The number of small covers over cubes and the product of at most three simplices up to equivariant cobordism

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Abstract: The equivariant cobordism class of a small cover over a simple convex polytope is determined by its tangential representation set. Since the tangential representation can be identified with the characteristic function of the simple convex polytope, by using characteristic functions we determine the number of small covers over cubes and the product of at most three simplices up to equivariant cobordism.

Key words: Cobordism; small cover; tangential representation.

1. Introduction. A small cover, defined by Davis and Januszkiewicz in [4], is a smooth closed manifold M^n with a locally standard $(\mathbf{Z}_2)^n$–action such that its orbit space is a simple convex polytope. This establishes a direct connection between equivariant topology and combinatorics. In recent years, several studies have attempted to enumerate the number of Davis-Januszkiewicz equivalence classes and equivariant homeomorphism classes of small covers over a specific polytope, see [1–3].

By $\mathcal{M}_n$ we denote the set of equivariant unoriented cobordism classes of all n-dimensional small covers. Let $\mathcal{M}_* = \sum_{n\geq 1}\mathcal{M}_n$. From [6, Theorems 1.4, 1.5, Corollary 5.8], $\mathcal{M}_*$ is generated by the classes of small covers over the products of simplices. Then we consider the following problem.

Problem. How can we determine the number of equivariant cobordism classes of small covers over the product of simplices?

When the dimension of each simplex is 1 or when the number of simplices is at most 3, we answer the above problem. By I^n we denote an n-cube. Let $\Delta_{n_1}, \Delta_{n_2}, \Delta_{n_3}$ be n_1-simplex, n_2-simplex and n_3-simplex respectively. The main results of this paper are stated as follows:

Theorem 1. *All small covers over I^n equivariantly bound.*

Theorem 2. *The number of equivariant cobordism classes of small covers over Δ_{n_1} is*

$$\begin{cases} \dfrac{\prod_{t=1}^{n_1}(2^{n_1}-2^{t-1})}{(n_1+1)!}, & n_1 \geq 3,\\ 1, & n_1 = 1,2. \end{cases}$$

Remark 1. An example of a small cover over Δ_{n_1} is $\mathbf{R}P^{n_1}$ with a standard action T_0 of $(\mathbf{Z}_2)^{n_1}$. In fact, when $n_1 \geq 2$, we may find $\frac{\prod_{t=1}^{n_1}(2^{n_1}-2^{t-1})}{(n_1+1)!}$ different small covers $(\mathbf{R}P^{n_1}, \sigma T_0)$ over Δ_{n_1} with $\sigma \in \mathrm{GL}(n_1, \mathbf{Z}_2)$ up to equivariant cobordism, see [5, Proposition 2.3].

Theorem 3. *The number of equivariant cobordism classes of small covers over $\Delta_{n_1}\times\Delta_{n_2}$ is*

$$\begin{cases} \dfrac{\prod_{t=1}^{n_1+n_2}(2^{n_1+n_2}-2^{t-1})}{(n_1+1)!(n_2+1)!}(2^{n_1}+2^{n_2}-1), & 2 \leq n_1 < n_2,\\ \dfrac{\prod_{t=1}^{2n_1}(2^{2n_1}-2^{t-1})}{2(n_1+1)!^2}(2^{n_1+1}-1), & 2 \leq n_1 = n_2,\\ \dfrac{\prod_{t=1}^{n_2+1}(2^{n_2+1}-2^{t-1})}{2(n_2+1)!}(2^{n_2}-1)+1, & 1 = n_1 < n_2. \end{cases}$$

Theorem 4. *When $2 \leq n_1 \leq n_2 \leq n_3$, the number of equivariant cobordism classes of small covers over $\Delta_{n_1}\times\Delta_{n_2}\times\Delta_{n_3}$ is*

