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The Multiple Hurwitz Zeta Function and a Generalization of Lerch's Formula

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Abstract. We investigate the multiple Hurwitz zeta function $\zeta_n(s_1, \cdots, s_n; a)$, in particular those values at non-positive integers. Then, as an application, we give a generalization of Lerch's formula.

1. Introduction and the statement of main results

In this paper we consider the following multiple Hurwitz zeta function for $a > 0$ and complex variables $s_1, \cdots, s_n$:

$$\zeta_n(s_1, \cdots, s_n; a) = \sum_{\substack{0 \le m_1 < \cdots < m_n \\ m_i \in \mathbf{Z}}} \frac{1}{(m_1 + a)^{s_1} \cdots (m_n + a)^{s_n}}, \tag{1.1}$$

and its specialization

$$\zeta_n(s; a) = \zeta_n(s, \cdots, s; a). \tag{1.2}$$

For convenience, we set

$$\zeta_0(s; a) = 1. \tag{1.3}$$

The right-hand side of (1.1) is absolutely convergent if $\mathrm{Re}(s_i) > 1$, $1 \le i \le n$. For $n = 1$, $\zeta_1(s; a) = \zeta(s; a)$ is the classical Hurwitz zeta function $\zeta(s, a)$, and for $a = 1$, $\zeta_n(s_1, \cdots, s_n; 1)$ is the Euler-Zagier multiple zeta function $\zeta_n(s_1, \cdots, s_n)$. When $s_1, \cdots, s_n$ are positive integers with $s_n \ge 2$, the values of $\zeta_n(s_1, \cdots, s_n)$ are called multiple zeta values, and they have been studied from the time of Euler. In recent years, many new relations among multiple zeta values were discovered by Arakawa, Hoffman, Kaneko, Ohno, and Zagier (cf. [4], [7], [8], [11], and [14]). There is a survey article by Arakawa and Kaneko ([6]).

Now we regard $\zeta_n(s_1, \cdots, s_n; a)$ as a complex variable function. Arakawa and Kaneko proved the analytic continuation of $\zeta_n(s_1, \cdots, s_n)$ with respect to the last variable s_n ([4]). The analytic continuation of $\zeta_n(s_1, \cdots, s_n)$ to $\mathbf{C}^n$ as a function of n variables was proved independently by Akiyama, Egami and Tanigawa ([1]), and Zhao ([15]). Akiyama and Ishikawa

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introduced a multiple Hurwitz zeta function which is slightly different from our (1.1) and proved its analytic continuation to $\mathbf{C}^n$ ([2]). Matsumoto and Tanigawa proved the analytic continuation of a wide class of multiple Dirichlet series and multiple Hurwitz zeta functions ([9] and [10]), and the analytic continuation of our multiple Hurwitz zeta function (1.1) is a special case of [9, Theorem 1].

It is an interesting problem to consider values of zeta functions at non-positive integers, but the multiple Hurwitz zeta function can have a point of indeterminacy there. Following [1] and [3], we consider three special limiting processes

$$\begin{aligned}
\zeta_n^{Reg}(s_1,\cdots,s_n;a) &= \lim_{t_1\to s_1}\cdots\lim_{t_n\to s_n}\zeta_n(t_1,\cdots,t_n;a)\,,\\
\zeta_n^{Rev}(s_1,\cdots,s_n;a) &= \lim_{t_n\to s_n}\cdots\lim_{t_1\to s_1}\zeta_n(t_1,\cdots,t_n;a)\,,\\
\zeta_n^{C}(s_1,\cdots,s_n;a) &= \lim_{\varepsilon\to 0}\zeta_n(s_1+\varepsilon,\cdots,s_n+\varepsilon;a)\,,
\end{aligned}\tag{1.4}$$

and call them regular values, reverse values, and central values, respectively. In [1] and [3], Akiyama, Egami and Tanigawa introduced them in order to investigate the Euler-Zagier multiple zeta function $\zeta_n(s_1,\cdots,s_n)$ and gave recurrence relations of regular values and reverse values using the Euler-Maclaurin summation formula. By the same method, we give recurrence relations of regular values and reverse values for our multiple Hurwitz zeta function in Theorem 2.2 stated in the next section.

