A result on the number of cyclic subgroups of a finite group

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Abstract: Let G be a finite group and $\alpha(G)=\frac{|C(G)|}{|G|}$, where $C(G)$ denotes the set of cyclic subgroups of G. In this short note, we prove that $\alpha(G)\leq\alpha(Z(G))$ and we describe the groups G for which the equality occurs. This gives some sufficient conditions for a finite group to be 4-abelian or abelian.

Key words: Finite groups; p-groups; number of cyclic subgroups.

1. Introduction. Let G be a finite group, $C(G)$ be the set of cyclic subgroups of G and $Z(G)$ be the center of G. In 2018, M. Garonzi and I. Lima introduced in their paper [1] the function

$$\alpha(G)=\frac{|C(G)|}{|G|}.$$

Since then many authors have studied the properties of this function and its relations with the structure of G. Note that we have

$$\alpha(G)=\frac{1}{|G|}\sum_{x\in G}\frac{1}{\varphi(o(x))},$$

where φ is Euler's totient function and $o(x)$ is the order of $x\in G$, showing that in fact $\alpha(G)$ depends only on the element orders of G. Denote by $o(G)$ the average order of G, that is

$$o(G)=\frac{1}{|G|}\sum_{x\in G}o(x).$$

By Lemma 2.7 of [4] (see also Corollary 2.6 of [6]), this satisfies the following beautiful inequality

$$o(G)\geq o(Z(G)).$$

Our main result shows that a reversed inequality holds for the function α and gives a description of finite groups G for which the equality occurs.

Theorem 1. *Let G be a finite group. Then*

$$\alpha(G)\leq\alpha(Z(G)),$$

and we have equality if and only if $G\cong G_1\times G_2$, where G_1 is a 2-group with $G_1=\Omega_{\{1\}}(G_1)Z(G_1)$ and G_2 is an abelian group of odd order.

The above condition $G_1=\Omega_{\{1\}}(G_1)Z(G_1)$ means that G_1 has a transversal modulo $Z(G_1)$ consisting only of elements of order 1 or 2. Important examples of such 2-groups are abelian 2-groups and almost extraspecial 2-groups (note that if G_1 is an almost extraspecial 2-group, then $\alpha(G_1)=\alpha(Z(G_1))=\frac{3}{4}$ by Theorem 4 of [5]).

Since a finite group G with $\alpha(G)=\alpha(Z(G))$ satisfies $\exp(G/Z(G))=2$, we infer the following corollary:

Corollary 2. *Every finite group G with $\alpha(G)=\alpha(Z(G))$ is 2-central.*

Also, Theorem 1 of [2] implies that:

Corollary 3. *Every finite group G with $\alpha(G)=\alpha(Z(G))$ is 4-abelian, namely, $(xy)^4=x^4y^4$ holds for all $x,y\in G$. Moreover, if $|G|$ is odd, then G is abelian.*

By $\Omega_{\{i\}}(G)$ we denote the set of elements of order at most p^i of a p-group G. The rest of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [3].

2. Proof of Theorem 1. Let $G/Z(G)=\{a_1Z(G)=Z(G),\ldots,a_mZ(G)\}$, where $m=[G:Z(G)]$. Then

$$|C(G)|=\sum_{x\in G}\frac{1}{\varphi(o(x))}=\sum_{i=1}^{m}\sum_{x\in Z(G)}\frac{1}{\varphi(o(a_ix))}. \tag{1}$$

We will prove that for every $i=2,\ldots,m$ we have

$$\sum_{x\in Z(G)}\frac{1}{\varphi(o(a_ix))}\leq\sum_{x\in Z(G)}\frac{1}{\varphi(o(x))}. \tag{2}$$

Let $k_i=\min\{o(y)\mid y\in a_iZ(G)\}$ and $y_i\in a_iZ(G)$ such that $o(y_i)=k_i$. We infer that

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$$o(y_i x) = \frac{k_i}{(k_i, o(x))}\, o(x) \vdots o(x), \ \forall\, x \in Z(G).$$

This leads to $\varphi(o(x)) \mid \varphi(o(y_i x))$ and so $\varphi(o(x)) \leq \varphi(o(y_i x)), \ \forall\, x \in Z(G)$. Since $a_i Z(G) = y_i Z(G)$, we obtain

$$\sum_{x \in Z(G)} \frac{1}{\varphi(o(a_i x))} = \sum_{x \in Z(G)} \frac{1}{\varphi(o(y_i x))} \leq \sum_{x \in Z(G)} \frac{1}{\varphi(o(x))},$$

as desired.

Clearly, (1) and (2) imply

$$|C(G)| \leq m \sum_{x \in Z(G)} \frac{1}{\varphi(o(x))} = m\,|C(Z(G))|,$$

that is

$$\alpha(G) \leq \alpha(Z(G)).$$

Next, we remark that the equality $\alpha(G) = \alpha(Z(G))$ holds if and only if

$$\varphi(o(y_i x)) = \varphi(o(x)), \ \forall\, i = 2, \ldots, m, \ \forall\, x \in Z(G).$$

By taking $x = 1$, we get $\varphi(k_i) = 1$, i.e. $k_i = 2$. This shows that $G/Z(G)$ is an elementary abelian 2-group. It follows that all Sylow p-subgroups of G for p odd are central and consequently

$$G \cong G_1 \times G_2,$$

where G_1 is a 2-group and G_2 is an abelian group of odd order. Obviously, we have

$$Z(G) \cong Z(G_1) \times G_2 \text{ and}$$
$$y_i \in \Omega_{\{1\}}(G_1), \ \forall\, i = 1, \ldots, m,$$

implying that $G_1 = \Omega_{\{1\}}(G_1) Z(G_1)$.

This completes the proof. □

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