n_\theta)$ which gives by restriction a group-homomorphism, again denoted $s\mapsto\sigma_s$, of T^n into the group $\mathrm{Aut}(\Omega(T^n_\theta))$ of automorphisms of the graded-involutive differential algebra $\Omega(T^n_\theta)$.

PROPOSITION 5.2.3. *The graded-involutive differential subalgebra* $\Omega(T^n_\theta)^\sigma$ *of* σ*-invariant elements of* $\Omega(T^n_\theta)$ *is in the graded center of* $\Omega(T^n_\theta)$ *and identified canonically with the graded-involutive differential subalgebra* $\Omega(T^n)^\sigma$ *of* σ*-invariant elements of* $\Omega(T^n)$.

In other words the subalgebra of σ-invariant elements of $\Omega(T^n_\theta)$ is not deformed (i.e. independent of θ). In fact one has $\Omega(T^n_\theta)^\sigma = \Omega(T^n)^\sigma\otimes 1$ ($\subset \Omega(T^n)\widehat{\otimes}C^\infty(T^n_\theta)$).

Now we compute the Hochschild dimension of T^n_θ. We first construct a continuous projective resolution of the left module $C^\infty(T^n_\theta)$ over $C^\infty(T^n_\theta)\,\widehat{\otimes}\,C^\infty(T^n_\theta)^{opp}$.

LEMMA 5.2.4. *There are continuous homomorphisms of left modules*

$$i_p : \Omega^p(T^n_\theta)\,\widehat{\otimes}\,C^\infty(T^n_\theta) \to \Omega^{p-1}(T^n_\theta)\,\widehat{\otimes}\,C^\infty(T^n_\theta)$$

over $C^\infty(T^n_\theta)\,\widehat{\otimes}\,C^\infty(T^n_\theta)^{opp}$ *for* $p \in \{1, \cdots, m\}$ *such that the sequence*

$$0 \to \Omega^m(T^n_\theta)\,\widehat{\otimes}\,C^\infty(T^n_\theta) \xrightarrow{i_m} \cdots \xrightarrow{i_1} C^\infty(T^n_\theta)\,\widehat{\otimes}\,C^\infty(T^n_\theta) \xrightarrow{\mu} C^\infty(T^n_\theta) \to 0$$

is exact, where μ *is induced by the product of* $C^\infty(T^n_\theta)$.

In fact one has continuous projective resolutions of $C^\infty(T^n)$ and of $C^\infty(T^n_\theta)$ of the form

$$0 \to \Omega^m(T^n)\,\widehat{\otimes}\,C^\infty(T^n) \xrightarrow{i^0_m} \cdots \xrightarrow{i^0_1} C^\infty(T^n)\,\widehat{\otimes}\,C^\infty(T^n) \xrightarrow{\mu} C^\infty(T^n) \to 0$$

$$0 \to \Omega^n(T^n_\theta)\,\widehat{\otimes}\,C^\infty(T^n_\theta) \xrightarrow{j_n} \cdots \xrightarrow{j_1} C^\infty(T^n_\theta)\,\widehat{\otimes}\,C^\infty(T^n_\theta) \xrightarrow{\mu} C^\infty(T^n_\theta) \to 0$$

which combine to give a continuous projective resolution of

$$C^\infty(T^n)\,\widehat{\otimes}\,C^\infty(T^n_\theta) = C^\infty(M \times T^n_\theta)$$

of the form

$$\begin{aligned} &0 \to \Omega^{m+n}(T^n \times T^n_\theta)\,\widehat{\otimes}\,C^\infty(T^n \times T^n_\theta) \xrightarrow{\tilde{i}_{m+n}} \cdots \\ &\xrightarrow{\tilde{i}_1} C^\infty(T^n \times T^n_\theta)\,\widehat{\otimes}\,C^\infty(T^n \times T^n_\theta) \xrightarrow{\mu} C^\infty(T^n \times T^n_\theta) \to 0 \end{aligned}$$

where $\Omega^p(T^n \times T^n_\theta) = \bigoplus_{p \geq k \geq 0} \Omega^k(T^n)\,\widehat{\otimes}\,\Omega^{p-k}(T^n_\theta)$ and where

$$\tilde{i}_p = \sum_k (i^0_k \otimes I + (-I)^k \otimes j_{p-k}) .$$

There is some freedom in the choice of the i^0_k, j_ℓ and one can choose them equivariantly (by choosing a σ-invariant metric on M, etc.) in such a way that the $\tilde{i}_p$ restrict as continuous homomorphisms

$$i_p : \Omega^p(T^n_\theta)\,\widehat{\otimes}\,C^\infty(T^n_\theta) \to \Omega^{p-1}(T^n_\theta)\,\widehat{\otimes}\,C^\infty(T^n_\theta)$$

of $C^\infty(T^n_\theta)\,\widehat{\otimes}\,C^\infty(T^n_\theta)^{opp}$-modules which gives the desired resolution of $C^\infty(T^n_\theta)$.

This shows that the Hochschild dimension n_θ of T^n_θ is $\leq n$.

Let $w \in \Omega^n(T^n)$ be a non-zero σ-invariant form of degree n on T^n (obtained by a straightforward local averaging). In view of Proposition 5.2.3, $w \otimes 1 = w_\theta$ is a σ-invariant element of $\Omega^n(T^n_\theta)$, i.e. $w_\theta \in \Omega^n(T^n_\theta)^\sigma$ which defines canonically a non-trivial invariant cycle v_θ in $Z_n(C^\infty(T^n_\theta), C^\infty(T^n_\theta))$. Thus one has $n_\theta \geq n$ and therefore the following result.

PROPOSITION 5.2.5.

$$\dim_H(T^n_\theta) = \dim(M)\,.$$

That is the Hochschild dimension n_θ of $C^\infty(T^n_\theta)$ coincides with the dimension n of T^n.

Combing Proposition 5.2.2 with this proposition, we finally conclude proof of Theorem 5.2.1.

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Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY,
KEIO UNIVERSITY,
HIYOSHI, KOHOKU-KU, YOKOHAMA, 223–8522 JAPAN.
e-mail: kamimura@math.keio.ac.jp