On the other hand, as for central values of $\zeta_n(s_1,\cdots,s_n)$, almost nothing is known except for the cases $n=2,3$ (cf. [1, §3, Remark 2]). In this paper, we give certain central values of the multiple Hurwitz zeta function by using harmonic products. It is easy to see that $\zeta_n(s;a)$ is equal to $\zeta_n^C(s,\cdots,s;a)$ if its value is determined.

THEOREM 1. *For $s\in\mathbf{C},\ s\notin\{\frac{1}{u}\mid u\in\mathbf{Z},\ 1\le u\le n\}$, the following identity holds*:

$$\zeta_n(s;a)=\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\zeta_{n-k}(s;a)\zeta_1(ks;a)\,.\tag{1.5}$$

In other words, we obtain the following identity:

$$\zeta_n^C(s,\cdots,s;a)=\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\zeta_{n-k}^C(s,\cdots,s;a)\zeta_1^C(ks;a)\,.\tag{1.6}$$

Further, as a corollary of Theorem 1, we obtain some central values explicitly. The following statement (ii) has been conjectured in [1, §3, Remark 2].

COROLLARY 2. (i) *The following identity holds*:

$$\zeta_n^C(0,\cdots,0;a)=\zeta_n(0;a)=\frac{(-1)^n}{n!}\prod_{k=1}^{n}\left(k+a-\frac{3}{2}\right).\tag{1.7}$$

(ii) *For any natural number u, the following identity holds*:

$$\zeta_n^C(-2u, \cdots, -2u) = \zeta_n(-2u; 1) = 0\,. \tag{1.8}$$

Here we recall a classical formula given by Lerch.

LERCH'S FORMULA (Lerch 1894, e.g. [13, p. 271]). *For $a > 0$, we have*

$$\zeta'(0, a) = \log \frac{\Gamma(a)}{\sqrt{2\pi}}\,, \tag{1.9}$$

where $\zeta'(s; a)$ denotes $\frac{d}{ds}\zeta(s; a)$ and $\Gamma(a)$ is the gamma function.

As an application of Theorem 1 and Corollary 2 (i), we obtain the following generalization of Lerch's formula.

THEOREM 3 (Multiple Lerch's formula). *For $a > 0$, we have*

$$\zeta_n'(0; a) = \frac{(-1)^{n-1}}{(n-1)!} \prod_{k=1}^{n-1} \left(k + a - \frac{1}{2}\right) \log \frac{\Gamma(a)}{\sqrt{2\pi}}\,, \quad (n \geq 1)\,, \tag{1.10}$$

where $\zeta_n'(s; a)$ denotes $\frac{d}{ds}\zeta_n(s; a)$ and an empty product means 1.

NOTATIONS. *We denote the set of rational integers and complex numbers by* $\mathbf{Z}$ *and* $\mathbf{C}$, *respectively. And we denote the set of integers not less than j (resp. not more than j) by $\mathbf{Z}_{\geq j}$ (resp. $\mathbf{Z}_{\leq j}$).*

2. Three kinds of limiting values of the multiple Hurwitz zeta function at non-positive integers

In this section, we consider values of the multiple Hurwitz zeta function at non-positive integers. In [9], Matsumoto introduced a wide class of multiple Hurwitz zeta function which contains our multiple Hurwitz zeta function (1.1), and proved their analytic continuation by using the Mellin-Barnes integral formula. In the notation in [9],

$$\zeta_n(s_1, \cdots, s_n; a) = \zeta_n((s_1, \cdots, s_n); (a, a+1, \cdots, a+n-1), (1, \cdots, 1))\,,$$

and we have the following theorem.

THEOREM 2.1 (Matsumoto [9, Theorem 1]). *The multiple Hurwitz zeta function $\zeta_n(s_1, \cdots, s_n; a)$ defined by (1.1) can be analytically continued to $\mathbf{C}^n$, and holomorphic except for the sets determined by*

$$s_n = 1 \quad \textit{and} \quad \sum_{i=1}^{j} s_{n-i+1} \in \mathbf{Z}_{\leq j} \quad (j = 2, 3, \cdots, n)\,. \tag{2.1}$$

In [1] and [3], Akiyama, Egami and Tanigawa gave recurrence relations of regular values and reverse values of the Euler-Zagier multiple zeta function. Similar to [1] and [3], we obtain the following recurrence relations for the multiple Hurwitz zeta function.

