Congruences for higher-order Euler numbers

By Guodong Liu

Department of Mathematics, Huizhou University, Huizhou, Guangdong, 516015, P. R. China
(Communicated by Shigefumi Mori, M. J. A., March 13, 2006)

Abstract: In this paper, we prove some congruences for higher-order Euler numbers.

Key words: Higher-order Euler numbers; Euler numbers, congruences.

1. Introduction and results. For an integer k, the Euler number $E_{2n}^{(k)}$ of order k (the index k may be negative) is defined by (see [2, 5])

$$(\sec x)^k = \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(k)} \frac{x^{2n}}{(2n)!}, \tag{1.1}$$

or equivalently

$$\left(\frac{2}{e^x + e^{-x}}\right)^k = \sum_{n=0}^{\infty} E_{2n}^{(k)} \frac{x^{2n}}{(2n)!}. \tag{1.2}$$

The numbers $E_{2n}^{(1)} = E_{2n}$ are the ordinary Euler numbers. By (1.1) or (1.2), we can get

(1.3)
$$E_{2n}^{(k)} = (2n)! \sum_{\substack{v_1+v_2+\cdots+v_k=n \\ v_1\geq 0, v_2\geq 0,\ldots,v_k\geq 0}} \frac{E_{2v_1}E_{2v_2}\cdots E_{2v_k}}{(2v_1)!(2v_2)!\cdots(2v_k)!}$$

when k is positive.

The Euler numbers E_{2n} satisfy the recurrence relation

$$E_0 = 1, \quad E_{2n} = -\sum_{k=0}^{n-1} \binom{2n}{2k} E_{2k}. \tag{1.4}$$

By induction, all the Euler numbers $E_0, E_2, E_4, \ldots$ are integers. By (1.3), we know $E_{2n}^{(k)}$ is an integer.

Recently, several researchers have studied the congruences for Euler numbers. For example:

In [7], Wenpeng Zhang obtained an interesting congruence for Euler numbers,

$$E_{p-1} \equiv 1 + (-1)^{(p+1)/2} \pmod{p}. \tag{1.5}$$

where p is any odd prime.

In [4], Guodong Liu obtained an congruence for Euler numbers,

$$\sum_{k=1}^{(p-1)/2} E_{2n+2k} \equiv -1 \pmod{p}. \tag{1.6}$$

where $n \geq 0$ is integer and p is any odd prime.

The following conjecture is on Euler numbers (see [1] B45).

Conjecture. *For any prime $p \equiv 1 \pmod 4$, the congruence $E_{(p-1)/2} \not\equiv 0 \pmod p$ holds.*

In [3], Guodong Liu proved the conjecture for $p \equiv 5 \pmod 8$. In [6], Pingzhi Yuan, using a result of [3] and the class number formula for the quadratic field with negative discriminant, proved the above conjecture.

The main purpose of this paper is to prove some congruences for higher-order Euler numbers. That is, we shall prove the following main conclusion.

Theorem 1. *Let $n \geq 0, r \geq 3$ be integers, p be any odd prime. If $r \equiv 2k+1 \pmod p$, where $1 \leq k \leq (p-1)/2$. Then*

$$\sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(r)} \equiv 0 \pmod{p}. \tag{1.7}$$

Theorem 2. *Let $n \geq 0, r \geq 2$ be integers, p be any odd prime. If $r \equiv 2k+2 \pmod p$, where $0 \leq k \leq (p-1)/2$. Then*

(1.8)
$$\sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(r)} \equiv \frac{(-1)^{(p+1)/2}}{2^{2k}} \binom{2k}{k} \pmod{p}.$$

Taking $r = 2$ in Theorem 2, we may immediately deduce the following

Corollary 1. *Let $n \geq 0$ be any integers, p be any odd prime. Then we have*

$$\sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(2)} \equiv (-1)^{(p+1)/2} \pmod{p}. \tag{1.9}$$

Setting $p = 3, 5, 7, 11$ in Corollary 1, we can get

2000 Mathematics Subject Classification. Primary 11B68; Secondary 11A07.

(1.10) $E_{2n+2}^{(2)} \equiv 1 \pmod 3$.

(1.11) $E_{2n+2}^{(2)} + E_{2n+4}^{(2)} \equiv -1 \pmod 5$.

(1.12) $E_{2n+2}^{(2)} + E_{2n+4}^{(2)} + E_{2n+6}^{(2)} \equiv 1 \pmod 7$.

(1.13) $E_{2n+2}^{(2)} + E_{2n+4}^{(2)} + E_{2n+6}^{(2)} + E_{2n+8}^{(2)} + E_{2n+10}^{(2)} \equiv 1 \pmod{11}$.