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$$\begin{cases} \dfrac{\prod_{t=1}^{n_1+n_2+n_3}(2^{n_1+n_2+n_3}-2^{t-1})}{(n_1+1)!(n_2+1)!(n_3+1)!} \times (2^{2n_1+n_2}+2^{n_1+2n_2}+2^{2n_1+n_3}+2^{n_1+2n_3} + 2^{2n_2+n_3}+2^{n_2+2n_3}-2^{2n_1}-2^{2n_2} - 2^{2n_3}-2^{n_1+n_2}-2^{n_1+n_3}-2^{n_2+n_3}+1), & 2 \le n_1 < n_2 < n_3, \\ \dfrac{\prod_{t=1}^{2n_2+n_3}(2^{2n_2+n_3}-2^{t-1})}{2(n_2+1)!^2(n_3+1)!} \times (2^{2n_2+n_3+1}+2^{n_2+2n_3+1}+2^{3n_2+1} - 3\cdot 2^{2n_2}-2^{2n_3}-2^{n_2+n_3+1}+1), & 2 \le n_1 = n_2 < n_3, \\ \dfrac{\prod_{t=1}^{n_1+2n_2}(2^{n_1+2n_2}-2^{t-1})}{2(n_1+1)!(n_2+1)!^2} \times (2^{2n_1+n_2+1}+2^{n_1+2n_2+1}+2^{3n_2+1}-2^{2n_1} - 3\cdot 2^{2n_2}-2^{n_1+n_2+1}+1), & 2 \le n_1 < n_2 = n_3, \\ \dfrac{\prod_{t=1}^{3n_2}(2^{3n_2}-2^{t-1})}{6(n_2+1)!^3}(6\cdot 2^{3n_2}-6\cdot 2^{2n_2}+1), & 2 \le n_1 = n_2 = n_3. \end{cases}$$

Theorem 5. *When $2 \le n_2 \le n_3$, the number of equivariant cobordism classes of small covers over $I \times \Delta_{n_2} \times \Delta_{n_3}$ is*

$$\begin{cases} \dfrac{\prod_{t=1}^{n_2+n_3+1}(2^{n_2+n_3+1}-2^{t-1})}{2(n_2+1)!(n_3+1)!} \times (2^{2n_2+n_3}+2^{n_2+2n_3}+2^{2n_2}+2^{2n_3} - 2^{n_2+1}-2^{n_3+1}-2^{n_2+n_3}+1)+1, & 2 \le n_2 < n_3, \\ \dfrac{\prod_{t=1}^{2n_2+1}(2^{2n_2+1}-2^{t-1})}{4(n_2+1)!^2}(2^{3n_2+1}+2^{2n_2}-2^{n_2+2}+1) + 1, & 2 \le n_2 = n_3. \end{cases}$$

Theorem 6. *When $n_3 \ge 2$, the number of equivariant cobordism classes of small covers over $I^2 \times \Delta_{n_3}$ is*

$$\frac{\prod_{t=1}^{n_3+2}(2^{n_3+2}-2^{t-1})}{8(n_3+1)!}(3\cdot 2^{2n_3}-2^{n_3+2}+1)+1.$$

The paper is organized as follows. In Section 2, we review some basic facts about small covers and the tangential representation. In Section 3, using characteristic functions and Stong homomorphism, we prove Theorem 1 and Theorem 3.

2. Preliminaries.

A convex polytope P^n of dimension n is said to be simple if every vertex of P^n is the intersection of exactly n facets (i.e. faces of dimension $(n-1)$). An n-dimensional smooth closed manifold M^n is said to be a small cover if it admits a smooth $(\mathbf{Z}_2)^n$–action such that the action is locally isomorphic to a standard action of $(\mathbf{Z}_2)^n$ on $\mathbf{R}^n$ and the orbit space $M^n/(\mathbf{Z}_2)^n$ is a simple convex polytope of dimension n.

Let P^n be a simple convex polytope of dimension n and $\mathcal{F}(P^n) = \{F_1, \cdots, F_\ell\}$ be the set of facets of P^n. Suppose that $\pi : M^n \to P^n$ is a small cover over P^n. Then there are ℓ connected submanifolds $\pi^{-1}(F_1), \cdots, \pi^{-1}(F_\ell)$. Each submanifold $\pi^{-1}(F_i)$ is fixed pointwise by a $\mathbf{Z}_2$–subgroup $\mathbf{Z}_2(F_i)$ of $(\mathbf{Z}_2)^n$, so that each facet F_i corresponds to the $\mathbf{Z}_2$–subgroup $\mathbf{Z}_2(F_i)$. Obviously, the $\mathbf{Z}_2$–subgroup $\mathbf{Z}_2(F_i)$ actually agrees with an element ν_i in $(\mathbf{Z}_2)^n$ as a vector space. For each face F of codimension u, since P^n is simple, there are u facets $F_{i_1}, \cdots, F_{i_u}$ such that $F = F_{i_1} \cap \cdots \cap F_{i_u}$. Then, the corresponding submanifolds $\pi^{-1}(F_{i_1}), \cdots, \pi^{-1}(F_{i_u})$ intersect transversally in the $(n-u)$-dimensional submanifold $\pi^{-1}(F)$, and the isotropy subgroup $\mathbf{Z}_2(F)$ of $\pi^{-1}(F)$ is a subtorus of rank u and is generated by $\mathbf{Z}_2(F_{i_1}), \cdots, \mathbf{Z}_2(F_{i_u})$ (or is determined by $\nu_{i_1}, \cdots, \nu_{i_u}$ in $(\mathbf{Z}_2)^n$). Thus, this actually gives a characteristic function [4]