THEOREM 2.2. *For $u_1, u_2, \cdots, u_n \in \mathbf{Z}_{\geq 0}$ with $n \geq 2$, we have*

$$\zeta_n^{Reg}(-u_1, \cdots, -u_n; a)$$
$$= \sum_{k=-1}^{u_n} (-u_n)_k \frac{B_{k+1}}{(k+1)!} \zeta_{n-1}^{Reg}(-u_1, \cdots, -u_{n-2}, -u_{n-1} - u_n + k; a)\,, \tag{2.2}$$

$$\zeta_n^{Rev}(-u_1, \cdots, -u_n; a)$$
$$= -\sum_{k=-1}^{u_1} (-u_1)_k \frac{B_{k+1}}{(k+1)!} \zeta_{n-1}^{Rev}(-u_1 - u_2 + k, -u_3, \cdots, -u_n; a) \tag{2.3}$$
$$- \zeta_{n-1}^{Rev}(-u_1 - u_2, -u_3, \cdots, -u_n; a) + \zeta(-u_1; a)\zeta_{n-1}^{Rev}(-u_2, \cdots, -u_n; a)\,.$$

Here we set

$$(s)_r = \begin{cases} s(s+1)\cdots(s+r-1) & (r = 1, 2, \cdots)\,, \\ 1 & (r = 0)\,, \\ 1/(s-1) & (r = -1)\,, \end{cases}$$

and B_n is the n-th Bernoulli number defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n\,.$$

REMARK 2.3. In particular, when $a = 1$, equations (2.2) and (2.3) are the same as those proved in [1, Eq. (6)] and [3, §7].

We quote here the Euler-Maclaurin summation formula, since we use it in the proof of Theorem 2.2.

LEMMA 2.4 (e.g. [12, I.0.2, Theorem 4]). *Let $\alpha, \beta, l \in \mathbf{Z}$, $0 \leq l$, $0 \leq \alpha \leq \beta$, and $f(x)$ be a $(l+1)$-times continuously differentiable function on $[\alpha, \beta]$. Then we have the following identity*:

$$\sum_{n=\alpha}^{\beta} f(n) = \int_{\alpha}^{\beta} f(x)dx + \frac{1}{2}(f(\alpha) + f(\beta)) + \sum_{k=1}^{l} \frac{B_{k+1}}{(k+1)!}(f^{(k)}(\beta) - f^{(k)}(\alpha))$$
$$- \frac{(-1)^{l+1}}{(l+1)!} \int_{\alpha}^{\beta} \tilde{B}_{l+1}(x) f^{(l+1)}(x)dx\,.$$

Here $\tilde{B}_n(x)$ is the n-th periodic Bernoulli polynomial defined as follows:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n\,, \qquad \tilde{B}_n(x) = B_n(\{x\})$$

where $\{x\}$ is the fractional part of x.

PROOF OF THEOREM 2.2. We set $f(x) = (x+a)^{-s}$ with $a > 0$, and assume that $\mathrm{Re}(s) > 1$. Then we have

$$f^{(k)}(x) = (-1)^k (s)_k (x+a)^{-s-k}\,,$$
$$\int_0^m f(x)dx = \frac{1}{s-1}\left(\frac{1}{a^{s-1}} - \frac{1}{(m+a)^{s-1}}\right).$$

Setting $\alpha = 0$ and $\beta = m$ in Lemma 2.4, we have

$$\begin{aligned}\sum_{n=0}^{m} \frac{1}{(n+a)^s} =& \frac{1}{s-1}\left(\frac{1}{a^{s-1}} - \frac{1}{(m+a)^{s-1}}\right) + \frac{1}{2}\left(\frac{1}{a^s} + \frac{1}{(m+a)^s}\right)\\ &+ \sum_{k=1}^{l} \frac{B_{k+1}}{(k+1)!}(s)_k\left(\frac{1}{a^{s+k}} - \frac{1}{(m+a)^{s+k}}\right)\\ &- \frac{(s)_{l+1}}{(l+1)!}\int_0^m \frac{\tilde{B}_{l+1}(x)}{(x+a)^{s+l+1}}dx\,.\end{aligned} \tag{2.4}$$

Letting $m \to \infty$, we have

$$\begin{aligned}\zeta_1(s;a) = \frac{1}{(s-1)a^{s-1}} + \frac{1}{2a^s} + \sum_{k=1}^{l} \frac{B_{k+1}}{(k+1)!}\frac{(s)_k}{a^{s+k}}\\ - \frac{(s)_{l+1}}{(l+1)!}\int_0^\infty \frac{\tilde{B}_{l+1}(x)}{(x+a)^{s+l+1}}dx\,.\end{aligned} \tag{2.5}$$