2. Some lemmas.

Lemma 1. *Let $n \geq 1$, $k \geq 1$ be integers. Then we have*

$$E_{2n}^{(k)} \equiv 0 \pmod k. \tag{2.1}$$

Proof. By (1.1), we have

$$\sum_{n=1}^{\infty}(-1)^n E_{2n}^{(k)} \frac{x^{2n-1}}{(2n-1)!} = k(\sec x)^k \tan x. \tag{2.2}$$

By $(\sec x)^2 = \sum_{n=0}^{\infty}(-1)^n E_{2n}^{(2)}(x^{2n}/((2n)!)) = \sum_{n=1}^{\infty}(-1)^{n-1} E_{2n-2}^{(2)}(x^{2n-2}/((2n-2)!))$, we get

$$\tan x = \sum_{n=1}^{\infty}(-1)^{n-1} E_{2n-2}^{(2)} \frac{x^{2n-1}}{(2n-1)!}. \tag{2.3}$$

By (2.2) and (2.3), we have

$$\begin{aligned} &\sum_{n=1}^{\infty}(-1)^n E_{2n}^{(k)} \frac{x^{2n-1}}{(2n-1)!} \\ &= k \sum_{n=0}^{\infty}(-1)^n E_{2n}^{(k)} \frac{x^{2n}}{(2n)!} \times \sum_{n=1}^{\infty}(-1)^{n-1} E_{2n-2}^{(2)} \frac{x^{2n-1}}{(2n-1)!} \\ &= k \sum_{n=1}^{\infty}(-1)^{n-1} \sum_{i=0}^{n-1}\binom{2n-1}{2i} \times E_{2i}^{(k)} E_{2n-2i-2}^{(2)} \frac{x^{2n-1}}{(2n-1)!}. \end{aligned} \tag{2.4}$$

Comparing the coefficients of x^{2n-1} on both sides of (2.4), we get

$$E_{2n}^{(k)} = -k \sum_{i=0}^{n-1}\binom{2n-1}{2i} E_{2i}^{(k)} E_{2n-2i-2}^{(2)} \equiv 0 \pmod k. \tag{2.5}$$

This completes the proof of Lemma 1. □

Lemma 2. *Let $n \geq 0, k \geq 1, m \geq 1$ be integers. Then we have*

$$E_{2n}^{(k+m)} \equiv E_{2n}^{(k)} \pmod m. \tag{2.6}$$

Proof. By (1.1), we have

$$\begin{aligned} &\sum_{n=0}^{\infty}(-1)^n E_{2n}^{(k+m)} \frac{x^{2n}}{(2n)!} = (\sec x)^{k+m} \\ &= (\sec x)^k (\sec x)^m \\ &= \left(\sum_{n=0}^{\infty}(-1)^n E_{2n}^{(k)} \frac{x^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty}(-1)^n E_{2n}^{(m)} \frac{x^{2n}}{(2n)!}\right) \\ &= \sum_{n=0}^{\infty}(-1)^n \sum_{j=0}^{n}\binom{2n}{2j} E_{2j}^{(k)} E_{2n-2j}^{(m)} \frac{x^{2n}}{(2n)!}. \end{aligned} \tag{2.7}$$

Comparing the coefficients of x^{2n} on both sides of (2.7), we get

$$\begin{aligned} E_{2n}^{(k+m)} &= \sum_{j=0}^{n}\binom{2n}{2j} E_{2n-2j}^{(k)} E_{2j}^{(m)} \\ &= E_{2n}^{(k)} + \sum_{j=1}^{n}\binom{2n}{2j} E_{2n-2j}^{(k)} E_{2j}^{(m)}. \end{aligned} \tag{2.8}$$

By (2.8) and Lemma 1, we have

$$E_{2n}^{(k+m)} \equiv E_{2n}^{(k)} \pmod m. \tag{2.9}$$

This completes the proof of Lemma 2. □

Lemma 3. *Let $n \geq 1, k \geq 1, m \geq 0$ be integers. Then we have*

$$E_{2n}^{(k)} \equiv \frac{1}{2^m} \sum_{i=0}^{m}\binom{m}{i}(m-2i)^{2n} \pmod{(m+k)}. \tag{2.10}$$

Proof. By (1.1), we have

$$\begin{aligned} &\sum_{n=0}^{\infty}(-1)^n E_{2n}^{(k)} \frac{x^{2n}}{(2n)!} = (\sec x)^k \\ &= (\sec x)^{m+k} (\sec x)^{-m} \\ &= \left(\sum_{n=0}^{\infty}(-1)^n E_{2n}^{(m+k)} \frac{x^{2n}}{(2n)!}\right) \times \left(\sum_{n=0}^{\infty}(-1)^n E_{2n}^{(-m)} \frac{x^{2n}}{(2n)!}\right) \\ &= \sum_{n=0}^{\infty}(-1)^n \sum_{j=0}^{n}\binom{2n}{2j} E_{2j}^{(m+k)} E_{2n-2j}^{(-m)} \frac{x^{2n}}{(2n)!}. \end{aligned} \tag{2.11}$$