$$\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbf{Z}_2)^n$$

defined by $\lambda(F_i) = \nu_i$ such that whenever the intersection $F_{i_1} \cap \cdots \cap F_{i_u}$ is non-empty, $\lambda(F_{i_1}), \cdots, \lambda(F_{i_u})$ are linearly independent in $(\mathbf{Z}_2)^n$.

In fact, Davis and Januszkiewicz gave a reconstruction process of a small cover by using a characteristic function $\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbf{Z}_2)^n$. Let $\mathbf{Z}_2(F_i)$ be the subgroup of $(\mathbf{Z}_2)^n$ generated by $\lambda(F_i)$. Given a point $p \in P^n$, by $F(p)$ we denote the minimal face containing p in its relative interior. Assume $F(p) = F_{i_1} \cap \cdots \cap F_{i_u}$ and $\mathbf{Z}_2(F(p)) = \bigoplus_{j=1}^u \mathbf{Z}_2(F_{i_j})$. Note that $\mathbf{Z}_2(F(p))$ is a u-dimensional subgroup of $(\mathbf{Z}_2)^n$. Let $M(\lambda)$ denote $P^n \times (\mathbf{Z}_2)^n/\sim$, where $(p, g) \sim (q, h)$ if $p = q$ and $g^{-1}h \in \mathbf{Z}_2(F(p))$. The free action of $(\mathbf{Z}_2)^n$ on $P^n \times (\mathbf{Z}_2)^n$ descends to an action on $M(\lambda)$ with quotient P^n. Thus $M(\lambda)$ is a small cover over P^n [4].

By $\Lambda(P^n)$ we denote the set of all characteristic functions on P^n. Then we have

Theorem 2.1. *Let $\pi : M^n \to P^n$ be a small cover over a simple convex polytope P^n. Then all small covers over P^n are given by $\{M(\lambda)|\lambda \in \Lambda(P^n)\}$ from the viewpoint of cobordism.*

Remark 2. Generally speaking, we can't make sure that there always exist small covers over a simple convex polytope P^n when $n \geq 4$. For example, see [4, Nonexample 1.22]. From [4, Example 1.1], we know that the n-dimensional torus T^n is a small cover over I^n. From [4, Example 1.2], there also exists a small cover $\mathbf{R}P^{n_i}$ over Δ_{n_i} for $i = 1, 2, 3$, and the product of these projective spaces is a small cover over the product of corresponding simplices.

Next we recall some results in [5]. Let $G = (\mathbf{Z}_2)^n$ and ρ_0 the trivial element in $\mathrm{Hom}(G, \mathbf{Z}_2)$ (the set of all homomorphisms from G to $\mathbf{Z}_2$). The irreducible real G-representations are all one-dimensional and correspond to all elements in $\mathrm{Hom}(G, \mathbf{Z}_2)$. Given an element β of $\mathcal{M}_n$, let (M^n, ϕ) be a representative of β such that M^n is a small cover. Take an isolated point p in the fixed point set $(M^n)^G$, then the G-representation at p can be written as $\tau_p(M^n) = \bigoplus_{\rho \neq \rho_0} \lambda_\rho^{q_\rho}$, where $\lambda_\rho : G \times \mathbf{R} \longrightarrow \mathbf{R}$, $(g, x) \mapsto \rho(g) \cdot x$ with $\rho \in \mathrm{Hom}(G, \mathbf{Z}_2)$ is the irreducible real G-representation and $\Sigma_{\rho \neq \rho_0} q_\rho = n$ and if $q_\rho \neq 0$, then $q_\rho = 1$. $\mathcal{N}_{M^n} = \{[\tau_p(M^n)] | p \in (M^n)^G\}$ is called the tangential representation set of (M^n, ϕ), where by $[\tau_p(M^n)]$ we denote the isomorphism class of $\tau_p(M^n)$.