Considering the difference of (2.4) and (2.5), we obtain

$$\begin{aligned}\sum_{n=m+1}^{\infty} \frac{1}{(n+a)^s} =& \frac{(m+a)^{1-s}}{s-1} - \frac{1}{2(m+a)^s}\\ &+ \sum_{k=1}^{l} \frac{B_{k+1}}{(k+1)!}\frac{(s)_k}{(m+a)^{s+k}} - \phi_l(m,s;a)\,,\end{aligned} \tag{2.6}$$

where

$$\phi_l(m,s;a) = \frac{(s)_{l+1}}{(l+1)!}\int_m^\infty \frac{\tilde{B}_{l+1}(x)}{(x+a)^{s+l+1}}dx\,. \tag{2.7}$$

By (2.6), we have

$$
\begin{aligned}
&\zeta_n(s_1,\cdots,s_n;a)\\
&\quad=\sum_{0\le m_1<\cdots<m_{n-1}}\frac{1}{(m_1+a)^{s_1}\cdots(m_{n-1}+a)^{s_{n-1}}}\sum_{m_n=m_{n-1}+1}^{\infty}\frac{1}{(m_n+a)^{s_n}}\\
&\quad=\sum_{0\le m_1<\cdots<m_{n-1}}\frac{1}{(m_1+a)^{s_1}\cdots(m_{n-1}+a)^{s_{n-1}}}\left\{\frac{(m_{n-1}+a)^{1-s_n}}{s_n-1}\right.\\
&\qquad\left.-\frac{1}{2(m_{n-1}+a)^{s_n}}+\sum_{k=1}^{l}\frac{B_{k+1}}{(k+1)!}\frac{(s_n)_k}{(m_{n-1}+a)^{s_n+k}}-\phi_l(m_{n-1},s_n;a)\right\},
\end{aligned}
$$

for $\mathrm{Re}(s_i)>1$, $1\le i\le n$. Hence we obtain

$$
\begin{aligned}
&\zeta_n(s_1,\cdots,s_n;a)\\
&\quad=\frac{\zeta_{n-1}(s_1,\cdots,s_{n-2},s_{n-1}+s_n-1;a)}{s_n-1}-\frac{\zeta_{n-1}(s_1,\cdots,s_{n-2},s_{n-1}+s_n;a)}{2}\\
&\qquad+\sum_{k=1}^{l}\frac{(s_n)_kB_{k+1}}{(k+1)!}\zeta_{n-1}(s_1,\cdots,s_{n-2},s_{n-1}+s_n+k;a) \qquad (2.8)\\
&\qquad-\sum_{0\le m_1<\cdots<m_{n-1}}\frac{\phi_l(m_{n-1},s_n;a)}{(m_1+a)^{s_1}\cdots(m_{n-1}+a)^{s_{n-1}}}.
\end{aligned}
$$

By definition (2.7), we see that $\phi_l(m,s;a)=O(m^{-\mathrm{Re}(s)-l})$. Hence the last summation of (2.8)

$$
\sum_{0\le m_1<\cdots<m_{n-1}}\frac{\phi_l(m_{n-1},s_n;a)}{(m_1+a)^{s_1}\cdots(m_{n-1}+a)^{s_{n-1}}} \tag{2.9}
$$

is convergent if

$$
\mathrm{Re}(s_n)+\mathrm{Re}(s_{n-1})+\sum_{\substack{1\le i\le n-2\\ \mathrm{Re}(s_i)<0}}\mathrm{Re}(s_i)>n-l-1\,. \tag{2.10}
$$