Comparing the coefficients of x^{2n} on both sides of (2.11), we get

$$E_{2n}^{(k)} = \sum_{j=0}^{n} \binom{2n}{2j} E_{2j}^{(m+k)} E_{2n-2j}^{(-m)} = E_{2n}^{(-m)} + \sum_{j=1}^{n} \binom{2n}{2j} E_{2j}^{(m+k)} E_{2n-2j}^{(-m)}. \tag{2.12}$$

By (2.12) and Lemma 1, we have

$$E_{2n}^{(k)} \equiv E_{2n}^{(-m)} \pmod{(m+k)}. \tag{2.13}$$

On the other hand, by (1.2), we have

$$\sum_{n=0}^{\infty} E_{2n}^{(-m)} \frac{x^{2n}}{(2n)!} = \left(\frac{e^x + e^{-x}}{2}\right)^m = 2^{-m} \sum_{i=0}^{m} \binom{m}{i} e^{(m-2i)x} = 2^{-m} \sum_{i=0}^{m} \binom{m}{i} \sum_{n=0}^{\infty} (m-2i)^n \frac{x^n}{n!}. \tag{2.14}$$

Comparing the coefficients of x^{2n} on both sides of (2.14), we get

$$E_{2n}^{(-m)} = 2^{-m} \sum_{i=0}^{m} \binom{m}{i} (m-2i)^{2n}. \tag{2.15}$$

By (2.13) and (2.15), we immediately obtain (2.10). This completes the proof of Lemma 3. □

3. Proof of the theorems.

Proof of Theorem 1. By Lemma 2 and Lemma 3, we have

$$\begin{aligned}
\sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(r)} &\equiv \sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(2k+1)} \\
&\equiv \frac{1}{2^{p-2k-1}} \sum_{j=0}^{p-2k-1} \binom{p-2k-1}{j} \times \sum_{i=1}^{(p-1)/2} (p-2k-1-2j)^{2n+2i} \\
&\equiv \frac{1}{2^{p-2k-1}} \sum_{j=0}^{p-2k-1} \binom{p-2k-1}{j} \times (p-2k-1-2j)^{2n+2} \\
&\quad \times \frac{(p-2k-1-2j)^{p-1}-1}{(p-2k-1-2j)^2-1} \equiv 0 \pmod{p}.
\end{aligned} \tag{3.1}$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. By Lemma 2 and Lemma 3, we have

$$\begin{aligned}
\sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(r)} &\equiv \sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(2k+2)} \\
&\equiv \frac{1}{2^{p-2k-2}} \sum_{j=0}^{p-2k-2} \binom{p-2k-2}{j} \times \sum_{i=1}^{(p-1)/2} (p-2k-2-2j)^{2n+2i} \\
&= \frac{1}{2^{p-2k-2}} \sum_{\substack{j=0 \\ p-2k-2-2j=\pm 1}}^{p-2k-2} \binom{p-2k-2}{j} \times \sum_{i=1}^{(p-1)/2} (p-2k-2-2j)^{2n+2i} \\
&\quad + \frac{1}{2^{p-2k-2}} \sum_{\substack{j=0 \\ p-2k-2-2j\neq \pm 1}}^{p-2k-2} \binom{p-2k-2}{j} \times \sum_{i=1}^{(p-1)/2} (p-2k-2-2j)^{2n+2i} \\
&= \frac{p-1}{2^{p-2k-1}} \times \left(\binom{p-2k-2}{(p-2k-1)/2} + \binom{p-2k-2}{(p-2k-3)/2}\right) \\
&\quad + \frac{1}{2^{p-2k-2}} \sum_{\substack{j=0 \\ p-2k-2-2j\neq \pm 1}}^{p-2k-2} \binom{p-2k-2}{j} \times (p-2k-2-2j)^{2n+2} \\
&\quad \times \frac{(p-2k-2-2j)^{p-1}-1}{(p-2k-2-2j)^2-1} \\
&\equiv -2^{2k} \left(\binom{p-2k-2}{(p-2k-1)/2} + \binom{p-2k-2}{(p-2k-3)/2}\right) \\
&\equiv -2^{2k} \binom{p-2k-1}{(p-2k-1)/2} \\
&\equiv \frac{(-1)^{(p+1)/2}}{2^{2k}} \binom{2k}{k} \pmod{p}.
\end{aligned} \tag{3.2}$$

This completes the proof of Theorem 2. □

Acknowledgements. The author would like to express his gratitude to the anonymous referee for valuable suggestions and corrections which have improved the original manuscript.

This work was supported by the Guangdong Provincial Natural Science Foundation of China (No. 05005928).

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