The homomorphisms $\rho_i : (g_1, \cdots, g_n) \longmapsto g_i$ form a standard basis of $\mathrm{Hom}(G, \mathbf{Z}_2)$. Let $R_n(G)$ denote the vector space over $\mathbf{Z}_2$ generated by the representation classes of dimension n. Then $R_*(G) = \sum_{n \geq 0} R_n(G)$ is isomorphic to the graded polynomial algebra $\mathbf{Z}_2[\rho_1, \cdots, \rho_n]$. Each $[\tau_p(M^n)]$ of $\mathcal{N}_{M^n}$ uniquely corresponds to a monomial of degree n in $\mathbf{Z}_2[\rho_1, \cdots, \rho_n]$ such that all n factors of the monomial form a basis of $\mathrm{Hom}(G, \mathbf{Z}_2)$. In [7], Stong showed that the natural homomorphism (Stong homomorphism) $\delta_n : \mathcal{M}_n \longrightarrow R_n(G)$ defined by

$$\delta_n([M^n, \phi]) = \sum_{p \in (M^n)^G} [\tau_p(M^n)]$$

is a monomorphism. This implies that for each β in $\mathcal{M}_n$, there exists a representative (M^n, ϕ) of β such that $\mathcal{N}_{M^n}$ is prime (i.e. either all elements of $\mathcal{N}_{M^n}$ are distinct or $\mathcal{N}_{M^n}$ is empty) and $\mathcal{N}_{M^n}$ is independent of the choice of representatives of β. Thus we can define $\mathcal{N}_\beta := \mathcal{N}_{M^n}$. Obviously we have $\beta_1 = \beta_2 \Longleftrightarrow \mathcal{N}_{\beta_1} = \mathcal{N}_{\beta_2}$, for $\beta_1, \beta_2 \in \mathcal{M}_n$.

Let $\pi : M^n \to P^n$ be a small cover over a simple convex polytope P^n. The set of the vertices of P^n is just the image of $(M^n)^G$ under the map π. Let E denote an edge (1-dimensional face) of P^n, then $\pi^{-1}(E)$ is a connected 1-dimensional G-submanifold of M^n by [4, Lemma 1.3]. For $p \in (M^n)^G$ and $\pi(p) \in E$, p is also a fixed point of this submanifold. We have a 1-dimensional real tangential representation $\tau_p(\pi^{-1}(E))$ of G at p. Suppose that $E_{i_1}, \cdots, E_{i_n}$ are the n edges that meet at $\pi(p)$. Then $\bigoplus_{k=1}^n \tau_p(\pi^{-1}(E_{i_k}))$ just gives $\tau_p(M^n)$. The isotropy group of $\pi^{-1}(E)$ is of rank n-1. Thus the tangential representation $\tau_p(\pi^{-1}(E))$ is determined by the vector orthogonal to the isotropy group (regarded as a subspace of $(\mathbf{Z}_2)^n$). Each edge is the intersection of n-1 facets. Suppose $E = \bigcap_{k=1}^{n-1} F_{j_k}$, where F_{j_k} denotes a facet. The vectors $\lambda(F_{j_k})$, $k = 1, \cdots, n-1$, span the isotropy group of $\pi^{-1}(E)$. So the characteristic function uniquely determines the tangential representation $\tau_p(M^n)$.

3. The number of small covers.

We only prove Theorem 1 and Theorem 3 because in the similar way we can give the proofs of Theorems 2, 4, 5. In fact, $I^n = \underbrace{I \times I \times \cdots \times I}_{n}$. Let a_{11}, a_{12} be two vertices of the first I-factor, a_{21}, a_{22} be two vertices of the second I-factor, $\cdots$, and a_{n1}, a_{n2} be two vertices of the last I-factor. Let $F_1 = a_{11} \times I^{n-1}, F_2 = a_{12} \times I^{n-1}$, $F_3 = I \times a_{21} \times I^{n-2}$, $F_4 = I \times a_{22} \times I^{n-2}, \cdots$, $F_{2n-1} = I^{n-1} \times a_{n1}$, and $F_{2n} = I^{n-1} \times a_{n2}$. Then $\mathcal{F}(I^n) = \{F_1, F_2, F_3, F_4, \cdots, F_{2n-1}, F_{2n}\}$. Using characteristic functions and Stong homomorphism, we first give the proof of Theorem 1.