In fact, by the inequality

$$\begin{aligned}
&\left| \sum_{0 \le m_1 < \cdots < m_{n-1}} \frac{\phi_l(m_{n-1}, s_n; a)}{(m_1 + a)^{s_1} \cdots (m_{n-1} + a)^{s_{n-1}}} \right| \\
&\qquad \le \frac{1}{|a^{s_1}|} \sum_{0 < m_2 < \cdots < m_{n-1}} \left| \frac{\phi_l(m_{n-1}, s_n; a)}{(m_2 + a)^{s_2} \cdots (m_{n-1} + a)^{s_{n-1}}} \right| \\
&\qquad\quad + \sum_{0 < m_1 < \cdots < m_{n-1}} \left| \frac{\phi_l(m_{n-1}, s_n; a)}{(m_1 + a)^{s_1} \cdots (m_{n-1} + a)^{s_{n-1}}} \right| ,
\end{aligned}$$

it follows that we have only to consider the summation

$$\sum_{0 < m_1 < \cdots < m_{n-1}} \frac{\phi_l(m_{n-1}, s_n; a)}{(m_1 + a)^{s_1} \cdots (m_{n-1} + a)^{s_{n-1}}} .$$

Considering the evaluation

$$\begin{aligned}
&\left| \sum_{0 < m_1 < \cdots < m_{n-1}} \frac{\phi_l(m_{n-1}, s_n; a)}{(m_1 + a)^{s_1} \cdots (m_{n-1} + a)^{s_{n-1}}} \right| \\
&\qquad \le \sum_{0 < m_2 < \cdots < m_{n-1}} \begin{cases} \left| \frac{\phi_l(m_{n-1}, s_n; a)}{(m_2 + a)^{s_2 - 1} \cdots (m_{n-1} + a)^{s_{n-1}}} \right| & (\text{if } \operatorname{Re}(s_1) \ge 0) \\ \left| \frac{\phi_l(m_{n-1}, s_n; a)}{(m_2 + a)^{s_1 + s_2 - 1} \cdots (m_{n-1} + a)^{s_{n-1}}} \right| & (\text{if } \operatorname{Re}(s_1) < 0) \end{cases}
\end{aligned}$$

and repeating this procedure, we obtain

$$\left| \sum_{0 < m_1 < \cdots < m_{n-1}} \frac{\phi_l(m_{n-1}, s_n; a)}{(m_1 + a)^{s_1} \cdots (m_{n-1} + a)^{s_{n-1}}} \right| \le \sum_{m_{n-1}=1}^{\infty} \frac{|\phi_l(m_{n-1}, s_n; a)|}{(m_{n-1} + a)^{F(s_1, \cdots, s_{n-1})}} , \qquad (2.11)$$

where

$$F(s_1, \cdots, s_{n-1}) = \operatorname{Re}(s_{n-1}) + \sum_{\substack{\operatorname{Re}(s_i) < 0 \\ 1 \le i \le n-2}} \operatorname{Re}(s_i) - (n - 2) . \qquad (2.12)$$

Hence (2.9) is convergent under the condition (2.10). Moreover, we see (2.9) is equal to 0 for $s_n = 0, -1, -2, \cdots, -l$ because $\phi_l(m, s; a) = 0$ for $s = 0, -1, -2, \cdots, -l$. Since $\zeta_n(s_1, \cdots, s_n; a)$ and $\zeta_{n-1}(s_1, \cdots, s_{n-1}; a)$ are analytically continued to the whole space, we can evaluate those regular values and obtain the recurrence relation (2.2).

Next we consider reverse values. We prove (2.3) by the same method as in [3, §7]. By (2.6), we have

$$\begin{aligned}\sum_{n=0}^{m-1}\frac{1}{(n+a)^s}&=\zeta(s;a)-\frac{(m+a)^{1-s}}{s-1}-\frac{1}{2(m+a)^s}\\&\quad-\sum_{k=1}^{l}\frac{B_{k+1}}{(k+1)!}\frac{(s)_k}{(m+a)^{s+k}}+\phi_l(m,s;a)\,,\end{aligned}\tag{2.13}$$