The proof of Theorem 1. Let $e_1, e_2, \cdots, e_n$ be the standard basis of $(\mathbf{Z}_2)^n$. We choose F_1, $F_3, \cdots, F_{2n-1}$ from $\mathcal{F}(I^n)$ such that they meet at one vertex of I^n. Without loss of generality, let $\lambda(F_{2h-1}) = e_h, 1 \leq h \leq n$. By the linear independence condition of characteristic functions, we have $\lambda(F_2) = e_1$ or $e_1 + e_{k_1} + \cdots + e_{k_i}$, $2 \leq k_1 < \cdots < k_i \leq n, 1 \leq i \leq n-1$. But when $\lambda(F_2) = e_1$, by Stong homomorphism, we see that the small cover constructed from such λ equivariantly bounds. Here we only consider non-bounding small covers. Thus $\lambda(F_2) = e_1 + e_{k_1} + \cdots + e_{k_i}$, $2 \leq k_1 < \cdots < k_i \leq n$, $1 \leq i \leq n-1$. Since $k_1 \geq 2$, Without loss of generality, suppose $k_1 = 2$ (if $k_1 > 2$, we may consider $\lambda(F_{2k_1})$ in the same way). In this case, by the linear independence condition of characteristic functions and Stong homomorphism, we have $\lambda(F_4) = e_2 + e_{t_1} + \cdots + e_{t_j}$, $3 \leq t_1 < \cdots < t_j \leq n$, $1 \leq j \leq n-2$. Since $t_1 \geq 3$, we may suppose $t_1 = 3$. In the similar way, we have $\lambda(F_6) = e_3 + e_{l_1} + \cdots + e_{l_m}$, $4 \le l_1 < \cdots < l_m \le n$, $1 \le m \le n-3$. We continue in the above way. Finally, we have $\lambda(F_{2n}) = e_n$. By Stong homomorphism, small covers constructed from these λ equivariantly bound. If we choose other basis of $(\mathbf{Z}_2)^n$, the result is the same as above. Thus, all small covers over I^n equivariantly bound. $\square$

For convenience, we introduce the following marks. By $F'_1, \cdots, F'_{n_1+1}$ we denote all facets of n_1-simplex Δ_{n_1}, and by $F'_{n_1+2}, \cdots, F'_{n_1+n_2+2}$ we denote all facets of n_2-simplex Δ_{n_2}. Set $\mathcal{F}' = \{F_i = F'_i \times \Delta_{n_2} | 1 \le i \le n_1+1\}$, and $\mathcal{F}'' = \{F_i = \Delta_{n_1} \times F'_i | n_1+2 \le i \le n_1+n_2+2\}$. Then $\mathcal{F}(\Delta_{n_1} \times \Delta_{n_2}) = \mathcal{F}' \bigcup \mathcal{F}''$.

The proof of Theorem 3. Let $e_1, \cdots, e_{n_1+n_2}$ be the standard basis of $(\mathbf{Z}_2)^{n_1+n_2}$. For $\Delta_{n_1} \times \Delta_{n_2}$ with $2 \le n_1 < n_2$, we choose $F_1, \cdots, F_{n_1}$ from $\mathcal{F}'$ and $F_{n_1+2}, \cdots, F_{n_1+n_2+1}$ from $\mathcal{F}''$ such that $F_1, \cdots, F_{n_1}$, $F_{n_1+2}, \cdots, F_{n_1+n_2+1}$ meet at one vertex of $\Delta_{n_1} \times \Delta_{n_2}$. Without loss of generality, let $\lambda(F_i) = e_i$, $1 \le i \le n_1; \lambda(F_i) = e_{i-1}$, $n_1+2 \le i \le n_1+n_2+1$. By the linear independence condition of characteristic functions, we have $\lambda(F_{n_1+n_2+2}) = e_{n_1+1} + \cdots + e_{n_1+n_2} + e_{k_1} + \cdots + e_{k_i}$, $1 \le k_1 < \cdots < k_i \le n_1$, $0 \le i \le n_1$. When $\lambda(F_{n_1+n_2+2}) = e_{n_1+1} + \cdots + e_{n_1+n_2}$, $\lambda(F_{n_1+1}) = e_1 + \cdots + e_{n_1} + e_{t_1} + \cdots + e_{t_j}$, $n_1+1 \le t_1 < \cdots < t_j \le n_1+n_2$, $0 \le j \le n_2$. When $\lambda(F_{n_1+n_2+2}) = e_{n_1+1} + \cdots + e_{n_1+n_2} + e_{k_1} + \cdots + e_{k_i}$, $1 \le k_1 < \cdots < k_i \le n_1$, $1 \le i \le n_1$, we have $\lambda(F_{n_1+1}) = e_1 + \cdots + e_{n_1}$. So the values of λ have $2^{n_1} + 2^{n_2} - 1$ possible choices. There are $\frac{\prod_{t=1}^{n_1+n_2}(2^{n_1+n_2} - 2^{t-1})}{(n_1+1)!(n_2+1)!}$ choices for a basis of $(\mathbf{Z}_2)^{n_1+n_2}$ if we consider equivariant cobordism classification by Stong homomorphism. Thus, there are $\frac{\prod_{t=1}^{n_1+n_2}(2^{n_1+n_2} - 2^{t-1})}{(n_1+1)!(n_2+1)!}(2^{n_1} + 2^{n_2} - 1)$ non-bounding small covers over $\Delta_{n_1} \times \Delta_{n_2}$ for $2 \le n_1 < n_2$.