for $\mathrm{Re}(s)>1$. By (2.13), we have

$$\begin{aligned}&\zeta_n(s_1,\cdots,s_n;a)\\&\quad=\sum_{0\le m_2<\cdots<m_n}\frac{1}{(m_2+a)^{s_2}\cdots(m_n+a)^{s_n}}\Bigg\{\zeta(s_1;a)+\frac{(m_2+a)^{1-s_1}}{1-s_1}\\&\qquad-\frac{1}{2(m_2+a)^{s_1}}-\sum_{k=1}^{l}\frac{B_{k+1}}{(k+1)!}\frac{(s_1)_k}{(m_2+a)^{s_1+k}}+\phi_l(m_2,s_1;a)\Bigg\}\\&\quad=\frac{\zeta_{n-1}(s_1+s_2-1,s_3,\cdots,s_n;a)}{1-s_1}-\frac{\zeta_{n-1}(s_1+s_2,s_3,\cdots,s_n;a)}{2}\\&\qquad-\sum_{k=1}^{l}\frac{B_{k+1}}{(k+1)!}(s_1)_k\zeta_{n-1}(s_1+s_2+k,s_3,\cdots,s_n;a)+\zeta(s_1;a)\zeta_{n-1}(s_2,\cdots,s_n;a)\\&\qquad+\sum_{0\le m_2<\cdots<m_n}\frac{\phi_l(m_2,s_1;a)}{(m_2+a)^{s_2}\cdots(m_n+a)^{s_n}}\,,\end{aligned}$$

for $\mathrm{Re}(s_i)>1$, $1\le i\le n$. The last summation is equal to

$$\begin{aligned}&\sum_{0\le m_3<\cdots<m_n}\frac{1}{(m_3+a)^{s_3}\cdots(m_n+a)^{s_n}}\left\{\sum_{m_2=0}^{\infty}\frac{\phi_l(m_2,s_1;a)}{(m_2+a)^{s_2}}-\sum_{m_2=m_3}^{\infty}\frac{\phi_l(m_2,s_1;a)}{(m_2+a)^{s_2}}\right\}\\&\quad=\zeta_{n-2}(s_3,\cdots,s_n;a)\sum_{m_2=0}^{\infty}\frac{\phi_l(m_2,s_1;a)}{(m_2+a)^{s_2}}-\sum_{0\le m_4<\cdots<m_n}\frac{1}{(m_4+a)^{s_4}\cdots(m_n+a)^{s_n}}\\&\qquad\left\{\sum_{m_3=0}^{\infty}\sum_{m_2=m_3}^{\infty}\frac{\phi_l(m_2,s_1;a)}{(m_2+a)^{s_2}(m_3+a)^{s_3}}-\sum_{m_3=m_4}^{\infty}\sum_{m_2=m_3}^{\infty}\frac{\phi_l(m_2,s_1;a)}{(m_2+a)^{s_2}(m_3+a)^{s_3}}\right\}.\end{aligned}$$

Repeating this procedure, we obtain

$\zeta_n(s_1,\cdots,s_n;a)$

$$
\begin{aligned}
&= \frac{\zeta_{n-1}(s_1+s_2-1, s_3, \cdots, s_n; a)}{1-s_1} - \frac{\zeta_{n-1}(s_1+s_2, s_3, \cdots, s_n; a)}{2} \\
&\quad - \sum_{k=1}^{l} (s_1)_k \frac{B_{k+1}}{(k+1)!} \zeta_{n-1}(s_1+s_2+k, s_3, \cdots, s_n; a) + \zeta(s_1; a)\zeta_{n-1}(s_2, \cdots, s_n; a) \\
&\quad + \zeta_{n-2}(s_3, \cdots, s_n; a) \sum_{m_2=0}^{\infty} \frac{\phi_l(m_2, s_1; a)}{(m_2+a)^{s_2}} \\
&\quad - \zeta_{n-3}(s_4, \cdots, s_n; a) \sum_{0 \le m_3 \le m_2} \frac{\phi_l(m_2, s_1; a)}{(m_2+a)^{s_2}(m_3+a)^{s_3}} \\
&\quad + \cdots + (-1)^{n-1} \zeta(s_n; a) \sum_{0 \le m_{n-1} \le \cdots \le m_2} \frac{\phi_l(m_2, s_1; a)}{(m_2+a)^{s_2} \cdots (m_{n-1}+a)^{s_{n-1}}} \\
&\quad + (-1)^n \sum_{0 \le m_n \le \cdots \le m_2} \frac{\phi_l(m_2, s_1; a)}{(m_2+a)^{s_2} \cdots (m_n+a)^{s_n}} \,.
\end{aligned}
$$

Then, in a way similar to the case of regular values, we obtain the recurrence relation (2.3) of reverse values. □

Now we come to the stage of proving Theorem 1 of §1.