For $\Delta_{n_1} \times \Delta_{n_2}$ with $2 \le n_1 = n_2$, let $\lambda(F_i) = e_i$, $1 \le i \le n_1; \lambda(F_i) = e_{i-1}$, $n_1+2 \le i \le n_1+n_2+1$. Using the above method, we see that the values of λ have $2^{n_1+1} - 1$ possible choices. But there are $\frac{\prod_{t=1}^{2n_1}(2^{2n_1} - 2^{t-1})}{2(n_1+1)!^2}$ choices for a basis of $(\mathbf{Z}_2)^{2n_1}$ in this case. Thus, there are $\frac{\prod_{t=1}^{2n_1}(2^{2n_1} - 2^{t-1})}{2(n_1+1)!^2}(2^{n_1+1} - 1)$ non-bounding small covers over $\Delta_{n_1} \times \Delta_{n_2}$ for $2 \le n_1 = n_2$.

For $\Delta_1 \times \Delta_{n_2}$ with $n_2 > 1$, $\mathcal{F}' = \{F_1, F_2\}, \mathcal{F}'' = \{F_3, \cdots, F_{n_2+3}\}$ and $\mathcal{F}(\Delta_1 \times \Delta_{n_2}) = \mathcal{F}' \bigcup \mathcal{F}''$. Let $\lambda(F_1) = e_1, \lambda(F_i) = e_{i-1}, 3 \le i \le n_2+2$. By the linear independence condition of characteristic functions, we have $\lambda(F_{n_2+3}) = e_2 + \cdots + e_{n_2+1}$ or $e_2 + \cdots + e_{n_2+1} + e_1$. When $\lambda(F_{n_2+3}) = e_2 + \cdots + e_{n_2+1}$, $\lambda(F_2) = e_1$ or $e_1 + e_{k_1} + \cdots + e_{k_i}$, $2 \le k_1 < \cdots < k_i \le n_2+1$, $1 \le i \le n_2$. But when $\lambda(F_{n_2+3}) = e_2 + \cdots + e_{n_2+1}$ and $\lambda(F_2) = e_1$, by Stong homomorphism, the small cover constructed from such λ equivariantly bounds. Here we only consider non-bounding small covers. Then when $\lambda(F_{n_2+3}) = e_2 + \cdots + e_{n_2+1}$, $\lambda(F_2) = e_1 + e_{k_1} + \cdots + e_{k_i}$, $2 \le k_1 < \cdots < k_i \le n_2+1$, $1 \le i \le n_2$. When $\lambda(F_{n_2+3}) = e_2 + \cdots + e_{n_2+1} + e_1$, we have $\lambda(F_2) = e_1$. Similarly using Stong homomorphism, the small cover constructed from such λ equivariantly bounds. So the values of λ have $2^{n_2} - 1$ possible choices. In this case, there are $\frac{\prod_{t=1}^{n_2+1}(2^{n_2+1} - 2^{t-1})}{2(n_2+1)!}$ choices for a basis of $(\mathbf{Z}_2)^{n_2+1}$ if we consider equivariant cobordism classification by Stong homomorphism. Thus, there are $\frac{\prod_{t=1}^{n_2+1}(2^{n_2+1} - 2^{t-1})}{2(n_2+1)!}(2^{n_2} - 1)$ non-bounding small covers over $\Delta_1 \times \Delta_{n_2}$. Adding the small cover that equivariantly bounds, we determine the number of small covers over $\Delta_1 \times \Delta_{n_2}$ up to equivariant cobordism for $n_2 > 1$. $\square$

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