PROOF OF THEOREM 1. The method of harmonic products is known in the theory of multiple zeta values (cf. [7] and [8]). In our case, the following identity holds for $n \ge 2$ when all the appearing multiple Hurwitz zeta functions are given by convergent infinite series;

$$
\begin{aligned}
\zeta(t; a)\zeta_{n-1}(s_1, \cdots, s_{n-1}; a) &= \sum_{i=1}^{n} \zeta_n(s_1, \cdots, s_{i-1}, t, s_i, \cdots, s_{n-1}; a) \\
&\quad + \sum_{j=1}^{n-1} \zeta_{n-1}(s_1, \cdots, s_{j-1}, s_j + t, s_{j+1}, \cdots, s_{n-1}; a) \,.
\end{aligned}
\tag{2.14}
$$

The right and left hands of (2.14) are analytically continued to $\mathbf{C}^n$, hence (2.14) holds on $\mathbf{C}^n$ except on the singularities. We assume that there appear no singularities of zeta functions for the moment. By (2.14), we have

$$
n\zeta_n(s, \cdots, s; a) = \zeta(s; a)\zeta_{n-1}(s, \cdots, s; a) - \sum_{i=1}^{n-1} \zeta_{n-1}(s, \cdots, \overset{i}{2s}, \cdots, s; a) \,.
$$

We use (2.14) again, and we have

$$
\sum_{i=1}^{n-1} \zeta_{n-1}(s, \cdots, \overset{i}{2s}, \cdots, s; a) = \zeta(2s; a)\zeta_{n-2}(s, \cdots, s; a) - \sum_{j=1}^{n-2} \zeta_{n-2}(s, \cdots, \overset{j}{3s}, \cdots, s; a) \,.
$$

By repeating this transformation, we obtain equation (1.5).

We consider the case that there may appear singularities of zeta functions. The possible singularities of $\zeta_n(s_1, \cdots, s_n; a)$ are given by Theorem 2.1, so we can take $\varepsilon > 0$ such that $(t_1+\varepsilon, \cdots, t_n+\varepsilon)$ is in the holomorphic region for a singularity $(t_1, \cdots, t_n)$. Taking suitable ε, therefore, we can avoid singularities and have

$$n\zeta_n(s+\varepsilon, \cdots, s+\varepsilon; a) = \sum_{k=1}^{n}(-1)^{k+1}\zeta_{n-k}(s+\varepsilon, \cdots, s+\varepsilon; a)\zeta(ks+k\varepsilon; a)\,,$$

just as the above calculations. When ε tends to 0, there appears the pole of the Hurwitz zeta function if $ks = 1$ $(k = 1, 2, \cdots, n)$. Thus we obtain equation (1.5) for $s \in \mathbf{C}$, $s \notin \{\frac{1}{u} \mid u \in \mathbf{Z},\ 1 \le u \le n\}$. $\square$

REMARK 2.5. At a point of indeterminacy, equation (2.14) does not hold in general even if one takes regular values, reverse values, or central values. But the identity

$$\zeta(s_1; a)\zeta(s_2; a) = \zeta_2^C(s_1, s_2; a) + \zeta_2^C(s_2, s_1; a) + \zeta(s_1+s_2; a)$$

holds except on the poles.

PROOF OF COROLLARY 2. (i) It is well known that $\zeta_1(0; a) = 1/2 - a$. By (2.14), we have

$$\zeta(0; a)\zeta_{n-1}(0; a) = n\zeta_n(0; a) + (n-1)\zeta_{n-1}(0; a)\,,$$

so $\zeta_n(0; a)$ satisfies the recurrence relation

$$\zeta_n(0; a) = \frac{1}{n}\left(\frac{3}{2} - a - n\right)\zeta_{n-1}(0; a)\,. \tag{2.15}$$

Hence we obtain

$$\begin{aligned}\zeta_n(0; a) &= \prod_{k=2}^{n}\frac{1}{k}\left(\frac{3}{2} - a - k\right)\zeta_1(0; a)\\ &= \prod_{k=1}^{n}\frac{1}{k}\left(\frac{3}{2} - a - k\right)\\ &= \frac{(-1)^n}{n!}\prod_{k=1}^{n}\left(k + a - \frac{3}{2}\right).\end{aligned}$$

(ii) Let u be a natural number. By Theorem 1, we have

$$n\zeta_n(-2u; 1) = \sum_{k=1}^{n}(-1)^{k+1}\zeta_{n-k}(-2u; 1)\zeta(-2ku; 1)\,. \tag{2.16}$$

The right-hand side of (2.16) becomes 0 since $\zeta(-2m; 1) = \zeta(-2m) = 0$ for a natural number m. $\square$

3. A generalization of Lerch's formula

In this section, we prove Theorem 3 of §1. In the proof, we use the identity

$$\frac{1}{n!}\prod_{k=1}^{n}\left(k+x-\frac{1}{2}\right)=\sum_{k=0}^{n}\frac{1}{k!}\prod_{m=0}^{k-1}\left(m+x-\frac{1}{2}\right) \tag{3.1}$$

for any $n\in\mathbf{Z}_{\geq 0}$, which is obtained by induction on n.

PROOF OF THEOREM 3. We define $f_n(a)$ by $\zeta_n'(0;a)=f_n(a)\log\frac{\Gamma(a)}{\sqrt{2\pi}}$ for $n\geq 0$. If $n=0$, then $f_0(a)=0$ since $\zeta_0(s;a)=1$. To obtain the result, it suffices to show the identity for $n\geq 1$:

$$f_n(a)=\frac{(-1)^{n-1}}{(n-1)!}\prod_{k=1}^{n-1}\left(k+a-\frac{1}{2}\right). \tag{3.2}$$

By Theorem 1, we have

$$\zeta_n'(s;a)=\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}(\zeta_{n-k}'(s;a)\zeta(ks;a)+k\,\zeta_{n-k}(s;a)\zeta'(ks;a)). \tag{3.3}$$

Then, by Corollary 2 (i) and Lerch's formula, we obtain

$$\begin{aligned}\zeta_n'(0;a)&=\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}(\zeta_{n-k}'(0;a)\zeta(0;a)+k\,\zeta_{n-k}(0;a)\zeta'(0;a))\\&=\frac{1}{n}\log\frac{\Gamma(a)}{\sqrt{2\pi}}\\&\quad\times\sum_{k=1}^{n}(-1)^{k+1}\left\{\left(\frac{1}{2}-a\right)f_{n-k}(a)+k\frac{(-1)^{n-k}}{(n-k)!}\prod_{m=1}^{n-k}\left(m+a-\frac{3}{2}\right)\right\}.\end{aligned}$$

Then we obtain

$$\begin{aligned}f_n(a)&=\frac{1}{n}\left\{\sum_{k=1}^{n}(-1)^{k+1}\left(\frac{1}{2}-a\right)f_{n-k}(a)\right.\\&\qquad\left.+(-1)^{n+1}\sum_{k=1}^{n}\frac{k}{(n-k)!}\prod_{m=1}^{n-k}\left(m+a-\frac{3}{2}\right)\right\}\\&=\frac{(-1)^n}{n}\sum_{k=0}^{n-1}\left\{(-1)^k\left(a-\frac{1}{2}\right)f_k(a)-\frac{n-k}{k!}\prod_{m=0}^{k-1}\left(m+a-\frac{1}{2}\right)\right\}.\end{aligned}$$

We prove (3.2) by induction on n. It holds for $n = 1$ by Lerch's formula $\zeta'(0, a) = \log \frac{\Gamma(a)}{\sqrt{2\pi}}$. Assume that (3.2) holds for $k \leq n - 1$, then we get

$$\begin{aligned} f_n(a) &= \frac{(-1)^n}{n} \sum_{k=1}^{n-1} \left\{ (-1)^k \left(a - \frac{1}{2}\right) \frac{(-1)^{k-1}}{(k-1)!} \prod_{m=1}^{k-1} \left(m + a - \frac{1}{2}\right) \right. \\ &\quad \left. - \frac{n-k}{k!} \prod_{m=0}^{k-1} \left(m + a - \frac{1}{2}\right) \right\} + \frac{(-1)^n}{n} \cdot (-n) \\ &= \frac{(-1)^n}{n} \sum_{k=1}^{n-1} \left\{ \frac{-n}{k!} \prod_{m=0}^{k-1} \left(m + a - \frac{1}{2}\right) \right\} + (-1)^{n-1} \\ &= (-1)^{n-1} \sum_{k=0}^{n-1} \frac{1}{k!} \prod_{m=0}^{k-1} \left(m + a - \frac{1}{2}\right). \end{aligned}$$

Then, by (3.1), we have

$$f_{n+1}(a) = \frac{(-1)^n}{n!} \prod_{k=1}^{n} \left(k + a - \frac{1}{2}\right),$$

which concludes the proof. □

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